ON $\varepsilon$-APPROXIMATION AND FIXED POINTS OF NONEXPANSIVE MAPPINGS IN METRIC SPACES

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Abstract. Using fixed point theory, B. Brosowski [2] proved that if $T$ is a nonexpansive linear operator on a normed linear space $X$, $C$ a $T$-invariant subset of $X$ and $x$ a $T$-invariant point, then the set $P_C(x)$ of best $C$-approximant to $x$ contains a $T$-invariant point if $P_C(x)$ is non-empty, compact and convex. Subsequently, many generalizations of the Brosowski’s result have appeared. We also obtain some results on invariant points of a nonexpansive mapping for the set of $\varepsilon$-approximation in metric spaces thereby generalizing and extending some known results including that of Brosowski, on the subject.

Using fixed point theory, the theorem of Meinardus [6] on invariant approximation was generalized by Brosowski [2] who proved that if $T$ is a nonexpansive linear operator on a normed linear space $X$, $C$ a $T$-invariant subset of $X$ and $x$ a $T$-invariant point, then the set $P_C(x)$ of best $C$-approximant to $x$ contains a $T$-invariant point if $P_C(x)$ is non-empty, compact and convex. Subsequently, various generalizations of Brosowski’s result have appeared (see e.g. [5]). In the present work we also obtain some results on invariant points of a nonexpansive mapping $T$ on the set of $\varepsilon$-approximation in metric spaces. Our results contain some of the results of [1], [2], [5], [6], [7], [8], [11] and [12].

To begin with, we recall a few definitions.

Let $G$ be a non-empty subset of a metric space $(X,d), x \in X$ and $\varepsilon > 0$. An element $g_o \in G$ is said to be (s.t.b.) an $\varepsilon$-approximation or $\varepsilon$-approximant to $x$ (respectively, $\varepsilon$-coapproximation or $\varepsilon$-coapproximant to $x$) if $d(x,g_o) \leq d(x,g) + \varepsilon$ (respectively, $d(g_o,g) + \varepsilon \leq d(x,g)$) for all $g \in G$, i.e. $(d(x,g_o) \leq d(x,G) + \varepsilon)$ (respectively, $d(g_o,g) + \varepsilon \leq d(x,G)$). We shall denote by $P_G(x,\varepsilon)$ (respectively, $R_G(x,\varepsilon)$) the set of all $\varepsilon$-approximant (respectively, $\varepsilon$-coapproximant) to $x$, i.e. $P_G(x,\varepsilon) = \{g_o \in G : d(x,g_o) \leq d(x,G) + \varepsilon\}$ (respectively, $R_G(x,\varepsilon) = \{g_o \in G : d(g_o,g) + \varepsilon \leq d(x,G)\}$). For $\varepsilon = 0$, the set $P_G(x,\varepsilon)$ (respectively, $R_G(x,\varepsilon)$) is the set of best approximations (respectively, best coapproximations) of $x$ in $G$.

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For $\varepsilon > 0$, the set $P_G(x, \varepsilon)$ is always a non-empty bounded set and is closed if $G$ is closed. In normed linear spaces, the elements of $\varepsilon$-approximation were introduced by R.C. Buck (who used the term ‘good approximation’ for such elements) and subsequently, the study was taken up by others (see, e.g. [10]).

A sequence $\langle g_n \rangle$ in $G$ is said to be $\varepsilon$-minimizing for $x$ if $\lim_{n \to \infty} d(x, g_n) \leq d(x, G) + \varepsilon$. The set $G$ is said to be $\varepsilon$-approximatively compact (see [7]) if for each $x \in X$, each $\varepsilon$-minimizing sequence has a subsequence converging to an element of $G$.

If a mapping $T : X \to X$ leaves subset $G$ of $X$ invariant, then the restriction of $T$ to $G$ is denoted by $T/G$.

If $G$ is a closed subset of $X$ then $T : G \to G$ is called a compact mapping [5] if for every bounded subset $A$ of $G$, $T(A)$ is compact in $G$.

A mapping $T : X \to X$ is s.t.b.

a) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$,

b) contraction if there exists $\alpha, 0 \leq \alpha < 1$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$.

