ON A CLASS OF MULTIVALENT FUNCTIONS DEFINED BY A MULTIPLIER TRANSFORMATION

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Abstract. In the present paper, the authors investigate starlikeness and convexity of a class of multivalent functions defined by multiplier transformation. As a consequence, a number of sufficient conditions for starlikeness and convexity of analytic functions are also obtained.

1. Introduction

Let \( A_p \) denote the class of functions of the form

\[ f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \]

\( p \in \mathbb{N} = \{1, 2, \ldots\} \), which are analytic in the open unit disc \( E = \{z : |z| < 1\} \).

We write \( A_1 = A \). A function \( f \in A_p \) is said to be \( p \)-valent starlike of order \( \alpha \) (\( 0 \leq \alpha < p \)) in \( E \) if

\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in E. \]

We denote by \( S^*_p(\alpha) \), the class of all such functions. A function \( f \in A_p \) is said to be \( p \)-valent convex of order \( \alpha \) (\( 0 \leq \alpha < p \)) in \( E \) if

\[ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in E. \]

Let \( K_p(\alpha) \) denote the class of all those functions \( f \in A_p \) which are multivalently convex of order \( \alpha \) in \( E \). Note that \( S^*_1(\alpha) \) and \( K_1(\alpha) \) are, respectively, the usual classes of univalent starlike functions of order \( \alpha \) and univalent convex functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), and will be denoted here by \( S^*(\alpha) \) and \( K(\alpha) \), respectively. We shall use \( S^* \) and \( K \) to denote \( S^*(0) \) and \( K(0) \), respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

For \( f \in A_p \), we define the multiplier transformation \( I_p(n, \lambda) \) as

\[ I_p(n, \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left( \frac{k + \lambda}{p + \lambda} \right)^n a_k z^k, \quad (\lambda \geq 0, \ n \in \mathbb{Z}). \]

AMS Subject Classification: 30C45

Keywords and phrases: Multivalent function; starlike function; convex function; multiplier transformation.
The operator $I_p(n, \lambda)$ has been recently studied by Aghalary et. al. [1]. Earlier, the operator $I_1(n, \lambda)$ was investigated by Cho and Srivastava [2] and Cho and Kim [3], whereas the operator $I_1(n, 1)$ was studied by Uralegaddi and Somanatha [9]. $I_1(n, 0)$ is the well-known Salagean [7] derivative operator $D_n$, defined as: $D_n f(z) = z + \sum_{k=2}^{\infty} k^n a_n z^n$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in A$.

A function $f \in A_p$ is said to be in the class $S_n(p, \lambda, \alpha)$ for all $z$ in $E$ if it satisfies

$$\text{Re} \left[ \frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \right] > \frac{\alpha}{p},$$

for some $\alpha$ ($0 \leq \alpha < p, p \in \mathbb{N}$). We note that $S_0(1,0,\alpha)$ and $S_1(1,0,\alpha)$ are the usual classes $S^*(\alpha)$ and $K(\alpha)$ of starlike functions of order $\alpha$ and convex functions of order $\alpha$, respectively.

In 1989, Owa, Shen and Obradović [6] obtained a sufficient condition for a function $f \in A$ to belong to the class $S_n(1,0,\alpha) = S_n(\alpha)$, say. In fact, they proved the following result:

**Theorem A.** For $n \in \mathbb{N}_0$, if $f \in A$ satisfies

$$\left| \frac{D_n^{n+1}f(z)}{D_n^{n+2}f(z)} - 1 \right|^{1-\beta} \left| \frac{D_n^{n+2}f(z)}{D_n^{n+1}f(z)} - 1 \right|^{\beta} < (1 - \alpha)^{1-2\beta}(1 - \frac{3}{2}\alpha + \alpha^2)^\beta, \quad z \in E,$$

for some real numbers $\alpha$ ($0 \leq \alpha \leq \frac{1}{2}$) and $\beta$ ($0 \leq \beta \leq 1$), then $f \in S_n(\alpha)$, i.e. $\text{Re} \left[ \frac{D_n^{n+1}f(z)}{D_n^{n+2}f(z)} \right] > \frac{\alpha}{p}$ in $E$.

