NOTES ON DOUBLY WARPED AND DOUBLY TWISTED
PRODUCT CR-SUBMANIFOLDS OF KAehler MANIFOLDS

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Abstract. Recently, B. Y. Chen studied warped product CR-submanifolds [7] and twisted
product CR-submanifolds [6] in Kaehler manifolds. In this paper, we have checked the existence
of other product CR-submanifolds such as doubly warped and doubly twisted products in Kaehler
manifolds.

1. Introduction

Warped product manifolds were introduced by R. L. Bishop and B. O’Neill in
[2] to construct new examples of negatively curved manifolds. Later, this notion
has been generalized in several ways. Let \((B, g_B)\) and \((F, g_F)\) be semi-Riemannian
manifolds of dimensions \(m\) and \(n\), respectively and let \(\pi : B \times F \to B\) and \(\sigma : B \times F \to F\) be the canonical projections. Also let \(b : B \times F \to (0, \infty)\), \(f : B \times F \to (0, \infty)\) be smooth functions. Then the doubly twisted product ([10], [15]) of \((B, g_B)\)
and \((F, g_F)\) with twisting functions \(b\) and \(f\) is defined to be the product manifold
\[ M = B \times F \]
with metric tensor \(g = f^2 g_B \oplus b^2 g_F\). We denote this kind manifolds
by \(fB \times_b F\). Denote by \(F(B)\) the algebra of smooth functions on \(B\) and by \(\Gamma(E)\)
the \(F(B)\) module of smooth sections of a vector bundle \(E\) (same notation for any
other bundle) over \(B\). If \(X \in \Gamma(TB)\) and \(V \in \Gamma(TF)\), then from Proposition 1 of
[10], we have

\[ \nabla_X V = V(\ln f)X + X(\ln b)V, \]

where \(\nabla\) denotes the Levi-Civita connection of the doubly twisted product \(fB \times_b F\)
of \((B, g_B)\) and \((F, g_F)\). In particular, if \(f = 1\), then \(B \times_b F\) is called the twisted
product of \((B, g_B)\) and \((F, g_F)\) with twisting function \(b\). We note that the notion
of twisted products was introduced in [5]. If \(M = B \times_b F\) is a twisted product
manifold, then (1.1) becomes

\[ \nabla_X V = X(\ln b)V. \]

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205
Moreover, if \( b \) only depends on the points of \( B \), then \( B \times_b F \) is called warped product of \((B, g_B)\) and \((F, g_F)\) with warping function \( b \). In this case, for \( X \in \Gamma(TB) \) and \( V \in \Gamma(TF) \), from Lemma 7.3 of [2], we have
\[
\nabla_X V = X(lnb)V, \tag{1.3}
\]
where \( b \) depends on the points of \( B \) and \( X \in \Gamma(TB), V \in \Gamma(TF) \).

As a generalization of the warped product of two semi-Riemannian manifolds, \( fB \times_b F \) is called the doubly warped product of semi-Riemannian manifolds \((B, g_B)\) and \((F, g_F)\) with warping functions \( b \) and \( f \) if only depend on the points of \( B \) and \( F \), respectively.

In [7], B. Y. Chen considered warped product CR-submanifolds of Kaehler manifolds and showed that there exist no warped product CR-submanifolds in the form \( M \perp \times_f M^T \), where \( M \perp \) is a totally real submanifold and \( M^T \) is a holomorphic submanifold of a Kaehler manifold \( \bar{M} \). Then he introduced CR-warped products which are warped product CR-submanifolds in the form \( M^T \times_f M \perp \) such that \( M^T \) is a holomorphic submanifold and \( M \perp \) is a totally real submanifold of \( \bar{M} \). CR-warped products (or CR-products) have been also studied in [3], [4], [8], [9], [11], [12], [13], [14].

B. Y. Chen [6] also introduced twisted product CR-submanifolds in Kaehler manifolds and showed that a twisted product CR-submanifold in the form \( M \perp \times_\lambda M^T \) is a CR-product. Then he considered twisted product CR-submanifolds in the form \( M^T \times_\lambda M \perp \) and established a general sharp inequality for twisted product CR-submanifolds in Kaehler manifolds.

