ASYMPTOTIC BEHAVIOUR OF DIFFERENTIATED
BERNSTEIN POLYNOMIALS

Heiner Gonska and Ioan Raşa

Abstract. In the present note we give a full quantitative version of a theorem of Floater dealing with the asymptotic behaviour of differentiated Bernstein polynomials. While Floater’s result is a generalization of the classical Voronovskaya theorem, ours generalizes a hardly known quantitative version of this theorem due to Videnskiǐ, among others.

1. Introduction

In a recent article Floater [2] proved the following

**Theorem 1.** If \( f \in C^{k+2}[0,1] \) for some \( k \geq 0 \), then

\[
\lim_{n \to \infty} n \left\{ (B_n f)^{(k)}(x) - f^{(k)}(x) \right\} = \frac{1}{2} \frac{d^k}{dx^k} \{x(1-x)f''(x)\},
\]

uniformly for \( x \in [0,1] \).

Here \( B_n \) is the Bernstein operator defined for a function \( f : [0,1] \to \mathbb{R} \) and \( x \in [0,1] \) by

\[
B_n f(x) = \sum_{i=0}^{n} f \left( \frac{i}{n} \right) p_{n,i}(x),
\]

where

\[
p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, \ldots, n.
\]

In the sequel we will also use the abbreviations X = x(1 − x) and \( e_i(x) = x^i \) for \( i = 0, 1, 2, \ldots \). Floater’s result is a generalization of the classical Voronovskaya theorem (see [10]) which is obtained for \( k = 0 \). In a recent paper [3] the latter
Theorem was given in quantitative form as follows, improving an earlier estimate by Videnskiǐ (see [9]).

**Theorem 2.** For \( f \in C^2[0, 1] \), \( x \in [0, 1] \) and \( n \in \mathbb{N} \) one has

\[
\left| n \cdot [B_n(f; x) - f(x)] - \frac{x(1 - x)}{2} \cdot f''(x) \right| \leq \frac{x(1 - x)}{2} \cdot \tilde{\omega}\left( f''; \sqrt{\frac{1}{n^2} + \frac{x(1 - x)}{n}} \right).
\]

Here \( \tilde{\omega} \) is the least concave majorant of \( \omega \), the first order modulus of continuity, satisfying

\[
\omega(f; \epsilon) \leq \tilde{\omega}(f; \epsilon) \leq 2\omega(f; \epsilon), \quad \epsilon \geq 0.
\]

The above inequality follows from a more general asymptotic statement which was inspired by results of Bernstein [1] and Mamedov [7]. This is given in

**Theorem 3.** Let \( q \in \mathbb{N}_0 \), \( f \in C^q[0, 1] \) and \( L: C[0, 1] \to C[0, 1] \) be a positive linear operator. Then

\[
\left| L(f; x) - \sum_{r=0}^{q} L((e_1 - x)^r; x) \cdot \frac{f^{(r)}(x)}{r!} \right| \leq \frac{L((e_1 - x)^q; x)}{q!} \tilde{\omega}\left( f^{(q)}; \frac{L((e_1 - x)^{q+1}; x)}{(q+1) L((e_1 - x)^q; x)} \right).
\]

It is the aim of this note to prove a quantitative version of Floater’s result. In doing so we will make essential use of a corollary of Theorem 3 for the case \( q = 2 \).

**Corollary 1.** Under the assumptions of Theorem 3 one has, for \( q = 2 \), the inequality

\[
\left| L(f; x) - f(x) \cdot L(e_0; x) - f'(x) \cdot L(e_1 - x; x) - \frac{1}{2} f''(x) \cdot L((e_1 - x)^2; x) \right| \leq \frac{1}{2} \cdot L((e_1 - x)^2; x) \cdot \tilde{\omega}\left( f''; \frac{1}{3} \sqrt{\frac{L((e_1 - x)^4; x)}{L((e_1 - x)^2; x)}} \right).
\]

The square root is obtained by using the Cauchy-Schwarz inequality for positive linear functionals.

### 2. An auxiliary result

An operator \( L: C[0, 1] \to C^k[0, 1] \) is said to be convex of order \( k - 1 \) if it preserves convexity of order \( k - 1 \), \( k \in \mathbb{N} \cup \{0\} \). This means that any function \( f \) with divided differences

\[
[x_0, \ldots, x_k; f] \geq 0 \quad \text{for any } x_0 < \cdots < x_k \in [0, 1]
\]

is mapped to a function \( Lf \) having the same property.
The Bernstein operator is an example of a mapping which is convex of all orders \( k \in \mathbb{N} \cup \{0\} \).