This result was, later on, extended by Li and Owa [5] for all $\alpha, 0 \leq \alpha < 1$ and $\beta \geq 0$. They proved

**Theorem B.** If $f \in A$ satisfies

$$\left| \frac{D_n^{n+1}f(z)}{D_n^{n+2}f(z)} - 1 \right|^{\gamma} \left| \frac{D_n^{n+2}f(z)}{D_n^{n+1}f(z)} - 1 \right|^{\beta} < \begin{cases} (1 - \alpha)^\gamma (\frac{3}{2} - \alpha)^\beta, & 0 \leq \alpha \leq 1/2, \\ 2^{\beta}(1 - \alpha)^{\beta+\gamma}, & 1/2 \leq \alpha < 1. \end{cases}$$

for some reals $\alpha$ ($0 \leq \alpha < 1$), $\beta \geq 0$ and $\gamma \geq 0$ with $\beta + \gamma > 0$, then $f \in S_n(\alpha)$, $n \in \mathbb{N}_0$.

In the present paper, our aim is to determine sufficient conditions for a function $f \in A_p$ to be a member of the class $S_n(p, \lambda, \alpha)$. As a consequence of our main result, we get a number of sufficient conditions for starlikeness and convexity of analytic functions.

### 2. Main result

To prove our result, we shall make use of the famous Jack’s lemma which we state below.
Lemma 2.1. (Jack [4]) Suppose \( w(z) \) be a nonconstant analytic function in \( E \) with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value at a point \( z_0 \in E \) on the circle \( |z| = r < 1 \), then \( z_0 w'(z_0) = mw(z_0) \), where \( m, m \geq 1 \), is some real number.

We, now, state and prove our main result.

Theorem 2.1. If \( f \in \mathbb{A}_p \) satisfies

\[
|I_p(n + 1, \lambda)f(z) - 1| \leq M(p, \lambda, \alpha, \beta, \gamma), \quad z \in E,
\]

(3)

for some real numbers \( \alpha, \beta \) and \( \gamma \) such that \( 0 \leq \alpha < p, \beta \geq 0, \gamma \geq 0, \beta + \gamma > 0 \),

then \( f \in S_n(p, \lambda, \alpha) \), where \( n \in \mathbb{N}_0 \) and

\[
M(p, \lambda, \alpha, \beta, \gamma) = \begin{cases} \left(1 - \frac{\alpha}{p}\right)^\gamma \left(1 - \frac{\alpha}{p} + \frac{1}{2(p+\lambda)}\right)^\beta, & 0 \leq \alpha \leq p/2, \\ \left(1 - \frac{\alpha}{p}\right)^{\gamma + \beta} \left(1 + \frac{1}{p+\lambda}\right)^\beta, & p/2 \leq \alpha < p. \end{cases}
\]

Proof. Case (i). Let \( 0 \leq \alpha \leq \frac{p}{2} \). Writing \( \frac{\alpha}{p} = \mu \), we see that \( 0 \leq \mu \leq \frac{1}{2} \).

Define a function \( w(z) \) as

\[
\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} = \frac{1 + (1 - 2\mu)w(z)}{1 - w(z)}, \quad z \in E.
\]

(4)

Then \( w \) is analytic in \( E \), \( w(0) = 0 \) and \( w(z) \neq 1 \) in \( E \). By a simple computation, we obtain from (4),

\[
\frac{z(I_p(n + 1, \lambda)f(z)')'}{I_p(n + 1, \lambda)f(z)} - \frac{z(I_p(n, \lambda)f(z)')'}{I_p(n, \lambda)f(z)} = \frac{2(1 - \mu)zw'(z)}{(1 - w(z))(1 + (1 - 2\mu)w(z))}
\]

(5)

By making use of the identity

\[(p + \lambda)I_p(n + 1, \lambda)f(z) = z(I_p(n, \lambda)f(z)')' + \lambda I_p(n, \lambda)f(z)\]

(6)

we obtain from (5)

\[
\frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)} = \frac{1 + (1 - 2\mu)w(z)}{1 - w(z)} + \frac{2(1 - \mu)zw'(z)}{(p + \lambda)(1 - w(z))(1 + (1 - 2\mu)w(z))}
\]

