ON so-METRIZABLE SPACES

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Abstract. In this paper, we give some new characterizations for so-metrizable spaces, which answers a question posed by Z. Li and generalize some results on so-metrizable spaces. As some applications of the above results, some mappings theorems on so-metrizable spaces are obtained.

1. Introduction

so-networks (i.e. sequentially-open networks) were introduced and investigated by S Lin in [15]. Spaces with a \( \sigma \)-locally finite so-network are called so-metrizable spaces, which lie between metrizable spaces and \( sn \)-metrizable spaces. In [16], S. Lin gave the following characterization for so-metrizable spaces (see [16, Corollary 2.9 and Theorem 3.15]).

**Theorem 1.1.** The following are equivalent for a space \( X \):

1. \( X \) is an so-metrizable space.
2. \( X \) is an \( \aleph \)-space and contains no closed subspace having \( S_2 \) or \( S_\omega \) as its sequential coreflection.

Note that there exist the following characterizations for metrizable spaces and \( sn \)-metrizable spaces respectively.

**Theorem 1.2.** [21, Corollary 9] The following are equivalent for a space \( X \):

1. \( X \) is a metrizable space.
2. \( X \) has a \( \sigma \)-discrete base.
3. \( X \) has a \( \sigma \)-hereditarily closure-preserving base.
4. \( X \) is a first countable space with a \( \sigma \)-hereditarily closure-preserving \( k \)-network.

**Theorem 1.3.** [9, Lemma 2.2] The following are equivalent for a space \( X \):

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(1) $X$ is an sn-metrizable space.
(2) $X$ has a $\sigma$-discrete sn-network.
(3) $X$ has a $\sigma$-hereditarily closure-preserving sn-network.
(4) $X$ is an snf-countable space with a $\sigma$-hereditarily closure-preserving k-network.

Z. Li posed the following question [13, Question 3.2].

**Question 1.4.** Whether there exist some characterizations for so-metrizable spaces, which are similar to Theorem 1.2 or Theorem 1.3?

In this paper, we answer the above question affirmatively and give some mappings theorems on so-metrizable spaces. Throughout this paper, all spaces are assumed to be regular $T_1$, and all mappings are continuous and onto. $\mathbb{N}$, $\omega$ and $\omega_1$ denote the set of all natural numbers, the first infinite ordinal and the first uncountable ordinal respectively. The sequence $\{x_n : n \in \mathbb{N}\}$ and the sequence $\{P_n : n \in \mathbb{N}\}$ of subsets are abbreviated to $\{x_n\}$ and $\{P_n\}$ respectively. Let $P$ be a subset of a space $X$ and $\{x_n\}$ be a sequence in $X$. $\{x_n\}$ converging to $x$ is eventually in $P$ if $\{x_n : n > k\} \cup \{x\} \subset P$ for some $k \in \mathbb{N}$; it is frequently in $P$ if $\{x_{n_k}\}$ is eventually in $P$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a collection of subsets of $X$ and $x \in X$. Then $(\mathcal{P})_x$ denotes the subcollection $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$, $\bigcup \mathcal{P}$ and $\bigcap \mathcal{P}$ denote the union $\bigcup \{P : P \in \mathcal{P}\}$ and the intersection $\bigcap \{P : P \in \mathcal{P}\}$ respectively.

### 2. Characterizations

**Definition 2.1.** [7,11] Let $X$ be a space.

1. Let $x \in P \subset X$. $P$ is called a sequential neighborhood of $x$ in $X$ if whenever $\{x_n\}$ is a sequence converging to $x$, then $\{x_n\}$ is eventually in $P$.
2. Let $P \subset X$. $P$ is called a sequentially-open subset in $X$ if $P$ is a sequential neighborhood of $x$ in $X$ for each $x \in P$. $F$ is called a sequentially-closed subset in $X$ if $X - F$ is sequentially-open in $X$.
3. $X$ is called a sequential space if each sequentially-open subset in $X$ is open in $X$.
4. $X$ is called a k-space, if $F \subset X$ is closed in $X$ iff $F \cap C$ is closed in $C$ for every compact subset $C$ in $X$.

