ON A UNIQUENESS THEOREM IN THE INVERSE STURM-LIOUVILLE PROBLEM

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Abstract. We introduce new supplementary data to the set of eigenvalues, to determine uniquely the potential and boundary conditions of the Sturm-Liouville problem. As a corollary we obtain extensions of some known uniqueness theorems in the inverse Sturm-Liouville problem.

1. Introduction and statement of the result

Let $L(q, \alpha, \beta)$ denote the Sturm-Liouville problem

\[
\ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0, \pi), \quad \mu \in \mathbb{C},
\]

\[
y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad \alpha \in (0, \pi],
\]

\[
y(\pi) \cos \beta + y'(\pi) \sin \beta = 0, \quad \beta \in [0, \pi),
\]

where $q$ is a real-valued, summable on $[0, \pi]$ function (we write $q \in L^1_{\mathbb{R}}[0, \pi]$). By $L(q, \alpha, \beta)$ we also denote the self-adjoint operator, generated by the problem (1.1)–(1.3). It is known, that the spectrum of $L(q, \alpha, \beta)$ is discrete and consists of simple eigenvalues (see [1], [2]), which we denote by $\mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \ldots$, emphasizing the dependence of $\mu_n$ on $q$, $\alpha$ and $\beta$.

Let $y = \varphi(x, \mu, \alpha, q)$ and $y = \psi(x, \mu, \beta, q)$ be the solutions of (1.1) with initial values

\[
\varphi(0, \mu, \alpha, q) = \sin \alpha, \quad \varphi'(0, \mu, \alpha, q) = -\cos \alpha
\]

\[
\psi(\pi, \mu, \beta, q) = \sin \beta, \quad \psi'(\pi, \mu, \beta, q) = -\cos \beta.
\]

The eigenvalues $\mu_n$ of $L(q, \alpha, \beta)$ are the solutions of the equation

\[
\chi(\mu) \overset{\text{def}}{=} \varphi(\pi, \mu, \alpha) \cos \beta + \varphi'(\pi, \mu, \alpha) \sin \beta
\]

\[
= -[\psi(0, \mu, \beta) \cos \alpha + \psi'(0, \mu, \beta) \sin \alpha] = 0. \quad (1.4)
\]

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It is easy to see, that \( \varphi_n(x) \overset{\text{def}}{=} \varphi(x, \mu_n(q, \alpha, \beta), \alpha, q) \) and \( \psi_n(x) \overset{\text{def}}{=} \psi(x, \mu_n(q, \alpha, \beta), \beta, q) \), \( n = 0, 1, 2, \ldots \), are the eigenfunctions, corresponding to the eigenvalue \( \mu_n(q, \alpha, \beta) \). The squares of the \( L^2 \)-norm of these eigenfunctions:

\[
a_n = a_n(q, \alpha, \beta) = \int_0^\pi \varphi_n^2(x) \, dx, \quad (1.5)
\]

are usually called the norming constants.

Since all eigenvalues are simple, there exist constants \( c_n = c_n(q, \alpha, \beta) \), \( n = 0, 1, 2, \ldots \), such that

\[
\varphi_n(x) = c_n \cdot \psi_n(x). \quad (1.6)
\]

The main result of this paper is the following “uniqueness” theorem (in inverse problem):

**Theorem 1.** If for all \( n = 0, 1, 2, \ldots \)

\[
\mu_n(q_1, \alpha_1, \beta_1) = \mu_n(q_2, \alpha_2, \beta_2), \quad \text{(A)}
\]

\[
c_n(q_1, \alpha_1, \beta_1) = c_n(q_2, \alpha_2, \beta_2), \quad \text{(B)}
\]

then \( \alpha_1 = \alpha_2, \beta_1 = \beta_2 \) and \( q_1(x) = q_2(x) \) almost everywhere (a.e.) on \([0, \pi]\).

