λ-FRACTIONAL PROPERTIES OF GENERALIZED JANOWSKI FUNCTIONS IN THE UNIT DISC

Mert Çağlar, Yaşar Polatğlu, Emel Yavuz

Abstract. For analytic function \( f(z) = z + a_2z^2 + \cdots \) in the open unit disc \( D \), a new fractional operator \( D_\lambda f(z) \) is defined. Applying this fractional operator \( D_\lambda f(z) \) and the principle of subordination, we give new proofs for some classical results concerning the class \( S_\lambda^*(A, B, \alpha) \) of functions \( f(z) \).

1. Introduction

Let \( \Omega \) be the family of functions \( w(z) \) regular in the open unit disc \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) and satisfying the conditions \( w(0) = 0, |w(z)| < 1 \) for all \( z \in \mathbb{D} \).

Let \( g(z) = z + b_2z^2 + \cdots \) and \( h(z) = z + c_2z^2 + \cdots \) be analytic functions in \( \mathbb{D} \). We say that \( g(z) \) is subordinate to \( h(z) \), written as \( g \prec h \), if

\[
g(z) = h(w(z)), \ w(z) \in \Omega, \quad \text{and for all } z \in \mathbb{D}.
\]

In particular if \( h(z) \) is univalent in \( \mathbb{D} \), then \( g \prec h \) if and only if \( g(0) = h(0) \), \( g(\mathbb{D}) \subset h(\mathbb{D}) \) ([1], [3]).

For arbitrary fixed numbers \( A, B, \alpha, -1 \leq B < A \leq 1, 0 \leq \alpha < 1 \), let \( \mathcal{P}(A, B, \alpha) \) denote the family of functions \( p(z) = 1 + p_1z + p_2z^2 + \cdots \) regular in \( \mathbb{D} \) and such that \( p(z) \in \mathcal{P}(A, B, \alpha) \) if and only if

\[
p(z) \prec \frac{1 + [(1 - \alpha)A + \alpha B]z}{1 + Bz} \iff p(z) = \frac{1 + [(1 - \alpha)A + \alpha B]w(z)}{1 + Bw(z)}
\]

for some function \( w(z) \) and all \( z \in \mathbb{D} \).

Using the fractional calculus, we define the fractional operator \( D^\lambda f(z) \) by

\[
D^\lambda f(z) = \Gamma(2 - \lambda)z^{\lambda}D_\lambda^zf(z),
\]

where \( D_\lambda^zf(z) \) is the fractional derivative of order \( \lambda \) which will be defined below.

AMS Subject Classification: 30C45

Keywords and phrases: Starlike, fractional integral, fractional derivative, distortion theorem.
Furthermore, let $S^*_\lambda(A, B, \alpha)$ denote the family of functions $f(z) = z + a_2z^2 + \cdots$ regular in $D$ and such that $f(z)$ is in $S^*_\lambda(A, B, \alpha)$ if and only if
\[ \frac{z}{D^\lambda f(z)}' = p(z) \]
for some $p(z)$ in $P(A, B, \alpha)$ and for all $z \in D$.

The fractional integral of order $\lambda$ is defined for a function $f(z) \in S^*_\lambda(A, B, \alpha)$, by
\[ D^{-\lambda}_z f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0), \]
where the function $f(z)$ is analytic in a simply connected region of the complex plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$ ([4], [5]).

The fractional derivative of order $\lambda$ is defined for a function $f(z) \in S^*_\lambda(A, B, \alpha)$, by
\[ D^{\lambda}_z f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1), \]
where the function $f(z)$ is analytic in a simply connected region of the complex plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in the definition of the fractional integral ([4], [5]).

Under the hypotheses of the fractional derivative of order $\lambda$, the fractional derivative of order $(n+\lambda)$ is defined for a function $f(z)$, by
\[ D^{n+\lambda}_z f(z) = \frac{d^n}{dz^n} D^{\lambda}_z f(z) \quad (0 \leq \lambda < 1, \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \]

By means of the definitions above, we see that
\begin{align}
D^{\lambda}_z z^k &= \Gamma(k+1) \frac{z^{k+\lambda}}{\Gamma(k+1+\lambda)} \quad (\lambda > 0), \\
D^{\lambda}_z z^{-k} &= \Gamma(k+1) \frac{z^{-k-\lambda}}{\Gamma(k+1-\lambda)} \quad (0 \leq \lambda < 1)
\end{align}
(1.1)
and
\[ D^{n+\lambda}_z z^k = \Gamma(k+1) \frac{z^{k-n-\lambda}}{\Gamma(k+1-n-\lambda)} \quad (0 \leq \lambda < 1, n \in \mathbb{N}_0). \]

Therefore, we conclude that, for any real $\lambda$
\[ D^{\lambda}_z z^k = \Gamma(k+1) \frac{z^{k-\lambda}}{\Gamma(k+1-\lambda)}. \]  
(1.2)

The following lemma, due to Jack [2], plays an important rôle in our proofs.