For an operator \( L \) being convex of order \( k-1 \) and satisfying \( L(\Pi_{k-1}) \subseteq \Pi_{k-1} \) consider

\[
I_k : C[0,1] \to C[0,1] \quad \text{given by} \quad (I_k f)(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t) \, dt.
\]

Let \( Q^k := D^k \circ L \circ I_k \) where \( D^k = \frac{d^k}{dx^k} \).

\( Q^k \) may be considered as a \( k \)-th order Kantorovich modification of \( L \). Since \( L \) is convex of order \( k-1 \), it follows that \( Q^k \) is a linear and positive (convex of order \(-1\)) operator. Since \( I_k D^k f - f \in \Pi_{k-1} \) and \( L(\Pi_{k-1}) \subseteq \Pi_{k-1} \), we have \( L(I_k D^k f - f) \in \Pi_{k-1} \). It follows \( D^k L I_k D^k f = D^k L f \), hence \( Q^k D^k f = D^k L f \), for all \( f \in C^k[0,1] \).

To our knowledge the latter construction is due to Sendov and Popov [8].

3. Main result

In this section we will prove the main result of this note by providing the following quantitative version of Floater’s convergence result.

**Theorem 4.** If \( f \in C^{k+2}[0,1] \) for some \( k \geq 0 \), then

\[
|n[(B_n f)^{(k)}(x) - f^{(k)}(x)] - \frac{1}{2} \cdot \frac{d^k}{dx^k} \{x(1-x)f''(x)\}| \\
\leq O\left(\frac{1}{n}\right) \cdot \max_{k \leq s \leq k+2} \{|f^{(s)}(x)|\} + O(1) \cdot \tilde{\omega} \left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right).
\]

Here \( O\left(\frac{1}{n}\right) \) and \( O(1) \) represent sequences of order \( O\left(\frac{1}{n}\right) \) and \( O(1) \), respectively, which depend on the fixed \( k \).

**Proof.** Put \( Q_k^n := D^k B_n I_k \). For this positive linear operator we apply Corollary 1 and write the left hand side of the inequality for \( f \in C^{k+2}[0,1] \) as

\[
|Q^n_k (f^{(k)}; x) - f^{(k)}(x) \cdot Q^n_k (e_0; x) - f^{(k+1)}(x) \cdot Q^n_k (e_1 - x; x) \\
- \frac{1}{2} \cdot f^{(k+2)}(x) \cdot Q^n_k ((e_1 - x)^2; x) |
\]

\[
= |D^k B_n (f; x) - f^{(k)}(x) + f^{(k)}(x)(1 - Q^n_k (e_0; x)) - f^{(k+1)}(x) \cdot Q^n_k (e_1 - x; x) \\
- \frac{1}{2} f^{(k+2)}(x) \cdot Q^n_k ((e_1 - x)^2; x) \cdot \frac{d^k}{dx^k} \{x(1-x) \cdot f''(x)\} |
\]

\[
+ \frac{1}{2n} \cdot \frac{d^k}{dx^k} \{x(1-x) \cdot f''(x)\} |
\]

\[
= |D^k B_n (f; x) - f^{(k)}(x) - \frac{1}{2n} \cdot \frac{d^k}{dx^k} \{x(1-x) \cdot f''(x)\} |
\]

\[
- \{(Q^n_k (e_0; x) - 1) \cdot f^{(k)}(x) + f^{(k+1)}(x) \cdot Q^n_k (e_1 - x; x) \\
+ \frac{1}{2} f^{(k+2)}(x) \cdot Q^n_k ((e_1 - x)^2; x) \cdot \frac{d^k}{dx^k} \{x(1-x) \cdot f''(x)\} |.
\]
Multiplying the inequality of Corollary 1 by \( n \) and using the (second) triangular inequality yields

\[
\left| n \cdot \{ D^k B_n(f; x) - f^{(k)}(x) \} - \frac{1}{2} \cdot \frac{d^k}{dx^k} \{ x(1-x) \cdot f''(x) \} \right| \\
\leq \left| n \cdot (Q_n^k(e_0; x) - 1) \cdot f^{(k)}(x) + f^{(k+1)}(x) \cdot n \cdot Q_n^k(e_1 - x; x) \right| \\
\quad + \frac{1}{2} \cdot f^{(k+2)}(x) \cdot n \cdot Q_n^k((e_1 - x)^2; x) - \frac{1}{2} \cdot \frac{d^k}{dx^k} \{ x(1-x) \cdot f''(x) \} \\
\quad + \frac{n}{2} \cdot Q_n^k((e_1 - x)^2; x) \cdot \omega \left( f^{(k+2)}; \frac{1}{3} \frac{Q_n^k((e_1 - x)^4; x)}{Q_n^k((e_1 - x)^2; x)} \right).
\]