In this paper, we investigate the existence of other product (doubly warped and doubly twisted product) CR-submanifolds in Kaehler manifolds. In fact, we show that there do not exist doubly warped and doubly twisted product CR-submanifolds in Kaehler manifolds.

2. Preliminaries

Let \( (\bar{M}, g) \) be a Kaehler manifold. This means that \( \bar{M} \) admits a tensor field \( J \) of type \((1,1)\) on \( \bar{M} \) such that, \( \forall X, Y \in \Gamma(T\bar{M}) \), we have
\[
J^2 = -I, \quad g(X, Y) = g(JX, JY), \quad (\nabla_X J)Y = 0 \tag{2.1}
\]
where \( g \) is Riemannian metric and \( \nabla \) is the Levi-Civita connection on \( \bar{M} \).

Let \( \bar{M} \) be a Kaehler manifold with complex structure \( J \) and \( M \) is a Riemannian manifold isometrically immersed in \( \bar{M} \). Then \( M \) is called holomorphic (complex) if \( J(T_p M) \subset T_p M \), for every \( p \in M \), where \( T_p M \) denotes the tangent space to \( M \) at the point \( p \). \( M \) is called totally real if \( J(T_p M) \subset T_p M^\perp \) for every \( p \in M \), where \( T_p M^\perp \) denotes the normal space to \( M \) at the point \( p \). In 1978, A. Bejancu introduced [1] a new class of submanifolds of Kaehler manifolds as follows. A submanifold \( M \) is called a CR-submanifold if there exists on \( M \) a differentiable distribution \( D : p \to D_p \subset T_p M \) such that \( D \) is invariant with respect to \( J \) and
its orthogonal complement $D^\perp$ is totally real distribution, i.e, $J(D^\perp_p) \subseteq T_pM^\perp$. Obviously, holomorphic and totally real submanifolds are CR-submanifolds having $D_p = T_pM$ and $D_p = 0$, respectively. A CR-submanifold is called proper if it is neither holomorphic nor totally real.

Let $M$ be a Riemannian manifold isometrically immersed in $\bar{M}$ and denote by the same symbol $g$ the Riemannian metric induced on $M$. Let $TM$ be the Lie algebra of vector fields in $M$ and $T^\perp M$ the set of all vector fields normal to $M$. Denote by $\nabla$ the Levi-Civita connection of $M$. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N$$

for any $X, Y \in TM$ and any $N \in T^\perp M$, where $\nabla^\perp$ is the connection in the normal bundle $T^\perp M$, $h$ is the second fundamental form of $M$ and $A_N$ is the Weingarten endomorphism associated with $N$. The second fundamental form and the shape operator $A$ are related by

$$g(A_N X, Y) = g(h(X, Y), N).$$

For Kaehler manifolds and their submanifolds, see [17].

3. Doubly warped and doubly twisted product CR-submanifolds

In this section, we consider CR-submanifolds which are doubly warped or doubly twisted products in the form $fM_T \times_b M^\perp$, where $M_T$ is a holomorphic submanifold and $M^\perp$ is a totally real submanifold of $\bar{M}$.

**Theorem 3.1.** Let $\bar{M}$ be a Kaehler manifold. Then there do not exist doubly warped product CR-submanifolds which are not (singly) warped product CR-submanifolds in the form $fM_T \times_b M^\perp$ such that $M_T$ is a holomorphic submanifold and $M^\perp$ is a totally real submanifold of $\bar{M}$.

**Proof.** Let us suppose that $M$ be a doubly warped product CR-submanifold of a Kaehler manifold $\bar{M}$. Then from (1.1) we have

$$g(\nabla_{JX} V, X) = V(\ln f) g(JX, X) = 0$$

for $X \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$. Using (2.2) we get $g(\nabla_{JX} V, X) = 0$. Since $D$ and $D^\perp$ are orthogonal, we obtain $g(\nabla_{JX} X, V) = 0$. Then from (2.1) we have $g(\nabla_{JX} JX, JV) = 0$. Thus from (2.2), we have

$$g(h(JX, JX), JV) = 0$$

(3.1)

for $X \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$. Now, substituting $X$ by $Y + Z$, $Z, Y \in \Gamma(D)$, in (3.1) and taking into account that $h$ is symmetric, we get