**Lemma 1.1** Let $w(z)$ be a non-constant function analytic in $D(r) = \{z \mid |z| < r \}$ with $w(0) = 0$. If
\[ |w(z_1)| = \text{Max} \{|w(z)| \mid |z| \leq |z_1|\} \quad (z_1 \in \mathbb{D}(r)), \]
then there exists a real number $k$ ($k \geq 1$), such that $z_1w'(z_1) = kw(z_1)$. 


2. Main Results

Lemma 2.1. Let \( f(z) = z + a_2 z^2 + \cdots \) be analytic in the open unit disc \( \mathbb{D} \). Then the \( \lambda \)-fractional operator \( D^\lambda f(z) \) satisfies the following equalities

\[
\begin{align*}
(i) \quad D^\lambda f(z) &= \Gamma(2-\lambda) z^\lambda D_1^\lambda f(z) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda) \Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n; \\
(ii) \quad for \lambda = 1, \quad Df(z) &= \lim_{\lambda \to 1} D^\lambda f(z) = zf'(z); \\
(iii) \quad for \lambda < 1, \delta < 1, \quad D^\lambda(D^\delta f(z)) &= D^{\lambda+\delta}(D^\lambda f(z)) \\
&= z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda) \Gamma(2-\delta) \Gamma(n+1)^2}{\Gamma(n+1-\lambda) \Gamma(n+1-\delta)} z^n; \\
(iv) \quad D(D^\lambda f(z)) &= z + \sum_{n=2}^{\infty} n a_n \frac{\Gamma(2-\lambda) \Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n = z(D^\lambda f(z))' \\
&= \Gamma(2-\lambda) z^\lambda (\lambda D^\lambda f(z) + zD^{\lambda+1} f(z)); \\
(v) \quad \frac{D(D^\lambda f(z))}{D^\lambda f(z)} - 1 &= \begin{cases} 
\frac{z f'(z)}{f(z)} - 1, & for \lambda = 0, \\
\frac{z f''(z)}{f'(z)}, & for \lambda = 1.
\end{cases}
\end{align*}
\]

Proof. Making use of the fractional derivative rules (1.1) and (1.2), we obtain

\[
D_1^\lambda f(z) = \frac{\Gamma(2)}{\Gamma(2-\lambda)} z^{1-\lambda} + a_2 \frac{\Gamma(3)}{\Gamma(3-\lambda)} z^{2-\lambda} + \cdots + a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^{n-\lambda} + \cdots
\]

wherefrom

\[
D^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_1^\lambda f(z) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda) \Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n.
\]

Other equalities follow directly from (2.1). \( \blacksquare \)

Lemma 2.2. Let \( f(z) = z + a_2 z^2 + \cdots \) and \( g(z) = z + b_2 z^2 + \cdots \) be analytic functions in the open unit disc \( \mathbb{D} \). Then the solution of the fractional differential equation

\[
D_1^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z)
\]

is

\[
f(z) = z + \sum_{n=2}^{\infty} b_n \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda) \Gamma(n+1)} z^n.
\]

Proof. Using the definition of fractional integral, fractional derivative and fractional calculus of order \((n+\lambda)\), we get

\[
D_1^\lambda f(z) = \frac{\Gamma(2)}{\Gamma(2-\lambda)} z^{1-\lambda} + a_2 \frac{\Gamma(3)}{\Gamma(3-\lambda)} z^{2-\lambda} + \cdots + a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^{n-\lambda} + \cdots
\]

\[
= \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z)
\]

\[
= \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + b_2 \frac{1}{\Gamma(2-\lambda)} z^{2-\lambda} + \cdots + b_n \frac{1}{\Gamma(2-\lambda)} z^{n-\lambda} + \cdots.
\]
Therefore, we have
\[
\frac{\Gamma(2)}{\Gamma(2-\lambda)} z^{1-\lambda} + a_2 \frac{\Gamma(3)}{\Gamma(3-\lambda)} z^{2-\lambda} + \cdots + a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^{n-\lambda} + \cdots
\]
\[
= \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + b_2 \frac{1}{\Gamma(2-\lambda)} z^{2-\lambda} + \cdots + b_n \frac{1}{\Gamma(2-\lambda)} z^{n-\lambda} + \cdots \quad (2.2)
\]
Comparing the coefficient of $z^{n-\lambda}$ in both sides of (2.2) we obtain
\[
a_n = \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda) \Gamma(n+1)} b_n. \quad \blacksquare
\]