The problem \( L(q, \alpha, \beta) \) is called “even” if \( \alpha + \beta = \pi \) and \( q(\pi - x) = q(x) \) a.e. on \([0, \pi]\).

**Corollary.** The problem \( L(q, \alpha, \beta) \) is even if and only if \( c_n(q, \alpha, \beta) = (-1)^n \).

The inverse Sturm-Liouville problems were stated and solved in different versions (see, for example, [3]–[18]). We will consider below the connections between some of the known uniqueness theorems and our Theorem 1 and its corollary (see §5, Theorems 1’, 2, 2’, 3).

2. Some preliminary results

**Lemma 1.** Let \((\alpha, \beta, q) \in (0, \pi] \times [0, \pi) \times L^1_{\mathbb{R}}[0, \pi]\). Then, for \( n \geq 1 \) (except \( \mu_0(\alpha, \beta, q) \))

\[
\mu_n(\alpha, \beta, q) = [n + \delta_n(\alpha, \beta)]^2 + [q] + r_n(\alpha, \beta, q) \quad (2.1)
\]

where \( [q] = \frac{1}{\pi} \int_0^\pi q(x) \, dx \),

\[
\delta_n(\alpha, \beta) = \frac{1}{\pi} \left[ \arccos \frac{\cos \alpha}{\sqrt{[n + \delta_n(\alpha, \beta)]^2 \sin^2 \alpha + \cos^2 \alpha}} \right. \\
\left. - \arccos \frac{\cos \beta}{\sqrt{[n + \delta_n(\alpha, \beta)]^2 \sin^2 \beta + \cos^2 \beta}} \right],
\]
and \( r_n = r_n(\alpha, \beta, q) = o(1) \), when \( n \to \infty \), uniformly by \( \alpha, \beta \in [0, \pi] \), and \( q \) from bounded subsets of \( L^1_k[0, \pi] \). The well-known asymptotics

\[
\mu_n(\alpha, \beta, q) = n^2 + \frac{2}{\pi} (\text{ctg} \beta - \text{ctg} \alpha) + [q] + \tilde{r}_n(\alpha, \beta, q), \quad \text{if } \sin \alpha \neq 0, \sin \beta \neq 0,
\]

(2.2)

\[
\mu_n(\pi, \beta, q) = \left(n + \frac{1}{2}\right)^2 + \frac{2}{\pi} \text{ctg} \beta + [q] + \tilde{r}_n(\beta, q), \quad \text{if } \sin \beta \neq 0 \quad (\beta \in (0, \pi)),
\]

(2.3)

\[
\mu_n(\alpha, 0, q) = \left(n + \frac{1}{2}\right)^2 - \frac{2}{\pi} \text{ctg} \alpha + [q] + \tilde{r}_n(\alpha, q), \quad \text{if } \sin \alpha \neq 0, \quad (\alpha \in (0, \pi)),
\]

(2.4)

\[
\mu_n(\pi, 0, q) = (n + 1)^2 + [q] + \tilde{r}_n(q),
\]

(2.5)

where \( \tilde{r}_n = o(1) \) (but this estimate is not uniform in \((\alpha, \beta) \in [0, \pi] \)), are the particular cases of (2.1). The sequence \( \{\delta_n(\alpha, \beta)\}_{n=1}^{\infty} \) has the limit

\[
\delta_\infty(\alpha, \beta) = \begin{cases} 
0, & \text{if } \alpha, \beta \in (0, \pi), \\
\frac{1}{2}, & \text{if } \alpha = \pi, \beta \in (0, \pi) \text{ or } \alpha(0, \pi), \beta = 0, \\
1, & \text{if } \alpha = \pi, \beta = 0.
\end{cases}
\]

(2.6)

For the proof and the details of Lemma 1 see paper [19].