A mapping $T : X \to X$ satisfies condition (A) (see [7]) if $d(Tx, y) \leq d(x, y)$ for all $x, y \in X$.

A family of maps $\{f_\alpha : \alpha \in G\}$ is s.t.b. a $G$-convex structure (see [3]), if

i. $f_\alpha : [0, 1] \to G$, i.e. $f_\alpha$ is a mapping from $[0, 1]$ into $G$ for each $\alpha \in G$;

ii. $f_\alpha(1) = \alpha$ for each $\alpha \in G$;

iii. $f_\alpha(t)$ is jointly continuous in $(\alpha, t)$, i.e. $f_\alpha(t) \to f_{\alpha_0}(t_0)$ for $\alpha \to \alpha_0$ in $G$ and $t \to t_0$ in $[0, 1]$, and

iv. $d(f_\alpha(t), f_\beta(t)) \leq \Phi(t)d(\alpha, \beta)$ where $\Phi : (0, 1) \to (0, 1)$.

For a metric space $(X, d)$, a continuous mapping $W : X \times X \times [0, 1] \to X$ is s.t.b. convex structure on $X$ if for all $x, y \in X$ and $\lambda \in [0, 1]$, we have

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $u \in X$. The metric space $(X, d)$ with convex structure is called a convex metric space [14].

A subset $K$ of a convex metric space $(X, d)$ is s.t.b. a convex set [14] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$.

The set $K$ is said to be starshaped (or $p$-starshaped) [4] if there exists a $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$.

Clearly, each convex set is starshaped but not conversely.

A convex metric space $(X, d)$ is said to satisfy Property (I) [4] if for all $x, y, p \in X$ and $\lambda \in [0, 1],

$$d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y).$$

A normed linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [14]). Property (I) is always satisfied in a normed linear space.
A more general class of sets containing the starshaped sets is called ‘contractive’.

A subset $K$ of a metric space $(X, d)$ is s.t.b. contractive if there exists a sequence $(f_n)$ of contraction mappings of $K$ into itself such that $f_n y \to y$ for each $y \in K$.

In a convex metric space $(X, d)$ satisfying Property (I), every starshaped set is contractive can be seen as below.

Suppose $K$ is starshaped with respect to $p \in K$. Define $f_n : K \to K$ as
\[
f_n(y) = W(y, p, 1 - \frac{1}{n}), n = 1, 2, 3, \ldots
\]
Consider, $d(y, f_n y) = d(y, W(y, p, 1 - \frac{1}{n})) \leq (1 - \frac{1}{n})d(y, y) + \frac{1}{n}d(y, p) \to 0$ as $n \to \infty$.

Thus $f_n y \to y$ for all $y \in K$. Moreover,
\[
d(f_n x, f_n y) = d(W(x, p, 1 - \frac{1}{n}), W(y, p, 1 - \frac{1}{n})) \leq (1 - \frac{1}{n})d(x, y)
\]
for all $x, y \in K$, i.e. $(f_n)$ is a sequence of contraction mappings.

The following result dealing with the structure of the set $P_G(x, \varepsilon)$ will be used in the sequel.

**Lemma.** If $G$ is an $\varepsilon$-approximatively compact set in a metric space $(X, d)$ then $P_G(x, \varepsilon)$ is a non-empty compact set.

**Proof.** By the definition of $d(x, G)$, we can find $g_0 \in G$ such that $d(x, g_0) \leq d(x, G) + \varepsilon$ and so $P_G(x, \varepsilon)$ is non-empty.

Let $(g_n)$ be a sequence in $P_G(x, \varepsilon)$, i.e. $d(x, g_n) \leq d(x, G) + \varepsilon$ for all $n = 1, 2, \ldots$ and so
\[
\lim_{n \to \infty} d(x, g_n) \leq d(x, G) + \varepsilon
\]
for all $x \in X$. Since $G$ is $\varepsilon$-approximatively compact, $(g_n)$ has a subsequence $(g_{n_i}) \to g_0 \in G$. So (1) implies $d(x, g_0) \leq d(x, G) + \varepsilon$, i.e. $g_0 \in P_G(x, \varepsilon)$ and so $P_G(x, \varepsilon)$ is compact.

The following result which deals with invariance of $\varepsilon$-approximations for nonexpansive mappings improves and generalizes Theorem 2.1 of [8].

