EQUITORSION CONFORM MAPPINGS OF GENERALIZED Riemannian Spaces

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Abstract. We define an equitorsion conform mapping of two generalized Riemannian spaces and obtain some invariant geometric objects of this mapping, generalizing the tensor of conform curvature.

0. Introduction

A generalized Riemannian space $GR_N$ in the sense of Eisenhart’s definition [5] is a differentiable $N$-dimensional manifold, equipped with nonsymmetric basic tensor $g_{ij}$.

The use of non-symmetric basic tensor and non-symmetric connection became especially actual after appearance of the works of A. Einstein [1]–[4] related to creation of the Unified Field Theory (UFT). Remark that at UFT the symmetric part $g_{ij}$ of the basic tensor $g_{ij}$ is related to the gravitation, and antisymmetric one $g_{ij}$ to the electromagnetism. M Prvanović [14] and S. Minčić [8] gave geometric interpretations of the torsion and curvature tensors of non-symmetric affine connection.

Consider two $N$-dimensional generalized Riemannian spaces $GR_N$ and $GR_N'$. Generalized Cristoffel’s symbols of the first kind of the space $GR_N$ and $GR_N'$ are given by

$$
\Gamma_{i,jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \quad \text{and} \quad \Gamma_{i,jk}' = \frac{1}{2}(g_{ji,k}' - g_{jk,i}' + g_{ik,j}') \quad (0.1)
$$

where, for example, $g_{ij,k} = \partial g_{ij}/\partial x^k$. Connection coefficients of these spaces are generalized Cristoffel’s symbols of the second kind $\Gamma_{jk}^i = g^{lp}_{jk} \Gamma_{p,ik}$ and $\Gamma_{jk}'^i = \frac{1}{2}(g_{jk,i}' - g_{ij,k}' + g_{ik,j}')$. 

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\[ \delta^i_j k, \] respectively, where \((g^i_j)^{(1)} = (g_{ij})^{-1}\) and \(i\) denote symmetrisation with division by indices \(i\) and \(j\). Generally it is \(\Gamma^i_j k \neq \Gamma^i_{k,j}\). We suppose that \(g = det(g_{ij}) \neq 0, \quad g = det(g^i_j) \neq 0, \quad g = det(g^{i,j}) \neq 0.\)

One says that a reciprocal one-valued mapping \(f : GR_N \to G\mathcal{R}_N\) is conform if for the basic tensors \(g_{ij}\) and \(\overline{g}_{ij}\) of these spaces the condition

\[ \overline{g}_{ij} = e^{2\psi} g_{ij} \quad (0.2) \]

is satisfied, where \(\psi\) is an arbitrary function of \(x\)'s, and the spaces are considered in the common by this mapping system of local coordinates \(x^i\). In this case for the Cristoffel's symbols of the first kind of the spaces \(GR_N\) and \(G\mathcal{R}_N\) the relation

\[ \Gamma^i_{j,k} = e^{2\psi}(\Gamma^i_{j,k} + g_{ij}\psi_{,k} - g_{jk}\psi_{,i} + g_{ik}\psi_{,j}) \quad (0.3) \]

holds true, and for the Cristoffel's symbols of the second kind

\[ \Gamma^i_{j,k} = \Gamma^i_{j,k} + g_{ij}^2 g_{jk} \psi_{,p} \psi_{,p} + g_{pk} \psi_{,j} \quad (0.4) \]

holds. Let us denote \(\psi_{,k} = \psi_{,k} = \partial \psi / \partial x^k\) and \(\psi^i = g_{ij}^2 \psi_{,j}\). Now from (0.4) we have

\[ \Gamma^i_{j,k} = \Gamma^i_{j,k} + g_{ij}^2 (g_{jp} \psi_{,k} - g_{jk} \psi_{,p} + g_{pk} \psi_{,j}) + g_{ij}^2 (g_{jp} \psi_{,k} - g_{jk} \psi_{,p} + g_{pk} \psi_{,j}), \]

i.e.