Let \( y_i(x, \mu, q), i = 1, 2 \), be the solutions of (1.1) with initial values

\[
y_1(0, \mu, q) = y'_2(0, \mu, q) = 1,
\]

\[
y'_1(0, \mu, q) = y_2(0, \mu, q) = 0.
\]

It is clear, that

\[
\varphi(x, \mu, \alpha, q) \equiv y_1(x, \mu, q) \sin \alpha - y_2(x, \mu, q) \cos \alpha.
\]

(2.7)

**Lemma 2.** 1) Let \( q \in L^1_{\mathbb{C}}[0, \pi] \). Then

\[
y_1(x, \lambda^2, q) = \cos \lambda x + \frac{\sin \lambda x}{2 \lambda} \int_0^x q(s) \, ds + \frac{1}{2 \lambda} \int_0^x q(s) \sin \lambda (x - 2s) \, ds + O\left(\frac{e^{\text{Im} \lambda |x|}}{\lambda^2}\right),
\]

(2.8)

\[
y_2(x, \lambda^2, q) = \sin \lambda x - \frac{\cos \lambda x}{2 \lambda^2} \int_0^x q(s) \, ds + \frac{1}{2 \lambda^2} \int_0^x q(s) \cos \lambda (x - 2s) \, ds + O\left(\frac{e^{\text{Im} \lambda |x|}}{\lambda^3}\right).
\]

(2.9)

In particular (for real \( \lambda \))

\[
y_1(\pi, \lambda^2, q) = \cos \lambda \pi + \frac{\sin \lambda \pi}{2 \lambda} \int_0^\pi q(s) \, ds + o\left(\frac{1}{\lambda}\right), \quad \lambda \to +\infty,
\]

(2.10)

\[
y_2(\pi, \lambda^2, q) = \sin \frac{\lambda \pi}{\lambda} - \frac{\cos \lambda \pi}{2 \lambda^2} \int_0^\pi q(s) \, ds + o\left(\frac{1}{\lambda^2}\right), \quad \lambda \to +\infty.
\]

(2.11)
Also
\[ y'_1(x, \lambda^2, q) = -\lambda \sin \pi x + O \left( e^{\text{Im} \lambda x} \right), \quad \text{(2.12)} \]
\[ y'_2(x, \lambda^2, q) = \cos \lambda x + O \left( \frac{e^{\text{Im} \lambda x}}{|\lambda|} \right), \quad \text{(2.13)} \]

2) For \( \mu = -t^2 = (it)^2 \rightarrow -\infty \) (\( t \rightarrow +\infty \))
\[ \chi(\mu) = \chi(-t^2) = \left\{ \begin{array}{ll}
\frac{t e^{\pi t}}{\pi} [\sin \alpha \cdot \sin \beta + O(\frac{1}{t})], & \text{if } \sin \alpha \neq 0, \ \sin \beta \neq 0, \\
\frac{t e^{\pi t}}{\pi} [\sin \beta + O(\frac{1}{t})], & \text{if } \sin \beta \neq 0, \ \alpha = \pi, \\
\frac{t e^{\pi t}}{\pi} [1 + O(\frac{1}{t})], & \text{if } \alpha = \pi, \ \beta = 0,
\end{array} \right. \quad \text{(2.14)} \]

3) Let \( q \in L^1_{\mathbb{R}}[0, \pi] \). Then
\[ \varphi(\pi, \mu, \alpha, q) = \sum_{n=0}^{\infty} \varphi(\pi, \mu_n, \alpha, q) \cdot \prod_{m \neq n, m=0}^{\infty} \frac{\mu_m - \mu}{\mu_m - \mu_n}. \quad \text{(2.15)} \]

**Proof.** 1) The asymptotic formulae (2.8)–(2.13) are proved in detail in [19], or they are corollaries of the results of [19] (see also [8]). For \( q \in L^2[0, \pi] \) they can be found in [10], [11] and other papers.

2) Relation (2.14) is the corollary of (1.4), (2.7) and (2.8)–(2.13).