**Theorem 1.** Let $T$ be a self mapping on a metric space $(X, d)$, $G$ a $T$-invariant subset of $X$ and $x$ a $T$-invariant point. If the set $D$ of $\varepsilon$-approximant to $x$ is a compact set with $D$-convex structure and $T$ is nonexpansive on $D \cup \{x\}$, then $D$ contains a $T$-invariant point.

**Proof.** Since $D = \{y \in G : d(x, y) \leq d(x, G) + \varepsilon\}, T : D \to D$. In fact if $y \in D$, then
\[
d(x, Ty) = d(Tx, Ty) \leq d(x, y) \leq d(x, G) + \varepsilon
\]
and so $Ty \in D$. 

Let \( \langle k_n \rangle, 0 \leq k_n < 1 \) be a sequence of real numbers such that \( k_n \to 1 \) as \( n \to \infty \). Define \( T_n \) as \( T_n z = f_{T z}(k_n), z \in D \). Since \( T(D) \subseteq D \) and \( 0 \leq k_n < 1 \), we have that each \( T_n \) is a well defined and maps \( D \) into \( D \). Moreover, for all \( y, z \in D \)
\[
d(T_n y, T_n z) = d(f_{T y}(k_n), f_{T z}(k_n)) \\
\leq \Phi(k_n)d(T y, T z) \leq \Phi(k_n)d(y, z),
\]
and so each \( T_n \) is a contraction mapping on \( D \). Since \( D \) is compact, it follows from Banach Contraction Principle that each \( T_n \) has a unique fixed point \( x_n \in D \), i.e. \( T_n x_n = x_n \) for each \( n \). Since \( D \) is compact, \( \langle x_n \rangle \) has a subsequence \( x_{n_i} \to \bar{x} \in D \). We claim that \( T\bar{x} = \bar{x} \). Consider
\[
x_{n_i} = T_{n_i} x_{n_i} = f_{T x_{n_i}}(k_{n_i}) \to f_{T \bar{x}}(1).
\]
As the family \( \{f_n\} \) is jointly continuous and \( T \) being nonexpansive, is continuous. Thus \( x_{n_i} \to T\bar{x} \). Therefore \( T\bar{x} = \bar{x} \) i.e. \( \bar{x} \in D \) is \( T \)-invariant.

For \( \varepsilon = 0 \), we have

**Corollary 1.** Let \( T \) be mapping on a metric space \( (X, d) \), \( G \) a \( T \)-invariant subset of \( X \) and \( x \) a \( T \)-invariant point. If the set \( D \) of best \( G \)-approximant to \( x \) is compact set with \( D \)-convex structure and \( T \) is nonexpansive on \( D \cup \{x\} \), then \( D \) contains a \( T \)-invariant point.

The above corollary improves and generalizes Theorem 2 of [7].

In view of the Lemma, we have

**Corollary 2.** Let \( T \) be mapping on a metric space \( (X, d) \), \( G \) an \( \varepsilon \)-approximatively compact (approximatively compact) and \( T \)-invariant subset of \( X \) and \( x \) a \( T \)-invariant point. If the set \( D \) of \( \varepsilon \)-approximant (best approximant) to \( x \) has convex structure and \( T \) is nonexpansive on \( D \cup \{x\} \), then \( D \) contains a \( T \)-invariant point.

**Theorem 2.** Let \( T \) be a self mapping on a metric space \( (X, d) \), \( G \) a \( T \)-invariant subset of \( X \) and \( x \) a \( T \)-invariant point. If the set \( D \) of \( \varepsilon \)-approximant to \( x \) is compact, contractive and \( T \) is nonexpansive on \( D \cup \{x\} \), then \( D \) contains a \( T \)-invariant point.

**Proof.** Since \( D = \{ y \in G : d(x, y) \leq d(x, G) + \varepsilon \}, T : D \to D \). In fact if \( y \in D \), then
\[
d(x, Ty) = d(Tx, Ty) \leq d(x, y) \leq d(x, G) + \varepsilon
\]
and so \( Ty \in D \). Since \( D \) is contractive, there exists a sequence \( \langle f_n \rangle \) of contraction mapping of \( D \) into itself such that \( f_n z \to z \) for every \( z \in D \).