**Remark 2.2.** The following are well known.

1. $P$ is a sequential neighborhood of $x$ in $X$ iff each sequence $\{x_n\}$ converging to $x$ is frequently in $P$.
2. The intersection of finitely many sequentially-open subsets of $x$ in $X$ is a sequentially-open subset of $x$ in $X$.
3. Sequential spaces $\implies$ k-spaces.

**Definition 2.3.** [4] Let $\mathcal{P}$ a collection of subsets of a space $X$. 
(1) \( P \) is called closure-preserving if \( \bigcup P' = \bigcup \{ P : P \in P' \} \) for each \( P' \subset P \).

(2) \( P \) is called hereditarily closure-preserving if any collection \( \{ H(P) : P \in P \} \) is closure-preserving, where every \( H(P) \subset P \in P \).

**Definition 2.4.** Let \( P = \bigcup \{ P_x : x \in X \} \) be a cover of a space \( X \), where \( P_x \subset (P)_x \).

(1) \( P \) is called a network of \( X \) [3], if whenever \( x \in U \) with \( U \) open in \( X \) there exists \( P \in P_x \) such that \( x \in P \subset U \), where \( P_x \) is called a network at \( x \) in \( X \).

(2) \( P \) is called a \( cs \)-network of \( X \) [19], if for every convergent sequence \( S \) converging to a point \( x \in U \) with \( U \) open in \( X \), \( S \) is eventually in \( P \subset U \) for some \( P \in P \).

(3) \( P \) is called a \( k \)-network of \( X \) [19], if for every compact subset \( K \subset U \) with \( U \) open in \( X \), there exists a finite \( F \subset P \) such that \( K \subset \bigcup F \subset U \).

**Definition 2.5.** Let \( P = \bigcup \{ P_x : x \in X \} \) be a cover of a space \( X \). Assume that \( P \) satisfies the following (a) and (b) for each \( x \in X \).

(a) \( P_x \) is a network at \( x \) in \( X \).

(b) If \( P_1, P_2 \in P_x \), then there exists \( P \in P_x \) such that \( P \subset P_1 \cap P_2 \).

(1) \( P \) is called an \( sn \)-network of \( X \) [16,19], if every element of \( P_x \) is a sequential neighborhood of \( x \) for each \( x \in X \), where \( P_x \) is called an \( sn \)-network at \( x \).

(2) \( P \) is called an \( so \)-network of \( X \) [15,16], if every element of \( P_x \) is a sequentially-open subset, where \( P_x \) is called an \( so \)-network at \( x \).

**Definition 2.6.** [16] Let \( X \) be a space. \( X \) is an \( sof \)-countable (resp. \( snf \)-countable) space if for each \( x \in X \), there exists an \( so \)-network (resp. \( sn \)-network) \( P_x \) at \( x \) in \( X \) such that \( P_x \) is countable.

**Definition 2.7.** Let \( X \) be a space.

(1) \( X \) is an \( so \)-metrizable space [13] if \( X \) has a \( \sigma \)-locally finite \( so \)-network.

(2) \( X \) is an \( sn \)-metrizable space [9] if \( X \) has a \( \sigma \)-locally finite \( sn \)-network.

(3) \( X \) is an \( \aleph \)-space [11] if \( X \) has a \( \sigma \)-locally finite \( k \)-network.

**Remark 2.8.** For a space, base \( \Rightarrow \) \( so \)-network \( \Rightarrow \) \( sn \)-network \( \Rightarrow \) \( cs \)-network. An \( so \)-network for a sequential space is a base. So the following hold:

(1) First-countable \( \Rightarrow \) \( sof \)-countable \( \Rightarrow \) \( snf \)-countable.

(2) First-countable \( \Leftrightarrow \) sequential and \( sof \)-countable.