Both summands of the r.h.s. will now be inspected separately. In order to do so, first observe that by Leibniz’ rule one has

\[
\frac{1}{2} \cdot \frac{d^k}{dx^k} \{ X f''(x) \} = \frac{1}{2} \left\{ f^{(k+2)}(x) \cdot X + k \cdot f^{(k+1)}(x) \cdot X' + \frac{k(k-1)}{2} f^{(k)}(x)(-2) \right\}.
\]

Note that this is correct also for \( k \in \{0, 1\} \). So the first summand can be estimated from above by

\[
\left| n \cdot (Q_n^k(e_0; x) - 1) + \frac{k(k-1)}{2} \right| \cdot |f^{(k)}(x)| + \left| n \cdot Q_n^k(e_1 - x; x) - \frac{k}{2} X' \right| \cdot |f^{(k+1)}(x)| \\
+ \left| \frac{n}{2} \cdot Q_n^k((e_1 - x)^2; x) - \frac{1}{2} X \right| \cdot |f^{(k+2)}(x)| \\
:= A_n^k \cdot |f^{(k)}(x)| + B_n^k \cdot |f^{(k+1)}(x)| + C_n^k \cdot |f^{(k+2)}(x)|.
\]

Now for \( n \geq 1 \) and \( k \geq 0 \), also noting that \( (n)_0 = 1 \), we have

\[
A_n^k = \left| n \cdot \left( \frac{(n)_k}{n^k} - 1 \right) + \frac{k(k-1)}{2} \right| \\
= \left| n \cdot \frac{(n)(n-1)\ldots(n-k+1) - n^k}{n^k} + \frac{k(k-1)}{2} \right| \\
= \left| \frac{(n)_k - n^k}{n^{k-1}} + \frac{k(k-1)}{2} \right| \leq \left| \frac{k(k-1)}{2} + \frac{k(k-1)}{2} \right| + O \left( \frac{1}{n} \right).
\]

In order to verify the last inequality it is only necessary to observe that

\[
\frac{1}{n^{k-1}} \left( (n)_k - n^k \right) = \frac{1}{n^{k-1}} \left\{ n^k - \frac{k(k-1)}{2} n^{k-1} + (\text{lower order terms in } n) - n^k \right\}.
\]

Note that \( A_n^k = 0 \) for \( k \in \{0, 1\} \).
Moreover (see [5], pp. 44–45), for \( n \geq 1 \) and \( k \geq 0 \) we get

\[
B_n^k = \left| n \cdot Q_n^k(e_1 - x; x) - \frac{k}{2} \cdot X \right| \cdot \left| n \cdot \frac{(n)_k}{n^k} \cdot \frac{k}{2n} X' - \frac{k}{2} \cdot X' \right|
\]

\[
= \frac{k}{2} |X'| \cdot \left| \frac{(n)_k}{n^k} - 1 \right| \leq \frac{k}{2} |X'| \cdot \frac{k(k-1)}{2k} = O \left( \frac{1}{n} \right).
\]

We also have \( B_n^k = 0 \) for \( k \in \{0, 1\} \). For the last factor we have (see [6], p. 26) for \( n \geq k + 2 \)

\[
C_n^k = \left| \frac{n}{2} \cdot Q_n^k((e_1 - x)^2; x) - \frac{1}{2} X \right|
\]

\[
= \frac{n}{2} \cdot \frac{(n)_k}{n^{k+2}} \left[ (n-k(k+1)) \cdot X + \frac{1}{12} k(3k+1) \right] - \frac{1}{2} X
\]

\[
= \frac{1}{2} X \left\{ \frac{(n)_k}{n^{k+1}}(n-k(k+1)) - 1 \right\} + \frac{1}{24} \cdot \frac{(n)_k}{n^{k+1}} k(3k+1)
\]

\[
\leq \frac{1}{2} X \left\{ \frac{(n)_k}{n^{k+1}}(n-k(k+1)) - 1 \right\} + O \left( \frac{1}{n} \right).
\]

It remains to consider the quantity inside \( \ldots \). The latter is equal to

\[
\frac{1}{n^{k+1}} \left( n^k - \frac{k(k-1)}{2} n^{k-1} + O(n^{k-2})(n-k(k+1)) \right) - 1
\]

\[
= 1 - \frac{k(k-1)}{2n} + O \left( \frac{1}{n^2} \right) - O \left( \frac{1}{n} \right) + O \left( \frac{1}{n^2} \right) - O \left( \frac{1}{n^3} \right) - 1 = O \left( \frac{1}{n} \right).
\]