\[ \Gamma^i_{j,k} = \Gamma^i_{j,k} + \delta^i_j \psi_{,k} + \delta^i_k \psi_{,j} - \psi^i g_{jk} + \xi^i_{jk}, \quad (0.5) \]

where

\[ \xi^i_{jk} = g_{ij}^2 (g_{jp} \psi_{,k} - g_{jk} \psi_{,p} + g_{pk} \psi_{,j}) = -\xi^i_{kj} \quad (0.5') \]

and \(ij\) denotes an antisymmetrisation with division. In the corresponding points \(M(x)\) and \(\overline{M}(x)\) of conform mapping we can put

\[ \Gamma^i_{j,k} = \Gamma^i_{j,k} + P^i_{j,k} \quad (i, j, k = 1, \ldots, N), \quad (0.6) \]

where \(P^i_{j,k}\) is the deformation tensor of the connection \(\Gamma\) of \(GR_N\) according to the conform mapping \(f : GR_N \to G\mathcal{R}_N\).

Notice that in \(GR_N\) we have

\[ \Gamma^i_{j,p} = 0, \quad (0.7) \]

(eq. (2.10) in [13]).

In a generalized Riemannian space one can define four kinds of covariant derivatives [10, 11]. For example, for a tensor \(a^i_j\) in \(GR_N\) we have

\[ a^i_{j,m} = a^i_{j,m} + \Gamma^i_{p,m} a^p_{j} - \Gamma^p_{j,m} a^i_{p}, \]

\[ a^i_{j,m} = a^i_{j,m} + \Gamma^i_{m,p} a^p_{j} - \Gamma^p_{m,j} a^i_{p}, \]

\[ a^i_{j,m} = a^i_{j,m} + \Gamma^i_{p,m} a^p_{j} - \Gamma^p_{m,j} a^i_{p}, \]

\[ a^i_{j,m} = a^i_{j,m} + \Gamma^i_{m,p} a^p_{j} - \Gamma^p_{j,m} a^i_{p}. \]
Denote by $\theta^\frac{\partial}{\partial \theta}$ a covariant derivative of the kind $\theta$ in $GR_N$ and $GR^\perp_N$ respectively. We have [7]
\[ g_{ij}\theta^m = 0. \]

In the case of the space $GR_N$, we have five independent curvature tensors [9] (in $\mathbb{R}^5$ is denoted by $\tilde{R}^i_{jmn}$):
\begin{align*}
R_1^{i jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm}, \\
R_2^{i jmn} &= \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp}, \\
R_3^{i jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} + \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn}(\Gamma^i_{pj} - \Gamma^i_{jp}), \\
R_4^{i jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} + \Gamma^p_{nj}\Gamma^i_{mp} + \Gamma^p_{mn}(\Gamma^i_{pj} - \Gamma^i_{jp}), \\
R_5^{i jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} + \Gamma^p_{mj}\Gamma^i_{np}) - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}).
\end{align*}

We use the conform mapping $f: GR_N \rightarrow GR^\perp_N$ to obtain tensors $\tilde{R}^i_{\theta jmn}$ ($\theta = 1, \ldots, 5$), where for example
\[ \tilde{R}_1^{i jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm}. \]

In the case of conform mapping $f: R_N \rightarrow \mathbb{R}^N$ of Riemannian spaces $R_N$ and $\mathbb{R}^N$ [6, 15] we have an invariant geometric object
\[ C_{jmn} = R_{jmn} + \delta^i_m P_{jn} - \delta^i_n P_{jm} + P^i_{m} g_{jn} - P^i_{n} g_{mj}. \]

where
\[ P_{jm} = \frac{1}{N - 2}(R_{jm} - \frac{1}{2(N - 1)}R g_{jm}), \]
and $R_{jmn}$ is Riemann-Cristoffel’s curvature tensor of the space $R_N$, $R_{jm}$ Ricci’s tensor and $R$ a scalar curvature.

The object $C_{jmn}^{i}$ is called a conform curvature tensor [6, 15]. Having a conform mapping of two generalized Riemannian spaces, we cannot find a generalization of the tensor of conform curvature as an invariant of conform mapping in general case. For that reason we define a special conform mapping.