(3) metrizable spaces \( \Rightarrow \) \( so \)-metrizable spaces \( \Rightarrow \) \( sn \)-metrizable spaces \( \Rightarrow \) \( \aleph \) spaces.

(4) metrizable spaces \( \Leftrightarrow \) \( k \)- and \( so \)-metrizable spaces.

The following example shows that “sequential” in Remarks 2.8(2) can not be relaxed to “\( k \)”.

**Example 2.9.** There exists a \( k \)-, \( sof \)-countable space \( X \) such that is not first-countable.
Proof. Let \( X \) be the Stone-Čech compactification \( \beta \mathbb{N} \) of \( \mathbb{N} \). Then \( X \) is compact, and so it is a \( k \)-space. Since each convergent sequence in \( \beta \mathbb{N} \) is trivial, \( \mathcal{P} = \{ \{ x \} : x \in X \} \) is an so-network of \( X \), so \( X \) is sof-countable. It is known that \( X \) is not first countable. \( \blacksquare \)

Definition 2.10. Let \( S = \{ 1/n : n \in \mathbb{N} \} \cup \{ 0 \} \) be a space with the usual topology induced from \( \mathbb{R} \). For each \( \alpha < \omega_1 \), let \( S_\alpha \) be a copy of \( S \). Then \( S_\omega \) denotes the quotient space obtained from the topological sum \( \oplus_{\alpha < \omega_1} S_\alpha \) by mapping all the nonisolated points into one point \([12]\).

Lemma 2.11. Let \( \mathcal{P} \) be a hereditarily closure-preserving collection of sequentially-open subsets of a space \( X \). Then \( \bigcap \mathcal{P} \) is a sequentially-open subset of \( X \).

Proof. Let \( x \in \bigcap \mathcal{P} \), and let \( \{ x_n \} \) be a sequence converging to \( x \). By Remark 2.2(1), we only need to prove that \( \{ x_n \} \) is frequently in \( \bigcap \mathcal{P} \). If \( x_n = x \) for infinitely many \( n \in \mathbb{N} \), then \( \{ x_n \} \) is frequently in \( \bigcap \mathcal{P} \). If \( x_n \neq x \) for all but finitely many \( n \in \mathbb{N} \), we may assume \( x_n \neq x \) for all \( n \in \mathbb{N} \), then \( \mathcal{P} \) is finite. Indeed, suppose \( \mathcal{P} \) is infinite. Then there exists an infinite subcollection \( \{ P_k : k \in \mathbb{N} \} \) of \( \mathcal{P} \), where \( P_k \neq P_l \) if \( k \neq l \). Since \( \{ x_n \} \) converges to \( x \) and \( P_k \) is sequentially-open for each \( k \in \mathbb{N} \), we can construct a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) such that \( x_{n_k} \in P_k \) for each \( k \in \mathbb{N} \). Note that \( \mathcal{P} \) is hereditarily closure-preserving and \( \{ x_{n_k} \} \) converges to \( x \). So \( x \in \overline{x_{n_k}} : k \in \mathbb{N} \} = \{ x_{n_k} : k \in \mathbb{N} \} \). This is a contradiction. So \( \mathcal{P} \) is finite. By Remark 2.2(2), \( \bigcap \mathcal{P} \) is sequentially-open. \( \blacksquare \)

Lemma 2.12. Let \( X \) be a space and \( x \in X \). If there exists a \( \sigma \)-hereditarily closure-preserving network at \( x \) in \( X \) such that its every element is a sequentially-open subset in \( X \), then there exists a countable and decreasing so-network at \( x \) in \( X \).