3) For \( q \in L^2[0, \pi] \) (2.15) is proved in [11] (more detailed proof is presented in [17]). For \( q \in L^1_{\mathbb{R}}[0, \pi] \) the proof is the same. ■

Now we establish some connections between spectral data. The following formula is well known (see, e.g., [18], (2.8))
\[ \int_0^\pi \varphi_n^2(x) \, dx = \varphi'(\pi, \mu_n) \cdot \varphi(\pi, \mu_n) = \varphi(\pi, \mu_n) \cdot \varphi(\pi, \mu_n) \]
\[ (f(x, \mu) = \frac{\partial}{\partial \mu} f(x, \mu)) \text{ which is equivalent to (see (1.4), (1.5, (1.6)))} \]
\[ a_n(q, \alpha, \beta) = -c_n(q, \alpha, \beta) \cdot \chi(\mu_n). \quad \text{(2.16)} \]
By definition (1.6) we have, that (\( \alpha \in (0, \pi] \))
\[ c_n(q, \alpha, \beta) = \frac{\varphi(\pi, \mu_n(q, \alpha, \beta), \alpha, q)}{\sin \beta}, \quad \sin \beta \neq 0 \ (\beta \neq 0) \quad \text{(2.17)} \]
and
\[ c_n(q, \alpha, 0) = -\varphi'(\pi, \mu_n(q, \alpha, 0), \alpha). \quad \text{(2.18)} \]
The normalized eigenfunctions \( h_n \) we define as
\[ h_n(x) = \frac{\varphi_n(x)}{\|\varphi_n\|}. \quad \text{(2.19)} \]
Now we present the definitions of spectral data \( \ell_n = \ell_n(q, \alpha, \beta) \), which were introduced in [10], [11] and [17] (as supplementary data to eigenvalues), and their connection with our spectral data \( c_n = c_n(q, \alpha, \beta) \), that follows from 1.6 and (2.17)–(2.19).

\[
\ell_n(q, \alpha, \beta) = \log \left[ (-1)^n \cdot \frac{b_n(\pi)}{b_n(0)} \right] = \log \left[ (-1)^n c_n(q, \alpha, \beta) \cdot \frac{\sin \beta}{\sin \alpha} \right],
\]
if \( \sin \alpha \neq 0, \sin \beta \neq 0, \) \hspace{1cm} (2.20)

\[
\ell_n(q, \pi, \beta) = \log \left[ (-1)^n \cdot \frac{b_n(\pi)}{b_n(0)} \right] = \log \left[ (-1)^n c_n(q, \pi, \beta) \cdot \sin \beta \right],
\]
if \( \sin \beta \neq 0, \alpha = \pi, \) \hspace{1cm} (2.21)

\[
\ell_n(q, \alpha, 0) = \log \left[ (-1)^{n+1} \cdot \frac{b_n'(\pi)}{b_n'(0)} \right] = \log \left[ (-1)^n c_n(q, \alpha, 0) \cdot \frac{1}{\sin \alpha} \right],
\]
if \( \sin \alpha \neq 0, \beta = 0, \) \hspace{1cm} (2.22)

\[
\ell_n(q, \pi, 0) = \log \left[ (-1)^n \cdot \frac{b_n'(\pi)}{b_n'(0)} \right] = \log \left[ (-1)^n c_n(q, \pi, 0) \right], \text{ if } \alpha = \pi, \beta = 0.
\] \hspace{1cm} (2.23)

3. The proof of Theorem 1

We prove Theorem 1 in 4 steps. At first we consider the case \( \alpha_1 = \pi, \beta_1 = 0. \) From condition (A), (2.1) and (2.5) we obtain \( (n = 0, 1, 2, \ldots) \)

\[
(n + 1)^2 + [q_1] + r_n(q_1, \pi, 0) = (n + \delta_n(\alpha_2, \beta_2))^2 + [q_2] + r_n(q_2, \alpha_2, \beta_2).
\]

It follows easily that \( \delta_n(\alpha_2, \beta_2) \to 1, \) when \( n \to \infty. \) According to (2.6), it is possible only if \( \alpha_2 = \pi, \beta_2 = 0. \) Then, from condition (B) and (2.23), we obtain \( \ell_n(q_1, \pi, 0) = \ell_n(q_2, \pi, 0) \) for \( n = 0, 1, 2, \ldots, \) and we can repeat the proof of Theorem 5, chapter III, of [10], page 62, to obtain \( q_1(x) = q_2(x), \) a.e.