A mapping $f: GR_N \rightarrow GR^\perp_N$ is an equitorsion conform mapping if the torsion tensors of the spaces $GR_N$ and $GR^\perp_N$ are equal. Then from (0.5) and (0.6) we have
\[ \xi_{jk} = 0. \]

In [12] we have investigated equitorsion geodesic mappings of generalized Riemannian spaces.
1. Equitorsion conform curvature tensor of the first kind

Using (0.6), we get a relation between the first kind curvature tensors of the spaces \( GR_N \) and \( \mathcal{G}R_N [12, 16] \)

\[
\mathcal{T}^i_{jmn} = R^i_{jmn} + P^i_{jmn} - P^i_{jmn} P^m_{j\nu} P^\nu_{jm} - P^m_{j\nu} P^\nu_{jm} + 2\Gamma^p_{mn} P^i_{jp}. 
\]

Substituting \( P \) with respect to (0.5, 6, 10), and using (0.7'), we obtain

\[
\mathcal{T}^i_{jmn} = R^i_{jmn} + \delta^i_j (\psi^m_{|1} - \psi_m |1) + \delta^i_m (\psi^j_{|1} - \psi_j |1)
\]

\[
- \delta^i_n (\psi^j_{|1} - \psi_j |1) g_{jm} + (\psi^j_{|1} - \psi_j |1) g_{jm}
\]

\[
- \delta^i_n \psi^m \psi^g_{jm} + \delta^i_m \psi^m \psi^g_{jn} + 2\delta^i_j \Gamma^{i}_{mn} \psi_p + 2\Gamma^{i}_{mn} \psi_j - 2\Gamma^{i}_{j,mn} \psi^i.
\]

Denoting

\[
\psi_{ij} = \psi_{1ij} - \psi_i \psi_j, \quad \psi^i_j = g^{ip} \psi_{p j}
\]

\[
\Delta_1 \psi = g^{pq} \psi_{p q} = \psi_p \psi^p
\]

and using the relation

\[
\psi_{mn} = \psi_{nm} = -2\Gamma^{i}_{mn} \psi_p
\]

in (1.1), we get

\[
\mathcal{T}^i_{jmn} = R^i_{jmn} + \delta^i_j \psi^m_{|1} - \psi^j_{|1} \psi^m + \psi^i_{m} g_{jm} - \psi^i_{n} g_{jm}
\]

\[
+ (\delta^i_m g_{jm} - \delta^i_n g_{jm}) \Delta_1 \psi + 2\Gamma^{i}_{mn} \psi_j - 2\Gamma^{i}_{j,mn} \psi^i.
\]

Further, let us denote

\[
\Delta^2 \psi = g^{pq} \psi_{p q}
\]

Then we have

\[
\psi_p = g^{pq} g^{pq} (\psi_{p q} - \psi_p \psi_q) g^{pq} = \Delta^2 \psi - \Delta_1 \psi.
\]

Contracting by indices \( i \) and \( n \) in (1.4) we get

\[
\mathcal{T}^i_{jmn} = R^i_{jmn} - (N - 2) \psi_{jm} - [\Delta^2 \psi + (N - 2) \Delta_1 \psi] g_{jm} - 2\Gamma^{i}_{j,mn} \psi^p.
\]

From (0.2) we get

\[
\mathcal{T}^i_{jmn} = e^{2\psi} g^{i j}.
\]

In (1.6) multiplying by \( g^{jm} \) and contracting by \( j \) and then by \( m \) we get

\[
e^{2\psi} \mathcal{T}^i_{1} = R - 2(N - 1) \Delta^2 \psi - (N - 1) (N - 2) \Delta_1 \psi,
\]
where $\tilde{R}_1^i = \tilde{g}^{pq} \tilde{R}_1^i_{pq}$, and $R = g^{pq} R_1^i_{pq}$ are scalar curvature of the first kind of the spaces $GR_N$ and $GR_N$ respectively. From (1.8) we have

$$\Delta_2 \psi = \frac{1}{2(N-1)} (R - e^{2\psi} \tilde{R}_1^i) - \frac{N-2}{2} \Delta_1 \psi. \quad (1.9)$$