Proof. Let \( \mathcal{P}' = \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \} \) is a network at \( x \) in \( X \), where \( \mathcal{P}_n \) is hereditarily closure-preserving for each \( n \in \mathbb{N} \) and every element of \( \mathcal{P}' \) is a sequentially-open subset in \( X \). We may assume each \( \mathcal{P}_n \subset \mathcal{P}_{n+1} \). For each \( n \in \mathbb{N} \), put \( P_n = \bigcap \mathcal{P}_n \), then \( P_{n+1} \subset P_n \) as \( \mathcal{P}_n \subset \mathcal{P}_{n+1} \). Put \( \mathcal{P} = \{ P_n : n \in \mathbb{N} \} \), then \( \mathcal{P} \) is countable and decreasing.

Claim 1. \( \mathcal{P} \) is a network at \( x \) in \( X \).

Let \( x \in U \) with \( U \) open in \( X \). Since \( \mathcal{P}' \) is a so-network, there exists \( P \in \mathcal{P}_n \) for some \( n \in \mathbb{N} \) such that \( x \in P \subset U \). Thus \( x \in P_n \subset P \subset U \). This proves that \( \mathcal{P} \) is a network at \( x \) in \( X \).

Claim 2. If \( P_i, P_j \in \mathcal{P} \), then \( P_k \subset P_i \cap P_j \) for some \( P_k \in \mathcal{P} \).

It is clear because \( \mathcal{P} \) is countable and decreasing.

Claim 3. \( P_n \) is a sequentially-open subset for each \( n \in \mathbb{N} \).

It holds from Lemma 2.11.

By the above three claims, \( \mathcal{P} \) is a countable and decreasing so-network at \( x \) in \( X \). \( \blacksquare \)
**Corollary 2.13.** Let a space $X$ have a $\sigma$-hereditarily closure-preserving so-network. Then $X$ is an sof-countable space.

**Lemma 2.14.** sof-countable space contains no copy of $S_{\omega_1}$.

**Proof.** Note that $S_{\omega_1}$ is a sequential space, but it is not first-countable. By Remark 2.8(2), $S_{\omega_1}$ is not sof-countable. Obviously, sof-countable spaces are hereditary to all subspaces. So sof-countable space contains no copy of $S_{\omega_1}$. $lacksquare$

**Lemma 2.15.** Let $X$ be an sof-countable space with a $\sigma$-hereditarily closure-preserving $k$-network. Then $X$ has a $\sigma$-discrete so-network.

**Proof.** Since $X$ is sof-countable, $X$ contains no copy of $S_{\omega_1}$ from Lemma 2.14. Note that a space is an $\aleph$-space iff it has a $\sigma$-hereditarily closure-preserving $k$-network, and contains no copy of $S_{\omega_1}$ [12, Theorem 2.6]. So $X$ is an $\aleph$-space. By [6, Theorem 4], $X$ has a $\sigma$-discrete cs-network $B$. For each $x \in X$, let $P'_x$ be a countable so-network at $x$ in $X$. By Remark 2.2(2), we can assume that each $P'_x$ is decreasing. For each $x \in X$, put $B_x = \{ B \in B : P \subset B \text{ for some } P \in P'_x \}$. By a similar way as in the proof of [18, Lemma 7(3)], $B_x$ is a network at $x$ in $X$. For each $B \in B_x$, choose $P_B \in P'_x$ such that $P_B \subset B$. Put $P_x = \{ P_B : B \in B_x \}$, and put $P = \bigcup_{x \in X} P_x$. It suffices to prove the following three claims.

**Claim 1.** $P$ is $\sigma$-discrete: It holds because $\bigcup_{x \in X} B_x$ is $\sigma$-discrete.

**Claim 2.** Every element of $P$ is sequentially-open: It is clear.

**Claim 3.** For each $x \in X$, $P_x$ is a network at $x$ in $X$: Let $x \in U$ with $U$ open in $X$. Since $B_x$ is a network at $x$ in $X$, $x \in B \subset U$ for some $B \in B_x$. By the construction of $P_x$, there exists $P_B \in P_x$ such that $x \in P_B \subset B \subset U$. So $P_x$ is a network at $x$ in $X$. $lacksquare$

Now we give the main theorem in this section, which answers Question 1.4 affirmatively.