Note that \( C_n^k = 0 \) for \( k = 0 \). So we know that

\[
\left| \frac{n}{2} \cdot \left\{ D^k B_n(f; x) - f^{(k)}(x) \right\} - \frac{1}{2} \cdot \frac{d^k}{dx^k} \{ x(1-x) \cdot f''(x) \} \right|
\]

\[
\leq O \left( \frac{1}{n} \right) \cdot \max \{|f^{(k)}(x)|, |f^{(k+1)}(x)|, |f^{(k+2)}(x)|\}
\]

\[
+ \frac{n}{2} \cdot Q_n^k((e_1 - x)^2; x) \cdot \omega \left( f^{(k+2)}; \frac{1}{3} \frac{Q_n^k((e_1 - x)^4; x)}{Q_n^k((e_1 - x)^2; x)} \right),
\]

where \( O \) depends only on \( k \).

For the factor in front of \( \omega(f^{(k+2)}; \ldots) \) we have already observed that

\[
\frac{n}{2} \cdot Q_n^k((e_1 - x)^2; x) = \frac{n}{2} \cdot \frac{(n)_k}{n^{k+2}} \left[ (n-k(k+1)) \cdot X + \frac{1}{12} k(3k+1) \right] = O(1).
\]

Hence it remains to consider the square root in \( \omega(f^{(k+2)}; \ldots) \). This is done in the following lemmas dealing with the moments of \( Q_n^k \).

**Lemma 1.** Suppose that \( L_n : \Pi \rightarrow \Pi, n \geq 1 \), is a linear operator mapping polynomials into polynomials and such that \( L_n(\Pi_j) \subset \Pi_j \), \( n \geq 1, j \geq 0 \). If we
define

\[ M_{n,m}(x) := \frac{1}{m!} L_n((e_1 - x)^m; x), n \geq 1, m \geq 0, \]

\[ R_{n,p}^k(x) := \frac{1}{p!} Q_n^k((e_1 - x)^p; x), n \geq 1, k \geq 0, p \geq 0, \]

then

\[ R_{n,p}^k(x) = \sum_{j=p}^{p+k} \binom{k}{p+k-j} M_{n,j}^{(j-p)}(x). \]

Proof. We will use the notation \((k)f = I_k f\) to denote a \(k\)-th antiderivative of \(f, f \in C[0, 1]\).

First observe that

\[ M_{n,m} \in \Pi_m \quad \text{for} \quad n \geq 1 \quad \text{and} \quad m \geq 0. \]

Now let \(k \geq 0, p \geq 0\) be fixed, \(f \in \Pi_p, x \in [0, 1]\). Then

\[(k)f(t) = \sum_{j=0}^{p+k} (k)_{(j)}(x) \frac{1}{j!} (t-x)^j,\]

and hence

\[ L_n((k)f)(x) = \sum_{j=0}^{p+k} (k)_{(j)}(x) M_{n,j}(x). \]

Thus

\[ Q_n^k f = \sum_{j=0}^{p+k} (k)_{(j)} M_{n,j}((k)_{(j)}) = \sum_{j=0}^{p+k} \binom{k}{j} (k)_{(j)} f^{(j)}(0) M_{n,j}^{(k-i)}. \]

Noting that \(M_{n,j}^{(k-i)} = 0\) if \(0 \leq i < k - j\), we may write

\[ Q_n^k f = \sum_{j=0}^{p+k} \binom{k}{j} f^{(j)}(0) M_{n,j}^{(k-i)}. \]

Substituting \(i + j = \ell, i = \ell + k - j\), the latter becomes

\[ = \sum_{j=0}^{p+k} \sum_{\ell = \max\{j-k,0\}}^{k} \binom{k}{\ell} f^{(j)}(0) M_{n,j}^{(\ell-i)} \]

\[ = \sum_{\ell=0}^{p+k} \min\{\ell+k,p+k\} \sum_{j=\ell}^{k} \binom{k}{\ell} f^{(j)}(0) M_{n,j}^{(\ell-i)}, f \in \Pi_p. \]

This is correct in view of \((a_{j,\ell} \in \mathbb{R})\)

\[ \sum_{j=0}^{p+k} \sum_{\ell=\max\{j-k,0\}}^{j} a_{j,\ell} = \sum_{\ell=0}^{p+k} \begin{cases} 
\sum_{j=\ell}^{k} a_{j,\ell}, \text{ if } k + \ell \leq k + p, \\
\sum_{j=\ell}^{k+p} a_{j,\ell}, \text{ if } k + \ell > k + p.
\end{cases} \]

\[ = \sum_{\ell=0}^{p+k} \min\{\ell+k,p+k\} \sum_{j=\ell}^{k} a_{j,\ell}. \]
Asymptotic behaviour of differentiated Bernstein polynomials

Note that the l.h.s. corresponds to “horizontal summation first, then vertical”, while the r.h.s. corresponds to the opposite.