Substituting (1.9) in (1.6) we get

$$(N - 2) \psi_{jm} = R_{jm} - \tilde{R}_1^i_{jm} - \frac{1}{2(N-1)} (R - e^{2\psi} \tilde{R}_1^i) g_{jm}$$

$$- \frac{N-2}{2} \Delta_1 \psi g_{jm} - 2 \Gamma_{j.m.p} \psi^p. \quad (1.10)$$

Let us denote in the space $GR_N$

$$P_{jm} = \frac{1}{N-2} (R_{jm} - \frac{1}{2(N-1)} R_{gjm}) \quad (1.10')$$

and analogously $\tilde{P}_{jm}$ in the space $GR_N$. In this case for $\psi_{jm}$ we obtain

$$\psi_{jm} = P_{jm} - \tilde{P}_{jm} - \frac{1}{2} \Delta_1 \psi g_{jm} - \frac{2}{N-2} \Gamma_{j.m.p} \psi^p. \quad (1.11)$$

Substituting (1.11) in (1.4), we get

$$\tilde{R}_1^i_{jmn} = R_1^i_{jmn} + \delta_m^i (P_{jm} - \tilde{P}_{jm}) - \delta_n^i (P_{jm} - \tilde{P}_{jm})$$

$$+ P_{jm}^i g_{jn} - \tilde{P}_{jm}^i g_{jn} - \frac{1}{N-2} (\delta_n^i \Gamma_{j.m.p} - \delta_m^i \Gamma_{j.n.p} + \Gamma_{im}^i \psi_{jm} - \Gamma_{jm}^i \psi_m) \psi^p$$

$$+ 2 \Gamma_{mn}^i \psi_j - 2 \Gamma_{j.m.n} \psi^i. \quad (1.12)$$

We can see that it follows from (0.2)

$$\psi_i = \frac{1}{2N} \left( \frac{\partial}{\partial x^i} \ln g - \frac{\partial}{\partial x^i} \ln \tilde{g} \right) \quad (1.13)$$

where $g = \det (g_{ij})$, $\tilde{g} = \det (\tilde{g}_{ij})$. From (0.10) and (1.13) we obtain

$$\Gamma_{j,m,n} \psi^i = \frac{1}{2N} \Gamma_{j,m,n} \tilde{g}_{pq} \frac{\partial}{\partial x^p} \ln \tilde{g} - \frac{1}{2N} \Gamma_{j,m,n} g_{pq} \frac{\partial}{\partial x^p} \ln g \quad (1.14)$$

and

$$\Gamma_{q,mj}^i \psi^q = \frac{1}{2N} \Gamma_{q,mj}^i \tilde{g}_{pq} \frac{\partial}{\partial x^p} \ln \tilde{g} - \frac{1}{2N} \Gamma_{q,mj}^i g_{pq} \frac{\partial}{\partial x^p} \ln g. \quad (1.15)$$

Taking into account (1.13, 14,15), we can write the relation (1.12) in the form

$$\tilde{C}_1^i_{jmn} = C_1^i_{jmn}, \quad (1.16)$$
where
\[
\begin{align*}
C^i_{j mn} &= R^i_{j mn} + \delta^i_m P^j_{m n} - \delta^i_n P^j_{m n} + P^i_{1 m} g_{j n} - P^i_{1 n} g_{j m} \\
&
+ \frac{1}{N(N - 2)} (\delta^i_m \Gamma_{j np} - \delta^i_n \Gamma_{j mp} + \Gamma^i_{m p g} g_{j n} - \Gamma^i_{n p g} g_{j m}) g^{p q} \frac{\partial}{\partial x^q} \ln g \\
&
+ \frac{1}{N} (\Gamma_{j mn p} g^p - \Gamma_{m n p} g^p) \frac{\partial}{\partial x^p} \ln g
\end{align*}
\]

and analogously for \( \overline{C}^i_{j mn} \). From (1.16) we see that the tensor \( C^i_{j mn} \) is an invariant of equitorsion conform mapping, and one can call it the equitorsion conform curvature tensor of the first kind. So, we have

**Theorem 1.** Let generalized Riemannian spaces \( GR_N \) and \( GR_N' \) be defined by virtue of their nonsymmetric basic tensors \( g_{ij} \) and \( \overline{g}_{ij} \) respectively. The equitorsion conform curvature tensor of the first kind \( C^i_{j mn} \) (1.17) is an invariant of the equitorsion conform mapping \( f : GR_N \rightarrow GR_N' \), defined by (0.2), (0.5), (0.10), i.e. (1.16) is in force, where the tensor \( P^{i}_{1} \) is given by (1.10).