**Theorem 2.16.** The following are equivalent for a space $X$:

1. $X$ has a $\sigma$-discrete so-network.
2. $X$ is an so-metrizable space.
3. $X$ has a $\sigma$-hereditarily closure-preserving so-network.
4. $X$ is an sof-countable space with a $\sigma$-hereditarily closure-preserving $k$-network.

**Proof.** (1) $\implies$ (2) $\implies$ (3): Obvious.

(3) $\implies$ (4): By Corollary 2.13, $X$ is sof-countable. Note that every $\sigma$-hereditarily closure-preserving so-network of a space is a $k$-network [20, Proposition 1.2(2)]. So $X$ has a $\sigma$-hereditarily closure-preserving $k$-network.

(4) $\implies$ (1): It holds by Lemma 2.15. $lacksquare$
3. Invariance and inverse invariance under mappings.

**Definition 3.1.** Let \( f : X \to Y \) be a mapping.

1. \( f \) is called a closed (resp. an open) mapping \([5]\) if \( f(B) \) is closed (resp. open) in \( Y \) for every closed (resp. open) subset \( B \) in \( X \).

2. \( f \) is called an \( sn \)-open mapping \([10]\) if there exists an \( sn \)-network \( P = \{ P_y : y \in Y \} \) of \( Y \) such that for each \( y \in Y \) and each \( x \in f^{-1}(y) \), whenever \( U \) is a neighborhood of \( x \), then \( P \subset f(U) \) for some \( P \in P_y \).

3. \( f \) is called a perfect mapping \([5]\) if \( f \) is closed and \( f^{-1}(y) \) is a compact subset of \( X \) for each \( y \in Y \).

**Remark 3.2.**

1. Open mappings \( \Rightarrow \) \( sn \)-open mappings.

2. It is easy to obtain a simple characterization for \( sn \)-open mappings: A mapping \( f : X \to Y \) is \( sn \)-open iff \( f(B) \) is a sequentially-open subset in \( Y \) for every open subset \( B \) in \( X \). (So more precisely, \( sn \)-open mappings should be called sequentially-open mappings).

**Definition 3.3.** A space \( X \) is said to have a \( G_\delta \)-diagonal \([11]\) if \( \{ (x, x) : x \in X \} \) is a \( G_\delta \)-set in \( X \times X \).

**Definition 3.4.** Let \( X \) be a space. Put \( \sigma = \{ P \subset X : P \) is sequentially open in \( X \} \), and endow \( X \) with the topology \( \sigma \). The space \( (X, \sigma) \) is called sequential coreflection of \( X \) \([16]\), and denoted by \( \sigma X \).

**Definition 3.5.**

1. Let \( L_0 = \{ a_n : n \in \mathbb{N} \} \) be a sequence converging to \( \infty \), where \( \infty \notin L_0 \). For each \( n \in \mathbb{N} \), let \( L_n \) be a sequence converging to \( b_n \), where \( b_n \notin L_n \). Put \( T_0 = L_0 \cup \{ \infty \} \) and \( T_n = L_n \cup \{ b_n \} \) for each \( n \in \mathbb{N} \). Let \( M \) be the topological sum of \( \{ T_n : n \geq 0 \} \). Then \( S_2 \) denotes the quotient space obtained from the topological sum \( M \) by identifying \( a_n \) with \( b_n \) for each \( n \in \mathbb{N} \) \([1]\).

2. Let \( S = \{ 1/n : n \in \mathbb{N} \} \cup \{ 0 \} \) be a space with the usual topology induced from \( \mathbb{R} \). For each \( \alpha < \omega \), let \( S_\alpha \) be a copy of \( S \). Then \( S_\omega \) denotes the quotient space obtained from the topological sum \( \oplus_{\alpha<\omega} S_\alpha \) by mapping all the nonisolated points into one point \([2]\).

It is easy to see that a closed image of a so-metrizable space need not be so-metrizable. Now we give a sufficient and necessary condition such that closed images of so-metrizable spaces are so-metrizable spaces.