**Theorem 2.3.** Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in the open unit disc $\mathbb{D}$. If $f(z)$ satisfies
\[
\left( \frac{D(D^\lambda f(z))}{D^\lambda f(z)} - 1 \right) F(z) = \begin{cases} \frac{(1-\alpha)(A-B)z}{1+Bz}, & B \neq 0, \\ (1-\alpha)A z = F_2(z), & B = 0, \end{cases}
\]
then $f(z) \in S^*_\alpha(A, B, \alpha)$ and this result is sharp as the function
\[
D^\lambda f(z) = \begin{cases} z(1+Bz), & B \neq 0, \\ ze^{(1-\alpha)Az}, & B = 0. \end{cases}
\]

**Proof.** We define the function $w(z)$ by
\[
\frac{D^\lambda f(z)}{z} = \begin{cases} (1+Bw(z)) e^{(1-\alpha)Aw(z)}, & B \neq 0, \\ e^{(1-\alpha)Aw(z)}, & B = 0, \end{cases}
\]
where $(1+Bw(z)) e^{(1-\alpha)Aw(z)}$ have the value 1 at the origin (we consider the corresponding Riemann branch). Then $w(z)$ is analytic in $\mathbb{D}$ and $w(0) = 0$. If we take the logarithmic derivative of the equality (2.4), simple calculations yield
\[
\left( \frac{D^\lambda f(z)}{z} \right)' = \begin{cases} \frac{(1-\alpha)(A-B)zw'(z)}{1+Bw(z)}, & B \neq 0, \\ (1-\alpha)Aw'(z), & B = 0. \end{cases}
\]
Now, it is easy to realize that the subordination (2.3) is equivalent to $|w(z)| < 1$ for all $z \in \mathbb{D}$. Indeed, assume the contrary; then, there exists $z_1 \in \mathbb{D}$ such that $|w(z_1)|$ attains its maximum value on the circle $|z| = r$ at the point $z_1$, that is $|w(z_1)| = 1$. Then, by I.S. Jack’s lemma, $z_1 w'(z_1) = kw(z_1)$ for some real $k \geq 1$. For such $z_1$ we have
\[
\left( \frac{D^\lambda f(z_1)}{z_1} \right)' = \begin{cases} \frac{(1-\alpha)(A-B)k w(z_1)}{1+Bw(z_1)} = F_1(w(z_1)) \notin F_1(\mathbb{D}), & B \neq 0, \\ (1-\alpha)Aw(z_1) = F_2(w(z_1)) \notin F_2(\mathbb{D}), & B = 0, \end{cases}
\]
because \(|w(z)| = 1\) and \(k \geq 1\). But this contradicts (2.3), so assumption is wrong, i.e., \(|w(z)| < 1\) for every \(z \in \mathbb{D}\).

The sharpness of the result follows from the fact that

\[
D^\lambda f(z) = \begin{cases} 
  z(1 + Bz) & B \neq 0, \\
  ze^{(1-\alpha)Az} & B = 0
\end{cases} \Rightarrow \frac{(1-\alpha)(A-B)}{B} D^\lambda f(z) \longrightarrow \begin{cases} 
  \frac{(1-\alpha)(A-B)}{1+Bz}, & B \neq 0, \\
  (1-\alpha)Az, & B = 0.
\end{cases}
\]

\[\left|\left(\frac{\Gamma(2-\lambda)D^\lambda f(z)}{z^{1-\lambda}}\right)^B - 1\right| < 1, \quad B \neq 0, \quad (2.5)\]

\[\left|\log \left(\frac{\Gamma(2-\lambda)D^\lambda f(z)}{z^{1-\lambda}}\right)^{1/(1-\alpha)A}\right| < 1, \quad B = 0. \quad (2.6)\]

**Proof.** This corollary is a simple consequence of Theorem 2.3. \(\blacksquare\)

**Remark 2.5.** We note that the inequalities (2.5) and (2.6) are the \(\lambda\)-fractional Marx-Strohhacker inequalities. Indeed, for \(A = 1\), \(B = -1\), \(\alpha = 0\), we have

\[\left|\left(\frac{\Gamma(2-\lambda)D^\lambda f(z)}{z^{1-\lambda}}\right)^B - 1\right| < 1,\]

which yields

a) \[\left|\sqrt{\frac{z}{f(z)}} - 1\right| < 1\] for \(\lambda = 0\): this is the Marx-Strohhacker inequality for starlike functions [1];

b) \[\left|\sqrt[1-\alpha]{\frac{1}{f(z)}} - 1\right| < 1\] for \(\lambda = 1\): this is the Marx-Strohhacker inequality for convex functions [1].