Thus, we have

\[
\left|\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} - 1\right|^\beta = \left|\frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)} - 1\right|^\beta
\]

\[
= \left|\frac{2(1 - \mu)w(z)}{1 - w(z)}\right|^\gamma \left|\frac{2(1 - \mu)w(z)}{1 - w(z)} + \frac{2(1 - \mu)zw'(z)}{(p + \lambda)(1 - w(z))(1 + (1 - 2\mu)w(z))}\right|^\beta
\]

\[
= \left|\frac{2(1 - \mu)w(z)}{1 - w(z)}\right|^{\gamma + \beta} \left|\frac{zw'(z)}{(p + \lambda)w(z)(1 + (1 - 2\mu)w(z))}\right|^\beta.
\]
Suppose that there exists a point \( z_0 \in E \) such that \( \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \). Then by Lemma 2.1, we have \( w(z_0) = e^{i\theta}, 0 < \theta \leq 2\pi \) and \( z_0w'(z_0) = mw(z_0), m \geq 1 \). Therefore
\[
\left| \frac{I_p(n + 1, \lambda)f(z_0)}{I_p(n, \lambda)f(z_0)} - 1 \right|^\beta = \left| \frac{2(1 - \mu)w(z_0)^\gamma w(z_0)}{1 - w(z_0)} \right| \geq (1 - \mu)^\beta (1 + \frac{1}{2(p + \lambda)(1 - \mu)})^\beta = (1 - \mu)^\gamma \left( 1 - \mu + \frac{1}{2(p + \lambda)} \right)^\beta
\]
which contradicts (3) for \( 0 \leq \alpha \leq \frac{p}{2} \). Therefore, we must have \( |w(z)| < 1 \) for all \( z \in E \), and hence \( f \in S_n(p, \lambda, \alpha) \).

Case (ii). When \( \frac{p}{2} \leq \alpha < p \). In this case, we must have \( \frac{1}{2} \leq \mu < 1 \), where \( \mu = \frac{\alpha}{p} \). Let \( w \) be defined by
\[
\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} = \frac{\mu}{\mu - (1 - \mu)w(z)}, \quad z \in E,
\]
where \( w(z) \neq \frac{\mu}{1 - \mu} \) in \( E \). Then \( w \) is analytic in \( E \) and \( w(0) = 0 \). Proceeding as in Case (i) and using identity (6), we obtain
\[
\left| \frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} - 1 \right|^\beta = \left| \frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)} - 1 \right|^\beta
\]
\[
\geq (1 - \mu)^\beta (1 + \frac{1}{2(p + \lambda)(1 - \mu)})^\beta
\]
which contradicts (3) for \( \frac{p}{2} \leq \alpha \leq p \). Therefore we must have \( |w(z)| < 1 \) for all \( z \in E \), and hence \( f \in S_n(p, \lambda, \alpha) \). This completes the proof of our theorem. 

3. Deductions

For \( p = 1 \), Theorem 2.1 reduces to the following result:

**Corollary 3.1.** If, for all \( z \in E \), a function \( f \in \mathcal{A} \) satisfies

\[
\left| \frac{I_1(n + 1, \lambda)f(z)}{I_1(n, \lambda)f(z)} - 1 \right| \geq \left| \frac{I_1(n + 2, \lambda)f(z)}{I_1(n + 1, \lambda)f(z)} - 1 \right|^\beta
\]

\[
\begin{cases} 
(1 - \alpha)^\gamma \left(1 - \alpha + \frac{1}{2(1 + \lambda)}\right)^\beta, & 0 \leq \alpha < 1/2, \\
(1 - \alpha)^{\gamma + \beta} \left(1 + \frac{1}{1 + \lambda}\right)^\beta, & 1/2 \leq \alpha < 1,
\end{cases}
\]

for some reals \( \alpha, \beta \), then \( f \in S_n(1, \lambda, \alpha) \), where \( n \in \mathbb{N}_0 \).