We claim that \( z_o \) is a fixed point of \( T \). Let \( \varepsilon > 0 \) be given. Since \( z_{n_i} \to z_o \) and \( f_n T z_o \to T z_o \), there exist a positive integer \( m \) such that for all \( n_i \geq m \)
\[
d(z_{n_i}, z_o) < \frac{\varepsilon}{2} \quad \text{and} \quad d(f_n T z_o, T z_o) < \frac{\varepsilon}{2}.
\]
Again,
\[ d(f_n Tz_n, f_n Tz_0) \leq d(z_n, z_0) < \frac{\varepsilon}{2}. \]
Hence,
\[ d(f_n Tz_n, Tz_0) \leq d(f_n Tz_n, f_n Tz_0) + d(f_n Tz_0, Tz_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \]
i.e.,
\[ d(f_n Tz_n, Tz_0) < \varepsilon \]
for all \( n \geq m \) and so \( f_n Tz_n \to Tz_0 \). But \( f_n Tz_n = z_{n_i} \to z_0 \)
and therefore \( Tz_0 = z_0 \). ■

Using the Lemma we have

**Corollary 3.** Let \( T \) be a self mapping on a metric space \( (X, d) \), \( G \) an \( \varepsilon \)-approximatively compact, \( T \)-invariant subset of \( X \) and \( x \) a \( T \)-invariant point. If the set \( D \) of \( \varepsilon \)-approximant to \( x \) is contractive and \( T \) is nonexpansive on \( D \cup \{x\} \),
then \( D \) contains a \( T \)-invariant point.

**Corollary 4.** Let \( T \) be mapping on a convex metric space \( (X, d) \) satisfying Property (I), \( G \) a \( T \)-invariant subset of \( X \) and \( x \) a \( T \)-invariant point. If the set \( D \)
of \( \varepsilon \)-approximant to \( x \) is compact, starshaped and \( T \) is nonexpansive on \( D \cup \{x\} \),
then \( D \) contains a \( T \)-invariant point.

**Proof.** As in Theorem 2, \( T \) is a self map on \( D \). Since \( D \) is non-empty and
starshaped, there exists \( p \in D \) such that \( W(z, p, \lambda) \in D \) for all \( z \in D \), \( \lambda \in I = [0, 1] \).
Let \( \langle k_n \rangle \), \( 0 \leq k_n < 1 \), be a sequence of real numbers such that \( k_n \to 1 \) as \( n \to \infty \).
Define \( T_n \) as \( T_n(z) = W(T_z, p, k_n) \), \( z \in D \). Since \( T \) is a self map on \( D \) and \( D \)
is starshaped, each \( T_n \) is a well defined and maps \( D \) into \( D \). Moreover,
\[ d(T_n y, T_n z) = d(W(Ty, p, k_n), W(Tz, p, k_n)) \leq k_n d(Ty, Tz) \leq k_n d(y, z), \]
i.e. each \( T_n \) is a contraction mapping on the compact set \( D \). So by Banach Contraction Principle each \( T_n \)
has a unique fixed point \( x_n \in D \), i.e. \( T_n x_n = x_n \) for each \( n \). Since \( D \) is compact, \( \langle x_n \rangle \)
has a subsequence \( x_n_i \to \bar{x} \in D \). We claim that \( T\bar{x} = \bar{x} \). Consider,
\[ d(x_n, T\bar{x}) = d(T_n x_n, T\bar{x}) = d(W(Tx_n, p, k_n), T\bar{x}) \]
\[ \leq k_n d(Tx_n, T\bar{x}) + (1 - k_n) d(p, T\bar{x}) \]
\[ \leq k_n d(x_n, \bar{x}) + (1 - k_n) d(p, T\bar{x}) \to 0, \]
and so \( x_n \to T\bar{x} \). Therefore \( T\bar{x} = \bar{x} \), i.e. \( \bar{x} \) is \( T \)-invariant. ■

**Remarks 1.** (i) Since in a convex metric space \( (X, d) \) satisfying Property (I)
every starshaped set is contractive, the result also follows from Theorem 2.


