CENTRALIZING AND COMMUTING GENERALIZED DERIVATIONS ON PRIME RINGS

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Abstract. Let \( R \) be a prime ring and \( d \) a derivation on \( R \). If \( f \) is a generalized derivation on \( R \) such that \( f \) is centralizing on a left ideal \( U \) of \( R \), then \( R \) is commutative.

A ring \( R \) is said to be prime if \( aRb = 0 \) implies that either \( a = 0 \) or \( b = 0 \). An additive mapping \( d: R \to R \) is said to be a derivation if \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in R \). A mapping \( f \) is said to be commuting on a left ideal \( U \) of \( R \) if \( [f(x), x] = 0 \) for all \( x \in U \) and \( f \) is said to be centralizing if \( [f(x), x] \in Z(R) \) for all \( x \in U \). There has been considerable interest in commuting, centralizing and related mappings in prime and semiprime rings (see [2] for a partial bibliography).

In this note we extend some results of Bell and Martindale [1] for generalized derivations. An additive mapping \( f: R \to R \) is said to be a generalized derivation on \( R \) if \( f(xy) = f(x)y + xd(y) \) for all \( x, y \in R \) (where \( d \) is a derivation on \( R \)). These mappings were introduced in [3].

Throughout this note \( R \) will represent a prime ring with \( Z(R) \) being its centre.

In the following we state a well known fact as

Remark 1. For a nonzero element \( a \in Z(R) \), if \( ab \in Z(R) \), then \( b \in Z(R) \).

In order to prove the main result, we find it necessary to establish the following Lemma.

Lemma 1. If \( f \) is an additive mapping from \( R \) to \( R \) such that \( f \) is centralizing on a left ideal \( U \) of \( R \), then for all \( x \in U \cap Z(R) \), \( f(x) \in Z(R) \).

Proof. Since \( f \) is centralizing on \( U \), we have \( [f(x + y), x + y] \in Z(R) \), for all \( x, y \in U \). This implies that

\[
[f(x), y] + [f(y), x] \in Z(R). \tag{1}
\]
Now if $x \in Z(R)$, then from equation (1), $[f(x), y] \in Z(R)$. Replacing $y$ by $f(x)y$, we get $f(x)f(x), y] \in Z(R)$. If $[f(x), y] = 0$, then $f(x) \in C_R(U)$, the centralizer of $U$ in $R$, and hence ([1, Identity IV]) belongs to $Z(R)$. But on the other hand if $[f(x), y] \neq 0$, it again follows from Remark 1 that $f(x) \in Z(R)$. ■

Next we prove the result which generalizes [1, Theorem 4].

**Theorem 1.** Let $R$ be a prime ring. Let $d: R \rightarrow R$ be a nonzero derivation and $f$ be a generalized derivation on a left ideal $U$ of $R$. If $f$ is commuting on $U$ then $R$ is commutative.

**Proof.** Since $f$ is commuting on $U$, we have $[f(x), x] = 0$ for all $x \in U$. Replacing $x$ by $x+y$, we get $[f(x), y] + [f(y), x] = 0$. Now by substituting $y = xy$ and simplifying we arrive at $[yd(x), x] = 0$. Replacing $y$ by $ry$, we get $[r, x]U\ d(x) = 0$ for all $x \in U$ and $r \in R$. Since $R$ is a prime ring, therefore either $[r, x] = 0$ or $d(x) = 0$ for all $t \in R$. So for any element $x \in U$, either $x \in Z(R)$ or $d(x) = 0$. Since $d$ is nonzero on $R$, then by [4, Lemma 2], $d$ is nonzero on $U$. Suppose $d(x) \neq 0$, for some $x \in U$, then $x \in Z(R)$. Suppose $z \in U$ is such that $z \notin Z(R)$, then $d(z) = 0$ and $x + z \notin Z(R)$. This implies $d(x + z) = 0$ and so $d(x) = 0$, a contradiction. This implies $z \in Z(R)$ for all $z \in U$. Thus $U$ is commutative and hence by [4, Lemma 3], $R$ is commutative. ■

Now we are ready to prove the result which involves centralizing generalized derivations on left ideals containing central elements.

**Theorem 2.** Let $U$ be a left ideal of a prime ring $R$ such that $U \cap Z(R) \neq 0$. Let $d$ be a nonzero derivation and $f$ be a generalized derivation on $R$ such that $f$ is centralizing on $U$. Then $R$ is commutative.

**Proof.** We assume that $Z(R) \neq 0$ because otherwise $f$ is commuting on $U$ and there is nothing left to prove. Now for a nonzero $z \in Z(R)$, we replace $x$ by $zy$ in (1) and get $[f(z), y]y + z[d(y), y] + z[f(y), y] \in Z(R)$. Now by Lemma 1, $f(z) \in Z(R)$ and therefore $z[d(y), y] + z[f(y), y] \in Z(R)$. But as $f$ is centralizing on $U$, we have $z[f(y), y] \in Z(R)$ and consequently $z[d(y), y] \in Z(R)$. Since $z$ is nonzero, it follows from Remark 1 that $[d(y), y] \in Z(R)$. This implies $d$ is centralizing on $U$ and hence by [1, Theorem 4], we conclude that $R$ is commutative. ■

**References**


(Received 23.11.2005)

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