**Remark 3.1.** Setting \( \lambda = 0 \) in Corollary 3.1, we obtain Theorem B.

Recently, Sivaprasad Kumar et. al. [8] proved the following result:

**Theorem C.** Let \( \psi \) be univalent in \( E \), \( \psi(0) = 1 \), \( \Re \psi(z) > 0 \) and \( \frac{z\psi'(z)}{\psi(z)} \) be starlike in \( E \). Suppose \( f \in \mathcal{A}_p \) satisfies

\[
\frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)} \prec \psi(z) + \frac{z\psi'(z)}{(p + \lambda)\psi(z)}.
\]

Then,

\[
\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec \psi(z).
\]

Set \( \psi(z) = \frac{1 + (1 - \frac{2\alpha}{p})z}{1 - z} \), \( 0 \leq \alpha < p \), \( p \in \mathbb{N} \), in Theorem C. Clearly \( \psi \) satisfies all the conditions of above theorem. Thus, we obtain the following result:

If \( f \in \mathcal{A}_p \) satisfies

\[
\frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)} \prec \frac{1 + (1 - \frac{2\alpha}{p})z}{1 - z} + \frac{(1 - \frac{2\alpha}{p})z}{(p + \lambda)(1 + (1 - \frac{2\alpha}{p})z)},
\]

then

\[
\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec \frac{1 + (1 - \frac{2\alpha}{p})z}{1 - z},
\]

i.e.

\[
\Re \left( \frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \right) > \frac{\alpha}{p}.
\]

Compare this result with the result below, which we get by writing \( \gamma = 0 \) and \( \beta = 1 \) in Theorem 2.1:

**Corollary 3.2.** If, for all \( z \in E \), a function \( f \in \mathcal{A}_p \) satisfies

\[
\left| \frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)} - 1 \right| \leq \begin{cases} 
1 - \frac{\alpha}{p} + \frac{1}{2(1 + \lambda)}, & 0 \leq \alpha \leq p/2, \\
(1 - \frac{\alpha}{p})(1 + \frac{1}{p + \lambda}), & p/2 \leq \alpha < p,
\end{cases}
\]

then \( \Re \left( \frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \right) > \frac{\alpha}{p} \), \( z \in E \).
Setting $p = 1$, $\lambda = 1$ and $n = 0$ in Theorem 2.1, we obtain the following result:

**Corollary 3.3.** If $f \in \mathcal{A}$ satisfies

$$\frac{|zf'(z) - 1|^\gamma}{|zf(z) - 1| - 1} \left| \frac{2zf'(z) + z^2f''(z)}{f(z) + zf'(z)} - 1 \right| < \left( \frac{3}{2} \right)^\beta (1 - \alpha)^{\gamma+\beta}, \quad 0 \leq \alpha < 1, \quad z \in E,$$

for some $\beta \geq 0$ and $\gamma \geq 0$ with $\beta + \gamma > 0$, then $f \in S^*(\alpha)$.

Setting $\alpha = 0$ in Corollary 3.3, we obtain the following criterion for starlikeness:

**Corollary 3.4.** For some non-negative real numbers $\beta$ and $\gamma$ with $\beta + \gamma > 0$, if $f \in \mathcal{A}$ satisfies

$$\frac{|zf'(z) - 1|^\gamma}{|zf(z) - 1| - 1} \left| \frac{2zf'(z) + z^2f''(z)}{f(z) + zf'(z)} - 1 \right| < \left( \frac{3}{2} \right)^\beta, \quad z \in E,$$

then $f \in S^*$.

In particular, for $\beta = 1$ and $\gamma = 1$, we obtain the following interesting criterion for starlikeness:

**Corollary 3.5.** If $f \in \mathcal{A}$ satisfies

$$\frac{|zf'(z) - 1|^\gamma}{|zf(z) - 1| - 1} \left| \frac{2zf'(z) + z^2f''(z)}{f(z) + zf'(z)} - 1 \right| < \frac{3}{2}, \quad z \in E,$$

then $f \in S^*$.

Setting $\lambda = 0$ and $n = 0$ in Theorem 2.1, we obtain the following sufficient condition for a function $f \in \mathcal{A}_p$ to be a $p$-valent starlike function of order $\alpha$.

**Corollary 3.6.** For all $z \in E$, if $f \in \mathcal{A}_p$ satisfies the condition

$$\frac{|zf'(z) - 1|^\gamma}{|zf(z) - 1| - 1} \left( 1 + \frac{|zf''(z)|}{|zf'(z)|} \right) - 1 \left| \frac{2zf'(z) + z^2f''(z)}{f(z) + zf'(z)} - 1 \right| < \left\{ \begin{array}{ll}
(1 - \alpha)^\gamma \left( \frac{1}{p} - \frac{1}{2} \right)^\beta, & 0 \leq \alpha < p/2, \\
(1 - \alpha)^\gamma + (1 + \frac{1}{p})^\beta, & p/2 \leq \alpha < p,
\end{array} \right.$$

for some real numbers $\alpha, \beta$ and $\gamma$ with $0 \leq \alpha < p$, $\beta \geq 0$, $\gamma \geq 0$, $\beta + \gamma > 0$, then $f \in S^*_p(\alpha)$.