We obtain $R_{n,p}^k(x)$ if we put $f = \frac{1}{p!}(e_1 - x)^p$ in $Q_{n}^k f$, also observing that $f^{(\ell)}(x) = 0$ for $\ell \in \{0, \ldots, p + k\} \setminus \{p\}$ and $f^{(p)}(x) = 1$. Hence

$$R_{n,p}^k(x) = \sum_{j=p}^{p+k} \binom{k}{p+k-j} M_{n,j}^{(j-p)}(x), \quad n \geq 1, k \geq 0, p \geq 0.$$ 

In order to come up with a description of the asymptotic behavior of the ratio in question, we investigate the quantities $R_{n,p}^k(x)$ further in case that $L_n = B_n$.

We have the following

**Lemma 2.** For the Bernstein operators $B_n$ we have

$$n \frac{R_{n,4}^k}{R_{n,2}^k} \leq A, \quad n \geq 1,$$

for some positive constant $A$.

**Proof.** For $B_n$ we have

$$M_{n,2j}(x) = \frac{X}{n^{2j-1}} \sum_{i=0}^{j-1} a_{ji}(n) X^i,$$

$$M_{n,2j+1}(x) = \frac{XX'}{n^{2j}} \sum_{i=0}^{j-1} b_{ji}(n) X^i,$$

where $a_{ji}(n)$ and $b_{ji}(n)$ are polynomials in $n$, of degree $i$.

By the previous lemma,

$$R_{n,4}^k = \sum_{j=4}^{k+4} \binom{k}{k+4-j} M_{n,j}^{(j-4)} = \frac{1}{n^2}(X^2 u_{k+1}(n) + X v_k(n) + w_{k-1}(n)),$$

where $u_{k+1}$, $v_k$ and $w_{k-1}$ are polynomials of degrees indicated by the corresponding indices. Analogously,

$$R_{n,2}^k = \frac{1}{n}(X q_{k+1}(n) + r_k(n)).$$

Now the claim of the lemma is a consequence of the latter two representations.

**Continuation of the proof of Theorem 4.** All we have to observe is that

$$\sqrt{\frac{Q_n^k((e_1 - x)^4; x)}{Q_n^k((e_1 - x)^2; x)}} = \sqrt{\frac{4! \cdot R_{n,4}^k(x)}{2! \cdot R_{n,2}^k(x)}} \leq \sqrt{6A \cdot \frac{1}{n}},$$

where $A$ is the uniform bound from Lemma 2. The final statement follows from the inequality

$$\tilde{\omega}(f; \epsilon) \leq (c + 1) \cdot \tilde{\omega}(f; \epsilon), \quad c, \epsilon \geq 0.$$
Remark 1. We noted earlier that $A_n^k = B_n^k = C_n^k = 0$ for $k = 0$. In this case
the inequality of Theorem 4 can be replaced by

$$|n[B_n f(x) − f(x)] − \frac{1}{2}x(1 − x) \cdot f''(x)| \leq O(1) \cdot \tilde{\omega} \left( f''; \frac{1}{\sqrt{n}} \right).$$

In fact, looking at the proof again shows that we even get the right hand side

$$\frac{n}{2} \cdot B_n((e_1 - x)^2; x) \cdot \tilde{\omega} \left( f''; \frac{1}{3 \sqrt{n}} \right) \leq \frac{x(1 − x)}{2} \cdot \tilde{\omega} \left( f''; \frac{1}{3 \sqrt{n}} \right).$$

This is the quantitative version of the classical Voronovskaya theorem given first in [4].

REFERENCES


(received 31.7.2008, in revised form 7.2.2009)

Heiner Gonska, University of Duisburg-Essen, Department of Mathematics, D-47048 Duisburg, Germany
E-mail: heiner.gonska@uni-due.de
Ioan Raşa, Technical University, Department of Mathematics, RO-400020 Cluj-Napoca, Romania
E-mail: Ioan.Rasa@math.utcluj.ro