(iii) Since a Banach space is a convex metric space with Property (I) and \( D \) is compact if \( G \) is \( \varepsilon \)-approximatively compact. Theorem 2.2 of [8] is a particular case of Corollary 4.

Clearly, \( f_n T \) is a contraction on the compact set \( D \) for each \( n \) and so by Banach contraction principle, each \( f_n T \) has a unique fixed point, say \( z_n \) in \( D \). Now the compactness of \( D \) implies that the sequence \( \langle z_n \rangle \) has a subsequence \( z_{n_i} \to z_0 \in D \).
For $\varepsilon = 0$, we derive the following known results as corollaries.

**Corollary 5.** Let $T$ be a self mapping on a convex metric space $(X, d)$ satisfying Property (I), $G$ a $T$-invariant subset of $X$ and $x$ a $T$-invariant point. If the set $D$ of best $G$-approximant to $x$ is non-empty compact and starshaped and $T$ is nonexpansive on $D \cup \{x\}$, then $D$ contains a $T$-invariant point.

**Corollary 6.** [12]. Let $T$ be a nonexpansive mapping on a normed linear space $X$. Let $G$ be a $T$-invariant subset of $X$ and $x$ a $T$-invariant point in $X$. If $D$, the set of best $G$-approximant to $x$ is non-empty compact and starshaped, then it contains a $T$-invariant point.

**Corollary 7.** [13] Let $X$ be a normed linear space and $T : X \rightarrow X$ be a nonexpansive mapping. Let $G$ be a finite-dimensional subspace $G$ of $X$ invariant. Then $T$ has a fixed point which is a best $G$-approximant to $x$ in $G$.

Since in this case the set $D$ is non-empty and compact, the result follows from Corollary 6.

**Theorem 3.** Let $T$ be a self mapping on a convex metric space $(X, d)$ satisfying Property (I). Suppose $G$ is a closed $T$-invariant subset of $X$, $T/G$ is compact and $x$ a $T$-invariant point. If the set $D$ of $\varepsilon$-approximant to $x$ is starshaped and $T$ is nonexpansive on $D \cup \{x\}$, then $D$ contains a $T$-invariant point.

**Proof.** As in the proof of Theorem 2, $D$ is $T$-invariant. Now $D$ is a bounded subset of $G$ and $T/G$ is compact so $T(D)$ is compact. Since $D$ is closed and starshaped, by Theorem 3 [1] $T$ has a fixed point in $D$. $\blacksquare$

For $\varepsilon = 0$, Theorem 3 improves Theorem 10 of [1] and also generalizes Theorem 4 of [5].

Now we give a result for $T$-invariant points in the set of $\varepsilon$-coapproximations in $G$ for a given element $x$ of a metric space $(X, d)$.

**Theorem 4.** Let $T$ be a self map satisfying condition (A) on a convex metric space $(X, d)$ satisfying Property (I), $G$ a subset of $X$ such that $R_G(x, \varepsilon)$ is non-empty compact, starshaped and $T$ is nonexpansive on $R_G(x, \varepsilon)$. Then there exists a $g_T \in R_G(x, \varepsilon)$ such that $Tg_T = g_T$.

**Proof.** Let $g_0 \in R_G(x, \varepsilon)$. Consider

$$d(Tg_0, g) + \varepsilon \leq d(g_0, g) + \varepsilon \leq d(x, G)$$

and so $Tg_0 \in R_G(x, \varepsilon)$ i.e. $T : R_G(x, \varepsilon) \rightarrow R_G(x, \varepsilon)$. Now proceeding as in Corollary 4, we shall get $g_T \in R_G(x, \varepsilon)$ which is a fixed point for $T$ $\blacksquare$

**Remarks 2.** (i) Taking $\varepsilon = 0$, we see that Theorem 4 improves and generalizes Theorem 4.1 of [8].
(ii) Proceeding as in Theorem 1, one can show that Theorem 4 holds if star-shapedness of \( R_G(x, \varepsilon) \) is replaced by the condition that \( R_G(x, \varepsilon) \) is a set with convex structure.

(iii) Results similar to those proved in the earlier part of the paper can be proved for the set of \( \varepsilon \)-coapproximations.

(iv) Theorem 4.2 of [8] on strong best coapproximation can also be proved for convex metric space under relaxed conditions as in Theorem 4.

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