Moreover, assigning special values to \(A\), \(B\), \(\alpha\) and \(\lambda\), we obtain Marx-Strohhacker inequalities for the all the subclasses \(S^\lambda_*(A, B, \alpha)\) of analytic functions in the unit disc where \(0 \leq \lambda < 1\), \(0 \leq \alpha < 1\), \(-1 \leq B < A \leq 1\).
Theorem 2.6. If \( f(z) \in S^*_\lambda(A, B, \alpha) \), then

\[
\frac{1}{\Gamma(2-\lambda)} r^{1-\lambda}(1-Br) \frac{(1-\alpha)(A-B)}{B} \leq |D_2^\lambda f(z)|
\]

\[
\leq \frac{1}{\Gamma(2-\lambda)} r^{1-\lambda}(1+Br) \frac{(1-\alpha)(A-B)}{B}, \quad B \neq 0,
\]

\[
\frac{1}{\Gamma(2-\lambda)} r^{1-\lambda}e^{-(1-\alpha)Ar} \leq |D_2^\lambda f(z)|
\]

\[
\leq \frac{1}{\Gamma(2-\lambda)} r^{1-\lambda}e^{(1-\alpha)Ar}, \quad B = 0.
\] (2.7)

These bounds are sharp, because the extremal function is the solution of the \( \lambda \)-fractional differential equation

\[ D_\lambda^z f(z) = \begin{cases} 
\frac{1-\lambda}{\Gamma(1-\lambda)}z^{1-\lambda}(1+Br) \frac{(1-\alpha)(A-B)}{B}, & B \neq 0, \\
\frac{1-\lambda}{\Gamma(1-\lambda)}z^{1-\lambda}e^{(1-\alpha)Az}, & B = 0.
\end{cases} \]

Proof. The set of the values \( \left( z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} \right) \) is the closed disc centered at

\[
\begin{cases} 
C(r) = \frac{1-B[(1-\alpha)A+\alpha B]}{1-Br^2}, & B \neq 0, \\
C(r) = (1,0), & B = 0,
\end{cases}
\]

with radius

\[
\begin{cases} 
\rho(r) = \frac{(1-\alpha)(A-B)r}{1-Br^2}, & B \neq 0, \\
\rho(r) = (1-\alpha)|A|r, & B = 0.
\end{cases}
\]

By using the definition of the class \( S^*_\lambda(A, B, \alpha) \) and the definition of the subordination we can write

\[
|z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} - \frac{1-B[(1-\alpha)A+\alpha B]}{1-Br^2}| \leq \frac{(1-\alpha)(A-B)r}{1-B^2r^2}.
\] (2.8)

After simple calculations from (2.8) we get

\[
\frac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]}{1-B^2r^2} \leq \text{Re} \left( z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} \right)
\]

\[
\leq \frac{1+(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]}{1-B^2r^2}, \quad B \neq 0,
\] (2.9)

\[
1-(1-\alpha)|A|r \leq \text{Re} \left( z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} \right) \leq 1+(1-\alpha)|A|r, \quad B = 0.
\]

On the other hand we have

\[
\text{Re} \left( z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} \right) = r \frac{\partial}{\partial r} \log |D^\lambda f(z)|, \quad |z| = r.
\] (2.10)
If we substitute (2.9) into (2.10) we get

\[
\begin{align*}
\frac{1}{r} - \frac{(1 - \alpha)(A - B)}{1 - Br} \leq \frac{\partial}{\partial r} \log |D^\lambda f(z)| & \leq \frac{1}{r} + \frac{(1 - \alpha)(A - B)}{1 + Br}, & B \neq 0, \\
\frac{1}{r} - (1 - \alpha)|A| \leq \frac{\partial}{\partial r} \log |D^\lambda f(z)| & \leq \frac{1}{r} + (1 - \alpha)|A|, & B = 0.
\end{align*}
\]

(2.11)

Integrating both sides (2.11) and substituting \( D^\lambda f(z) = \Gamma(2 - \lambda)z^\lambda D^\lambda_2 f(z) \) into the result of integration we obtain (2.7).

**Remark 2.7.** Similarly, if we give special values to \( A, B, \alpha \) and \( \lambda \) we obtain the distortions of the subclasses \( S^*_\lambda(A, B, \alpha) \).

**Acknowledgement.** The authors would like to express sincerest thanks to the referee for suggestions.

**References**


(received 11.04.2007, in revised form 24.03.2008)

Department of Mathematics and Computer Science, İstanbul Kültür University, 34156 İstanbul, Turkey

E-mail: m.caglar@iku.edu.tr, y.polatoglu@iku.edu.tr, e.yavuz@iku.edu.tr