2. Equitorsion conform curvature tensor of the second kind

For the second kind curvature tensors of the spaces \( GR_N \) and \( GR_N' \) we get the relation [12, 16]
\[
\overline{R}^i_{j mn} = R^i_{j mn} + \delta^i_{m} P^j_{m} - P^i_{n m} - P^i_{n j} + 2 \Gamma^i_{m n} P_{j}, \tag{2.1}
\]
i.e., using (0.5,6,10) one obtains
\[
\overline{R}^i_{j mn} = R^i_{j mn} + \delta^i_{m} \psi_{j n} - \delta^i_{n} \psi_{j m} + \psi^i_{n} g_{j m} - \psi^i_{m} g_{j n} \\
+ (\delta^i_{m} g_{j m} - \delta^i_{n} g_{j n}) \Delta \psi + 2 \Gamma^i_{m n} \psi_{j} - 2 \Gamma_{m n} \psi_{i} g_{j},
\]
where
\[
\psi_{i j} = \psi_{i j} - \psi_{j i}, \quad \psi^i_{2 j} = g^{p q} \psi_{i q}, \quad \Delta \psi = g^{p q} \psi_{p} \psi_{q} \tag{2.2}
\]
Now, analogously to previous case, we get the invariant object of the equitorsion conform mapping \( f : GR_N \rightarrow GR_N' \)
\[
C^i_{j mn} = R^i_{j mn} + \delta^i_{m} P^j_{2 m} - \delta^i_{n} P^j_{2 n} + P^i_{2 m} g_{j n} - P^i_{2 n} g_{j m} \\
+ \frac{1}{N(N - 2)} (\delta^i_{m} \Gamma_{j np} - \delta^i_{n} \Gamma_{j mp} + \Gamma^i_{m p g} g_{j n} - \Gamma^i_{n p g} g_{j m}) g^{p q} \frac{\partial}{\partial x^q} \ln g \\
+ \frac{1}{N} (\Gamma_{j mn p} g^p - \Gamma_{m n p} g^p) \frac{\partial}{\partial x^p} \ln g
\]
\( \tag{2.3} \)
where
\[ P_{jm}^2 \equiv \frac{1}{N - 2} \left( R_{jm}^2 - \frac{1}{2(N - 1)} R g_{mj} \right), \quad (2.4) \]

\( R_{jm}^2 \) is Ricci’s curvature tensor of the second kind and \( R^2 \) is a scalar curvature tensor of the second kind. The object \( C_{jm}^2 \) is a tensor and we call it equitorsion conform curvature tensor of the second kind. Accordingly, we have

**Theorem 2.** Starting from the curvature tensor \( R_{jm}^2 \), under conditions as in Theorem 1, one obtains an invariant tensor \( C_{jm}^2 \) (2.3) of the equitorsion conform mapping of generalized Riemannian spaces, where \( P^2 \) is given according to (2.4).

### 3. Equitorsion conform curvature tensor of the third kind

In the case of the third kind curvature tensors of the spaces \( GR_N \) and \( G\bar{R}_N \) we get the relation [12, 16]
\[ R_{jm}^3 = R_{jm}^3 + P_{jm}^3 - P_{nj}^3 P_{jm}^3 + P_{np}^3 P_{jm}^3 - P_{nj}^3 P_{pm}^3 \]
\[ + 2P_{nm}^3 \Gamma_{pj}^i + 2P_{nm}^3 \Gamma_{pj}^3 \]
i.e., because of (0.5,6,10), (1.2a,b) and (2.2),
\[ \mathcal{R}_{jm}^3 = R_{jm}^3 + \delta^i_{jm} \psi_{jn} - \delta^i_{nj} \psi_{jm} + \psi^i_{gm} g_{mj} \]
\[ + (\delta^i_{nj} g_{mj} - \delta^i_{mj} g_{nj}) \Delta_1 \psi + 2\psi_{m} \Gamma_{nj}^i + 2\psi_{n} \Gamma_{mj}^i - 2\psi^i g_{nm} \Gamma_{pj}^i \]
\[ \quad \text{(3.1)} \]