Remark. The uniqueness theorems in [10], [11] and [17] are proved under condition \( q_1, q_2 \in L^2_{\mathbb{R}}[0, \pi], \) but they are true also for \( q_1, q_2 \in L^2_{\mathbb{R}}[0, \pi], \) because the asymptotic formulae and estimates (see (2.8)–(2.13)) for solutions of (1.1) (which are used particularly to prove that some contour integrals tend to zero) are true also for \( q \in L^1[0, \pi], \) as it is proved in details in [20].

Secondly, we consider the case \( \alpha_1 = \pi, \beta \in (0, \pi). \) Then condition (A) gives us

\[
\left(n + \frac{1}{2}\right)^2 + \frac{2}{\pi} \cot \beta + [q_1] + r_n(q_1, \pi, \beta_1) = [n + \delta_n(\alpha_2, \beta_2)]^2 + [q_2] + r_n(q_2, \alpha_2, \beta_2)
\]

by (2.1) and (2.3). It easy to prove from (3.1), that \( \lim_{n \to -\infty} \delta_n(\alpha_2, \beta_2) = \frac{1}{2}, \) and by (2.6) it is possible only if \( \alpha_2 = \pi, \beta_2 \in (0, \pi) \) or \( \alpha_2 \in (0, \pi), \beta_2 = 0. \)

In the case \( \alpha_2 = \pi, \beta_2 \in (0, \pi) \) we have

\[
\frac{2}{\pi} \cot \beta_1 + [q_1] = \frac{2}{\pi} \cot \beta_2 + [q_2]
\]
by (3.1) and (2.3). Also
\[\frac{y_2(\pi, \mu_n, q_1)}{\sin \beta_1} = \frac{y_2(\pi, \mu_n, q_2)}{\sin \beta_2}\]
by condition (B) and (2.17). Together with (A) and (2.15) we obtain
\[\frac{y_2(\pi, \mu, q_1)}{\sin \beta_1} = \frac{y_2(\pi, \mu, q_2)}{\sin \beta_2}\]
for all \(\mu \in \mathbb{C}\). Substituting \(\mu = \left(n + \frac{1}{2}\right)^2\) in (3.2), by (2.11) we obtain
\[
\frac{y_2\left(\pi, \left(n + \frac{1}{2}\right)^2, q_1\right)}{\sin \beta_1} = \frac{1}{\sin \beta_1} \left[\frac{(-1)^n}{n + \frac{1}{2}} + \frac{o(1)}{(n + \frac{1}{2})^2}\right] = \frac{1}{\sin \beta_2} \left[\frac{(-1)^n}{n + \frac{1}{2}} + \frac{o(1)}{(n + \frac{1}{2})^2}\right].
\]
It follows that \(\sin \beta_1 - \sin \beta_2 = \frac{o(1)}{n + \frac{1}{2}}\), i.e. \(\sin \beta_1 = \sin \beta_2\). Then, by (2.21), we have \(\ell_n(q_1, \pi, \beta_1) = \ell_n(q_2, \pi, \beta_2)\), \(n = 0, 1, 2\ldots\), and we can repeat the proof of Theorem 3 in [17] to obtain \(\beta_1 = \beta_2\) and \(q_1(x) = q_2(x)\), a.e.