$$g(h(JY, JZ), JV) = 0$$

(3.2)
for \( Y, Z \in \Gamma(D) \) and \( V \in \Gamma(D^\perp) \). On the other hand, from (2.2) we have
\[
g(h(JY, JZ), JV) = g(\nabla_{JY} JZ, JV).
\]
Using (2.1) we obtain
\[
g(h(JY, JZ), JV) = -g(\bar{\nabla} JY JZ, JV)
\]
for \( Y, Z \in \Gamma(D) \) and \( V \in \Gamma(D^\perp) \). Then from (1.1) we get
\[
g(h(JY, JZ), JV) = -V(\ln f)g(JY, Z)
\]
for \( Y, Z \in \Gamma(D) \) and \( V \in \Gamma(D^\perp) \). Thus using (3.2) in (3.3), we obtain
\[
V(\ln f)g(JY, Z) = 0.
\]
Hence, for \( Z = JY \), we get
\[
V(\ln f)g(Y, Y) = 0.
\]
Then, we derive \( V(\ln f) = 0 \) due to \( g(Y, Y) \neq 0 \). \( V(\ln f) = 0 \) implies that \( f \) is constant. Thus \( M \) is a warped product CR-submanifold in the form \( M_T \times_b M_\perp \) which is called CR-warped product (see [7]). Thus, proof is complete.

Theorem 3.1 tells us that there exist no doubly warped product CR-submanifolds in Kaehler manifolds other than warped product CR-submanifolds. In the rest of this section, we investigate the existence of doubly twisted product CR-submanifolds in Kaehler manifolds.

**Theorem 3.2.** Let \( \bar{M} \) be a Kaehler manifold. Then there do not exist doubly twisted product CR-submanifolds of \( \bar{M} \) which are not (singly) twisted product CR-submanifolds in the form \( fM_T \times_b M_\perp \) such that \( M_T \) is a holomorphic submanifold and \( M_\perp \) is a totally real submanifold of \( \bar{M} \).

**Proof.** From (1.1), we have
\[
g(\nabla_X V, Y) = V(\ln f)g(X, Y)
\]
for \( X, Y \in \Gamma(D) \) and \( V \in \Gamma(D^\perp) \). Since \( D \) and \( D^\perp \) are orthogonal, we get
\[
-h(\nabla_X Y, V) = V(\ln f)g(X, Y).
\]
Using (2.2) and (2.1), we obtain
\[
-g(\bar{\nabla} X JY, JV) = V(\ln f)g(X, Y).
\]
Thus, from (2.2), we derive
\[
-g(h(X, JY), JV) = V(\ln f)g(X, Y)
\]
for \( X, Y \in \Gamma(D) \) and \( V \in \Gamma(D^\perp) \). On the other hand, from (2.2), we have
\[
g(h(X, JY), JV) = g(\nabla_{JY} X, JV)
\]
for \( X, Y \in \Gamma(D) \) and \( V \in \Gamma(D^\perp) \). Using (2.1), we get
\[
g(h(X, JY), JV) = -g(\bar{\nabla} JY JX, V).
\]
Hence, (2.2) implies that
\[
g(h(X, JY), JV) = g(JX, \nabla_{JY} V).
\]
Thus from (1.1), we obtain
\[ g(h(X, JY), JV) = V(\ln f)g(JX, JY). \]
Using (2.1) we arrive at
\[ g(h(X, JY), JV) = V(\ln f)g(X, Y) \] (3.5)
for \( X, Y \in \Gamma(D) \) and \( V \in \Gamma(D^\perp) \). Then from (3.4) and (3.5) we obtain
\[ V(\ln f)g(X, Y) = 0. \]
Since \( D \) is Riemannian, we get \( V(\ln f) = 0 \). This implies that \( f \) only depends the points of \( M_T \). Thus we can write
\[ g = \tilde{g}_{M_T} \oplus b^2g_{M_\perp}, \text{ where } \tilde{g}_{M_T} = f^2g_{M_T}. \]
Thus it follows that \( M \) is a twisted product CR-submanifold in the form \( M = M_T \times_b M_\perp \), (see [6] for twisted product CR-submanifolds). Hence, we conclude that there are no doubly twisted product CR-submanifolds in Kaehler manifolds, other than twisted product CR-submanifolds. 

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