Also, the following is satisfied
\[ \psi_{mn} = \psi_{mn} + 2\Gamma_{nm}^p \psi_{np}, \quad \psi_{mn}^i = \psi_{mn}^i + 2g^{ip} \Gamma_{mn}^q \psi_{pq}. \quad \text{(3.2)} \]

From (3.1), (3.2) and (10) we get
\[ \mathcal{R}_{jm}^3 = R_{jm}^3 + \delta^i_{jm} \psi_{jn} - \delta^i_{nj} \psi_{jm} + \psi^i_{gm} g_{mj} \]
\[ + (\delta^i_{nj} g_{mj} - \delta^i_{mj} g_{nj}) \Delta_1 \psi + 2\psi_{m} \Gamma_{nj}^i + 2\psi_{n} \Gamma_{mj}^i - 2\psi^i g_{nm} \Gamma_{pj}^i \]
\[ + 2\delta^i_{m} \Gamma_{pj}^p \psi_{np} - 2g^{ip} \Gamma_{mj}^q \psi_{pq} g_{jm}. \quad \text{(3.3)} \]

Contracting (3.3) with respect to \( i \) and \( n \), and using (1.5), we get
\[ \mathcal{R}_{jm}^3 = R_{jm}^3 - (N - 2) \psi_{jm} - [\Delta_2 \psi + (N - 2) \Delta_1 \psi] g_{jm} - \psi^p \Gamma_{m,pj}. \quad \text{(3.4)} \]

Multiplying (3.4) by \( \mathcal{R}_{jm}^3 \) and using (2), we get
\[ \Delta_1 \psi = \frac{1}{2(N - 1)} \left( R_{jm}^3 - e^{2\psi} \Gamma_{jm}^3 \right) - \frac{N - 2}{2} \Delta_1 \psi. \quad \text{(3.5)} \]
Substituting (3.5) in (3.4) and denoting
\[ P_{3jm} = \frac{1}{N-2} \left( R_{3jm} - \frac{1}{2(N-1)} R g_{jm} \right) \]  
(3.6)
in \( GR_N \) and analogously in \( G\overline{R}_N \), in this case for \( \psi_{1jm} \) we obtain
\[ \psi_{1jm} = P_{3jm} - \frac{1}{2} \Delta_1 \psi g_{jm} - \frac{2}{N-2} \Gamma_{m,pj} \psi^p. \]  
(3.7)
Substituting (3.7) in (3.3) and using (1.14,15) we get
\[ C_{3}^{ijmn} = C_{3}^{ijmn} \]  
(3.8)
where
\[ C_{3}^{ijmn} = R_{3}^{ijmn} + \delta^i_m P_{3j}^m - \delta^i_n P_{3j}^m + P_{3}^{i,m} g_{nj} - P_{3}^{i,m} g_{jm} \]
\[ + \frac{1}{N(N-2)} (\delta^i_m \Gamma_{n,pj} - \delta^i_n \Gamma_{m,pj}) g^{pq} \frac{\partial}{\partial x^q} \ln g \]
\[ + \frac{1}{N} \left( g^{ip} \Gamma_{m,jn} - \delta^i_m \Gamma_{j}^n - \delta^i_n \Gamma_{j}^m \right) g^{pq} \frac{\partial}{\partial x^q} \ln g \]
\[ + \Gamma_{pj}^i g_{mn} g^{pq} - \delta^i_m \Gamma_{j}^n \frac{\partial}{\partial x^q} \ln g \]
(3.9)
And analogously for \( C_{3}^{ijmn} \) of the space \( G\overline{R}_N \). From (3.8) we can see that the tensor \( C_{3}^{ijmn} \) is an invariant of equitorsion conform mapping, and one can call it the equitorsion conform curvature tensor of the third kind. Now we have

**Theorem 3.** From the curvature tensor \( R_{3}^{ijmn} \), under the conditions as in Theorem 1, we obtain an invariant tensor \( C_{3}^{ijmn} \) (3.9) of the equitorsion conform mapping \( f : GR_N \rightarrow G\overline{R}_N \), where \( P_{3} \) is given according to (3.6).