In the case \(\alpha_2 \in (0, \pi)\), \(\beta_2 = 0\) from condition (B), according to (2.17) and (2.18) \(\frac{y_2(\pi, \mu_n, q_1)}{\sin \beta_1} = -\varphi'(\pi, \mu_n, \alpha_2, q_2)\) and by (2.11), (2.7), (2.12) and (2.13) we obtain
\[
\frac{1}{\sin \beta_1} \left\{\sin \sqrt{\mu_n} \pi - \cos \sqrt{\mu_n} \pi \int_0^\pi q_1(s) ds + \frac{o(1)}{\mu_n}\right\} =\\
= (-\sqrt{\mu_n} \sin \sqrt{\mu_n} \pi + O(1)) \sin \alpha_2 + \left\{\cos \sqrt{\mu_n} \pi + O\left(\frac{1}{\sqrt{\mu_n}}\right)\right\} \cos \alpha_2.
\]
Since \(\sin \beta_1 \neq 0\) and \(\sin \alpha_2 \neq 0\), the last equality is impossible (the left-hand side tends to zero, while the right-hand side does not). Thus in the case \(\alpha_1 = \pi, \beta_1 \in (0, \pi)\), Theorem 1 is also proved.

The third case is \(\alpha_1 \in (0, \pi)\), \(\beta_1 = 0\). In this case from condition (A), (2.1) and (2.4) we obtain
\[
\left(n + \frac{1}{2}\right)^2 - \frac{2}{\pi} \cot \alpha_1 + [q_1] + r_n(q_1, \alpha_1, 0) = [n + \delta_n(\alpha_2, \beta_2)]^2 + [q_2] + r_n(q_2, \alpha_2, \beta_2)
\]
From this equality it follows easily that \(\lim_{n \to \infty} \delta_n(\alpha_2, \beta_2) = \frac{1}{2}\), and therefore, either \(\alpha_2 = \pi, \beta_2 \in (0, \pi)\) (as proved above, this case is impossible), or \(\alpha_2 \in (0, \pi), \beta_2 = 0\). Similarly to the second case, we prove that \(\sin \alpha_1 = \sin \alpha_2\) and by (2.16) we obtain that \(\ell_n(q_1, \alpha_1, 0) = \ell_n(q_2, \alpha_2, 0)\). According to Theorem 4 of [17] we get \(\alpha_1 = \alpha_2\) and \(q_1(x) = q_2(x), a.e.\)

The fourth and the last case is \(\sin \alpha_1 \neq 0\) and \(\sin \beta_1 \neq 0\), i.e. \(\alpha_1, \beta_1 \in (0, \pi)\). The cases \(\alpha_2 = \pi\) or \(\beta_2 = 0\) are impossible, since they reduce to cases I, II or III. Therefore \(\alpha_2, \beta_2 \in (0, \pi)\). It follows from (A) and (2.2) that \(\lim_{n \to \infty} (\mu_n(q_1, \alpha_1, \beta_1) - n^2) = \frac{2}{\pi} (\cot \alpha_1 - \cot \beta_1) + \frac{1}{\pi} \int_0^\pi q_1(t) dt =\)
\[ \lim_{n \to \infty} (\mu_n(q_2, \alpha_2, \beta_2) - n^2) = \frac{2}{\pi} \left( \text{ctg} \alpha_2 - \text{ctg} \beta_2 \right) + [q_2]. \]

Also we have by (A), (B) and (2.17)
\[ \frac{\varphi(\pi, \mu_n, \alpha_1, q_1)}{\sin \beta_1} = \frac{\varphi(\pi, \mu_n, \alpha_2, q_2)}{\sin \beta_2}. \]

Then, by (2.15) we obtain
\[ \frac{\varphi(\pi, \mu, \alpha_1, q_1)}{\sin \beta_1} = \frac{\varphi(\pi, \mu, \alpha_2, q_2)}{\sin \beta_2} \]
for all \( \mu \in \mathbb{C} \). Now, by (2.7), (2.10) and (2.11) for \( \mu = n^2 \) we have
\[ \frac{\varphi(\pi, n^2, \alpha_1, q_1)}{\sin \beta_1} = \frac{\sin \alpha_1}{\sin \beta_1} \cdot \frac{(-1)^n + o(1)}{n} \]
\[ = \frac{\sin \alpha_2}{\sin \beta_2} \cdot \frac{(-1)^n + o(1)}{n} = \frac{\varphi(\pi, n^2, \alpha_2, q_2)}{\sin \beta_2} \]
and it follows easily that \( \frac{\sin \alpha_1}{\sin \beta_1} = \frac{\sin \alpha_2}{\sin \beta_2} \).