**Lemma 3.6.** Let \( f : X \to Y \) be a closed mapping, and let \( X \) have a \( \sigma \)-hereditarily closure-preserving \( k \)-network. Then \( Y \) is so-metrizable iff \( Y \) is sof-countable.

**Proof.** Necessity is obvious. We only need to prove sufficiency. Let \( Y \) be sof-countable. Note that closed mappings preserve \( \sigma \)-hereditarily closure-preserving \( k \)-networks. So \( Y \) has a \( \sigma \)-hereditarily closure-preserving \( k \)-network. By theorem 2.16, \( Y \) is so-metrizable. \( \blacksquare \)
We immediately obtain the following result by the above lemma.

**Theorem 3.7.** A closed image of an so-metrizable space is so-metrizable iff it is sof-countable.

Perfect mappings preserve metrizable spaces. However, we do not know even whether finite-to-one, closed mappings preserve so-metrizable spaces. As an applications to Theorem 3.7, we give some partial answers to this question.

**Lemma 3.8.** Let \( f : X \to Y \) be an sn-open, closed mapping and each point in \( X \) be a \( G_\delta \)-set. If \( P \) is a sequentially-open subset in \( X \), then \( f(P) \) is a sequentially-open subset in \( Y \).

**Proof.** Let \( P \) be a sequentially-open subset in \( X \) and \( y \in f(P) \). Let \( \{y_k\} \) be a sequence converging to \( y \). It suffices to prove that \( \{y_k\} \) is frequently in \( f(P) \). Without loss of generality, we assume that \( y_i \neq y_j \) for all \( i \neq j \) and \( y_k \neq y \) for all \( k \). Pick \( x \in P \) such that \( f(x) = y \), then \( \{x\} \) is a \( G_\delta \)-set in \( X \). Let \( \{W_n : n \in \mathbb{N}\} \) be a sequence of open neighborhoods of \( x \) such that \( \overline{W}_{n+1} \subseteq W_n \) for each \( n \in \mathbb{N} \) and \( \bigcap_{n \in \mathbb{N}} W_n = \{x\} \). For each \( n \in \mathbb{N} \), \( f(W_n) \) is a sequentially-open subset of \( Y \) by Remark 3.2(2). So \( \{y_k\} \) is eventually in \( f(W_n) \). Thus there exists \( k_n \in \mathbb{N} \) such that \( y_{k_n} \in f(W_n) \). Pick \( x_n \in W_n \cap f^{-1}(y_{k_n}) \). By this method, we construct a sequence \( \{x_n\} \) such that \( x_n \in W_n \) and \( f(x_n) = y_{k_n} \) for each \( n \in \mathbb{N} \). Here, we can assume that \( \{f(x_n)\} = \{y_{k_n}\} \) is a subsequence of \( \{y_k\} \). Now we prove that \( \{x_n\} \) converges to \( x \).

If \( \{x_n\} \) does not converge to \( x \), then there exists a neighborhood \( U \) of \( x \) such that \( \{x_n\} \) is not eventually in \( U \). So there exists a subsequence \( \{x_{n_i}\} \) such that \( x_{n_i} \not\in U \) for each \( i \in \mathbb{N} \). Put \( L = \{x_{n_i} : i \in \mathbb{N}\} \), then \( L \) is an infinite subset of \( X \) and \( x \) is not a cluster point of \( L \). On the other hand, \( f(L) = f(\mathbb{N}) \) since \( f \) is closed. Thus \( y \in f(L) \) and \( y \not\in f(L) \), so \( L \) has a cluster point \( z \neq x \). Because \( \{x\} = \bigcap_{n \in \mathbb{N}} W_n = \bigcap_{n \in \mathbb{N}} \overline{W}_n \), \( z \in X - \overline{W}_n \) for some \( n \in \mathbb{N} \). Note that \( X - \overline{W}_n \) is a neighborhood and only contains finitely many points of \( L \). This contradicts that \( z \) is a cluster point of \( L \). Thus we prove that \( \{x_n\} \) converges to \( x \).