4. Equitorsion conform curvature tensor of the fourth kind

For curvature tensors of the fourth kind we get [12, 16]
\[ \bar{R}_{4}^{ijmn} = R_{4}^{ijmn} + P_{4}^{i,m} |_n - P_{4}^{j,m} |_n + P_{4}^{i} P_{4}^{j} P_{4}^{m} P_{4}^{n} - P_{4}^{m} P_{4}^{n} P_{4}^{i} P_{4}^{j} \]
\[ + 2 P_{4}^{m} \Gamma_{pj}^{i} + 2 P_{4}^{m} \Gamma_{pj}^{i} \]
\[ + \Gamma_{pj}^{i} g_{mn} g^{pq} - \Gamma_{pj}^{i} \frac{\partial}{\partial x^q} \ln g \]
i.e.
\[ \bar{R}_{4}^{ijmn} = R_{4}^{ijmn} + \delta^i_m \psi_{jn} - \delta^i_j \psi_{jm} + \psi_{n}^i g_{nj} - \psi_{j}^i g_{jm} \]
\[ + (\delta^i_m g_{nj} - \delta^i_n g_{jm}) \Delta_1 \psi + 2 \psi_{n}^i \Gamma_{jm}^{i} + 2 \psi_{m}^i \Gamma_{nj}^{i} - 2 \psi^p g_{mn} \Gamma_{pj}^{i} \]
\[ + 2 \delta^i_m \Gamma_{pj}^{i} \psi_p - 2 g^{ip} \Gamma_{pj}^{i} \psi_q g_{jm}. \]
In this case, analogously to previous case, we get an invariant object of the equitorsion conform mapping in the form

\[
C^i_{jmn} = R^i_{jmn} + \delta^i_m P_{jpn} - \delta^i_n P_{jpm} + P^i_{jm} g_{nj} - P^i_{jn} g_{mj} \\
+ \frac{1}{N(N-2)} (\delta^i_m \Gamma_{n,pj} - \delta^i_n \Gamma_{m,pj}) g_{pq} \frac{\partial}{\partial x^q} \ln g \\
+ \frac{1}{N} (g_{ip} \Gamma^q_{pn} g_{jm} - \delta^i_m \Gamma^q_{nj} - \delta^i_n \Gamma^q_{mj}) \\
+ \Gamma_{pj} g_{nm} g_{pq} (\delta^i_m \Gamma^q_{nj} - \delta^i_n \Gamma^q_{mj}) \frac{\partial}{\partial x^q} \ln g,
\]

where \( R^i_{jmn} \) is Ricci’s curvature tensor of the fourth kind and \( R \) a scalar curvature of the fourth kind. The object \( C^i_{jmn} \) is a tensor and we call it equitorsion conform curvature tensor of the fourth kind of the equitorsion conform mapping. So, the next theorem is valid.

**Theorem 4.** From the curvature tensor \( R^i_{jmn} \), under the conditions as in Theorem 1, one obtains an invariant tensor \( C^i_{jmn} \) of the equitorsion conform mapping of generalized Riemannian spaces, where \( P \) is given with respect to \( R \).