Thus, by (2.20) we obtain
\[ \ell_n(q_1, \alpha_1, \beta_1) = \ell_n(q_2, \alpha_2, \beta_2), \quad n = 0, 1, 2, \ldots, \]
and by the uniqueness theorem of [11] we have that
\( \alpha_1 = \alpha_2, \beta_1 = \beta_2 \) and \( q_1(x) = q_2(x), \text{ a.e.} \)
The proof of Theorem 1 is complete. ■

4. Proof of the Corollary

Let \( q^*(x) \equiv q(\pi - x) \). It is easily verified that (see [11])
\[ \varphi(\pi - x, \mu, \alpha, q^*) \equiv \psi(x, \mu, \pi - \alpha, q) \] (4.1)
and
\[ \mu_n(q, \alpha, \beta) = \mu_n(q^*, \pi - \beta, \pi - \alpha), \quad n = 0, 1, 2, \ldots. \] (4.2)

**Lemma 3.** For all \( n = 0, 1, 2, \ldots, \alpha \in (0, \pi] \) and \( \beta \in [0, \pi) \) the equality
\[ c_n(q, \alpha, \beta) \cdot c_n(q^*, \pi - \beta, \pi - \alpha) = 1 \] (4.3)
is true.

**Proof.** By (4.1), (4.2) and (1.6)
\[ \psi(x, \mu_n(q, \alpha, \beta, \beta, q) = \varphi(\pi - x, \mu_n(q, \alpha, \beta), \pi - \beta, q^*) \]
\[ \equiv \varphi(\pi - x, \mu_n(q^*, \pi - \beta, \pi - \alpha), \pi - \beta, q^*) \]
\[ \equiv c_n(q^*, \pi - \beta, \pi - \alpha) \psi(\pi - x, \mu_n(q^*, \pi - \beta, \pi - \alpha), \pi - \beta, q^*) \]
\[ \equiv c_n(q^*, \pi - \beta, \pi - \alpha) \cdot c_n(q, \alpha, \beta) \cdot \psi(x, \mu_n(q, \alpha, \beta), \beta, q). \]

It follows that (4.3) holds true.

To prove the sufficiency we note that if \( c_n(q, \alpha, \beta) = (-1)^n \), then
\[ c_n(q^*, \pi - \beta, \pi - \alpha) = (-1)^n \] by (4.3) and since \( \mu_n(q, \alpha, \beta) = \mu_n(q^*, \pi - \beta, \pi - \alpha) \),
then \( q(x) = q^*(x) \) and \( \alpha = \pi - \beta \) by Theorem 1.

If problem \( L(q, \alpha, \beta) \) is even, i.e. \( q(\pi - x) = q(x) \) and \( \alpha + \beta = \pi \), then \( c_n(q, \alpha, \beta) = 1 \) by (4.3). Since the roots \( \mu_n \) of function \( \chi(\mu) \) are simple, then \( \chi(\mu_n) \)
and \( \chi(\mu_{n+1}) \) have the different sign and since \( a_n > 0 \), it follows that \( c_n \) and \( c_{n+1} \)
have the different sign by $a_n = -c_n \cdot \chi(\mu_n)$ (see (2.16)). If we show that $\chi(\mu_0) < 0$, we will obtain that $c_0(q, \alpha, \beta) = 1 = (-1)^0$ and therefore $c_n(q, \alpha, \beta) = (-1)^n$.