The substitution $p = 1$ in Corollary 3.6, yields the following result:

**Corollary 3.7.** If $f \in \mathcal{A}$ satisfies

$$\frac{|zf'(z) - 1|^\gamma}{|zf(z) - 1| - 1} \left| \frac{2zf'(z) + z^2f''(z)}{f(z) + zf'(z)} - 1 \right| \left( 1 + \frac{|zf''(z)|}{|zf'(z)|} \right) - 1 \left| \frac{2zf'(z) + z^2f''(z)}{f(z) + zf'(z)} - 1 \right| < \left\{ \begin{array}{ll}
(1 - \alpha)^\gamma \left( \frac{1}{2} - \alpha \right)^\beta, & 0 \leq \alpha \leq 1/2, \\
(1 - \alpha)^\gamma + (1 + \frac{1}{2})^\beta, & 1/2 \leq \alpha < 1,
\end{array} \right.$$

where $z \in E$ and $\alpha, \beta, \gamma$ are real numbers with $0 \leq \alpha < 1$, $\beta \geq 0$, $\gamma \geq 0$, $\beta + \gamma > 0$, then $f \in S^*(\alpha)$. 

In particular, writing $\beta = 1, \gamma = 1$ and $\alpha = 0$ in Corollary 3.7, we obtain the following result of Li and Owa [5]:

**Corollary 3.8.** If $f \in A$ satisfies
\[
\left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{3}{2}, \quad z \in \mathbb{E},
\]
then $f \in S^*$. 

Taking $\lambda = 0$ and $n = 1$ in Theorem 2.1, we get the following interesting criterion for convexity of multivalent functions:

**Corollary 3.9.** If, for all $z \in \mathbb{E}$, a function $f \in A_p$ satisfies
\[
\left| \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\gamma - 1 \right| \left| \frac{1}{p} \left( 1 + \frac{zf''(z) + z^2f'''(z)}{f'(z) + f''(z)} \right) - 1 \right|^\beta < \begin{cases} 
(1 - \frac{\alpha}{p})^\gamma \left( 1 - \frac{\alpha}{p} + \frac{1}{2p} \right)^\beta, & 0 \leq \alpha \leq p/2, \\
(1 - \frac{\alpha}{p})^\gamma + \beta \left( 1 + \frac{1}{p} \right)^\beta, & p/2 \leq \alpha < p,
\end{cases}
\]
for some real numbers $\alpha, \beta$ and $\gamma$ with $0 \leq \alpha < p$, $\beta \geq 0$, $\gamma \geq 0$, $\beta + \gamma > 0$, then $f \in K_p(\alpha)$.

Taking $p = 1$ in Corollary 3.9, we obtain the following sufficient condition for convexity of univalent functions:

**Corollary 3.10.** For some non-negative real numbers $\alpha$, $\beta$ and $\gamma$, with $\beta + \gamma > 0$ and $\alpha < 1$, if $f \in A$ satisfies
\[
\left| \frac{zf''(z) + z^2f'''(z)}{f'(z) + f''(z)} \right| \beta < \begin{cases} 
(1 - \alpha)^\gamma (1 - \alpha + \frac{1}{2})^\beta, & 0 \leq \alpha \leq 1/2, \\
(1 - \alpha)^\gamma + \beta 2^\beta, & 1/2 \leq \alpha < 1,
\end{cases}
\]
for all $z \in \mathbb{E}$, then $f \in K(\alpha)$.

In particular, writing $\beta = 1, \gamma = 1$ and $\alpha = 0$ in Corollary 3.10, we obtain the following sufficient condition for convexity of analytic functions:

**Corollary 3.11.** If $f \in A$ satisfies
\[
\left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z) + z^2f'''(z)}{f'(z) + f''(z)} \right) \right| < \frac{3}{2}, \quad z \in \mathbb{E},
\]
then $f \in K$.

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(received 15.01.2007, in revised form 19.03.2008)

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