### 5. Equitorsion conform curvature tensor of the fifth kind

For the curvature tensors of the fifth kind of the spaces \( GR_N \) and \( G\bar{R}_N \) we find the relation \([12, 16]\)

\[
\overline{R}^i_{jmn} = R^i_{jmn} + \frac{1}{2} (P^i_{jm} - P^i_{jn} + \delta^i_m P^m_{jn} - \delta^i_n P^n_{jm} + P^i_{pm} P^p_{jm} - P^i_{pj} P^j_{pm})
\]

i.e.

\[
\overline{R}^i_{jmn} = R^i_{jmn} + \frac{1}{2} \left[ \delta^i_m (\psi_{j4n} + \psi_{j3n} - 2\psi_{j} \psi_n) - \delta^i_n (\psi_{j3m} + \psi_{j4m} - 2\psi_{j} \psi_m) \right. \\
\left. + (\psi^i_{j3m} + \psi^i_{j4m} - 2\psi_{j} \psi^i_m) g_{jn} - (\psi^i_{j3n} + \psi^i_{j4n} - 2\psi_{j} \psi^i_n) g_{jm} \right] + 2(\delta^i_m g_{jn} - \delta^i_n g_{jm}) \psi_p \psi^p.
\]

Let us denote

\[
\psi_{jn} = \frac{1}{2} (\psi_{j3n} + \psi_{j4n} - 2\psi_{j} \psi_n), \quad \psi^i_{jn} = g^{ip}_{j34} \psi_{jp}, \quad \Delta_1 \psi = g^{pq} \psi_p \psi^q.
\]

Then

\[
\overline{R}^i_{jmn} = R^i_{jmn} + \delta^i_m \psi_{jn} - \delta^i_n \psi_{jm} + \psi^i_{jn} g_{jm} - \psi^i_{jm} g_{jn} \\
+ (\delta^i_m g_{jn} - \delta^i_n g_{jm}) \Delta_1 \psi.
\]
Contracting by indices \(i, n\) and denoting

\[ R_{5 \ jmn}^p = R_{5 \ jmn}, \quad R_{5 \ jmp}^p = R_{5 \ jmn}, \quad \Delta_{34} \psi = \frac{1}{2} g_{5 \ljm}(\psi_{p\ljq} + \psi_{p\ljq}), \quad (5.4) \]

we obtain

\[ R_{5 \ jm} = R_{5 \ jm} - (N - 2) \psi_{34} - [\Delta_{34} \psi + (N - 2) \Delta_1 \psi] g_{jm}, \quad (5.5) \]

wherefrom, multiplying by \( g_{jm} = e^{-2\psi} g_{jm} \) and contracting by \( j \) and then by \( m \) one obtains

\[ \Delta_{34} \psi = \frac{1}{2} (N - 1) (R_{5 \ jm} - e^{2\psi} R_{5 \ jm}) - \frac{N - 2}{2} \Delta_1 \psi. \quad (5.6) \]

From (5.5) and (5.6) we get

\[ \psi_{34} = P_{5 \ jm} - P_{5 \ jm} - \frac{1}{2} \Delta_1 \psi g_{jm} \quad (5.7) \]

where we denoted

\[ P_{5 \ jm} = \frac{1}{N - 2} (R_{5 \ jm} - \Delta_{34} \psi g_{jm}) \quad (5.8) \]

in \( GR_N \) and analogously \( P_{5 \ jm} \) in \( G\overline{R}_N \).

Analogously to previous cases eliminating \( \psi_{34} \) from (5.3) we can write

\[ C_{5 \ jmn}^i = C_{5 \ jmn}^i, \quad (5.9) \]

where we denoted

\[ C_{5 \ jmn}^i = R_{5 \ jmn}^i + \delta_m^i P_{5 \ jm} - \delta_n^i P_{5 \ jm} + P_{5 \ im} g_{nj} - P_{5 \ jm} g_{in}. \quad (5.10) \]

The object \( C_{5 \ jmn}^i \) is an invariant of the equitensor conform mapping. We call it equitensor conform curvature tensor of the fifth kind. So, we have

**Theorem 5.** Starting from the curvature tensor \( R_{5 \ jmn}^i \), under the conditions as in the Theorem 1, we obtain an invariant tensor \( C_{5 \ jmn}^i \) (5.10) of the equitensor mapping \( f : GR_N \rightarrow G\overline{R}_N \), where \( P_{5 \ jm} \) is given according to (5.8).

If \( GR_N(G\overline{R}_N) \) reduces to \( R_N(G\overline{R}_N) \), then the objects \( C_{5 \ jmn}^i (\theta = 1, \ldots, 5) \) reduces to the conform curvature tensor (0.9).

**REFERENCES**


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