Really, it follows from (2.14) that when $\mu$ changes from $-\infty$ to $\mu_0$, $\chi(\mu)$ changes from $+\infty$ to $0$, i.e. $\chi(\mu_0) < 0$. The proof of corollary is complete.

### 5. Some extensions

Following reasons, very similar to the proof of Theorem 1, we see that the following holds.

**Theorem 2.** Let $(\alpha_i, \beta_i, q_i) \in (0, \pi] \times [0, \pi] \times L^1_\mathbb{R}[0, \pi], i = 1, 2$. If $\mu_n(q_1, \alpha_1, \beta_1) = \mu_n(q_2, \alpha_2, \beta_2)$ and $\ell_n(q_1, \alpha_1, \beta_1) = \ell_n(q_2, \alpha_2, \beta_2)$ for all $n = 0, 1, 2, \ldots$, then $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $q_1(x) = q_2(x)$, a.e.

If, following [11], we introduce the set

$$M(p, \alpha_0, \beta_0) = \{(q, \alpha, \beta) \in L^1_\mathbb{R}[0, \pi] \times (0, \pi] \times [0, \pi] : \mu_0(q, \alpha, \beta) = \mu_n(p, \alpha_0, \beta_0), n \geq 0\},$$

then we can formulate next theorem (in terms of [11]), which follows from Theorem 1 and its Corollary.

**Theorem 1’. (i) The mapping**

$$(\alpha, \beta, q) \in (0, \pi] \times (0, \pi] \times L^1_\mathbb{R}[0, \pi] \mapsto (\mu_n(q, \alpha, \beta), c_n(q, \alpha, \beta) n \geq 0)$$

is one to one. Equivalently, the mapping

$$(q, \alpha, \beta) \in M(p, \alpha_0, \beta_0) \mapsto (c_n(q, \alpha, \beta) n \geq 0)$$

is one to one.

(ii) The mapping

$$(q, \alpha, \beta) \in L^1_\mathbb{R}[0, \pi] \times (0, \pi] \times [0, \pi] \mapsto (\mu_n(q, \alpha, \beta); n \geq 0),$$

is one to one when restricted to the subset of even points (i.e. $\alpha + \beta = \pi$, $q(\pi - x) = q(x)$) in $L^1_\mathbb{R}[0, \pi] \times (0, \pi] \times [0, \pi]$.

If in Theorem 1’ we change $c_n(q, \alpha, \beta)$ to $\ell_n(q, \alpha, \beta)$ we obtain a proposition (call it Theorem 2’), which follows from Theorem 2 and its Corollary (see [11]): $L(q, \alpha, \beta)$ even if and only if $\ell_n(q, \alpha, \beta) = 0, n \geq 0$, and which not only joins the uniqueness theorems of [10], [11] and [17], but also extend them.

Also the connection (2.16) shows that Theorem 1 is equivalent to

**Theorem 3.** Let $(\alpha_i, \beta_i, q_i) \in (0, \pi] \times [0, \pi] \times L^1_\mathbb{R}[0, \pi], i = 1, 2$. If $\mu_n(q_1, \alpha_1, \beta_1) = \mu_n(q_2, \alpha_2, \beta_2)$ and $\alpha_n(q_1, \alpha_1, \beta_1) = \alpha_n(q_2, \alpha_2, \beta_2)$ for all $n = 0, 1, 2, \ldots$, then $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $q_1(x) = q_2(x)$, a.e.

Of course, it is a variant of the Theorem of Marchenko [8] for finite intervals, which is usually ([9], [16], [21]) formulated for $\alpha_i, \beta_i \in (0, \pi)$, with condition

$$\frac{a_n(q_1, \alpha_1, \beta_1)}{\sin^2 \alpha_1} = \frac{a_n(q_2, \alpha_2, \beta_2)}{\sin^2 \alpha_2}$$

instead of $a_n(q_1, \alpha_1, \beta_1) = a_n(q_2, \alpha_2, \beta_2)$. 
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