Since \( P \) is a sequentially-open subset in \( X \) and \( x \in P \), \( \{x_n\} \) is eventually in \( P \), and so \( \{f(x_n)\} = \{y_{k_n}\} \) is eventually in \( f(P) \). This shows that \( \{y_n\} \) is frequently in \( f(P) \). \( \blacksquare \)

**Theorem 3.9.** Let \( f : X \to Y \) be an sn-open, closed mapping. If \( X \) is so-metrizable, then \( Y \) is so-metrizable.

**Proof.** Let \( X \) be so-metrizable. By theorem 3.7, we need to prove that \( Y \) is sof-countable. Let \( \mathcal{P} \) be a \( \sigma \)-hereditarily closure-preserving so-network of \( X \). Put \( \mathcal{F} = \{f(P) : P \in \mathcal{P}\} \), then \( \mathcal{F} \) is \( \sigma \)-hereditarily closure-preserving because closed mappings preserve \( \sigma \)-hereditarily closure-preserving collections. Let \( y \in Y \), put \( \mathcal{F}_y = \{f(P) : P \in \mathcal{P}, x \in f^{-1}(y) \cap P \} \), then \( \mathcal{F}_y \subset \mathcal{F} \) is \( \sigma \)-hereditarily closure-preserving. Since \( X \) is so-metrizable, each point in \( X \) is a \( G_\delta \)-set. By Lemma 3.8, every element of \( \mathcal{F}_y \) is a sequentially-open subset in \( Y \). It suffices to prove that
exists a closed subspace \( S \), having an \( \aleph \)-space. By Lemma 3.11, there exists a subsequence \( \{\mathbf{n}_m\} \) converging to some \( \mathbf{x} \in X \). Note that \( \mathbf{x} \notin X - B \) for each \( \mathbf{n} \in \mathbb{N} \). □

**Lemma 3.12.** If \( X \) is an \( \aleph \)-space that contains no closed subspace having an \( \aleph \)-, non-metrizable space as its sequential coreflection, then \( X \) is so-metrizable.

**Proof.** Let \( X \) be an \( \aleph \)-space that contains no closed subspace having an \( \aleph \)-, non-metrizable space as its sequential coreflection. \( S_2 \) and \( S_\omega \) are \( \aleph \)-, non-metrizable spaces [17, Example 1.8.6 and Example 1.8.7], so \( X \) contains no closed subspace having \( S_2 \) or \( S_\omega \) as its sequential coreflections. By Theorem 1.1, \( X \) is so-metrizable.

**Theorem 3.13.** Let \( f : X \to Y \) be a perfect mapping and \( Y \) be so-metrizable. Then \( X \) is so-metrizable iff \( X \) has a \( G_\delta \)-diagonal.

**Proof.** Necessity is obvious. We only need to prove sufficiency.

Let \( X \) have a \( G_\delta \)-diagonal. By Remark 2.8(3) and [14, Theorem 3.4], \( X \) is an \( \aleph \)-space. By Lemma 3.12, it suffices to prove that \( X \) contains no closed subspace having an \( \aleph \)-, non-metrizable space as its sequential coreflection. If not, then there exists a closed subspace \( S \) of \( X \) such that \( \sigma S \) is homeomorphic to an \( \aleph \)-, non-metrizable spac \( T \). Put \( g : \sigma S \to \sigma f(S) \), where \( g = f|_{\sigma S} \) is the restriction of \( f \) on \( \sigma S \).

(a) \( g \) is a closed mapping: Let \( F \) is a closed subset of \( \sigma S \). Then \( F \) is a sequentially-closed subset of \( S \). It is clear that \( S \) has a \( G_\delta \)-diagonal and \( f(S) :
$S \rightarrow f(S)$ is a closed mapping. By Lemma 3.11, $f|_S(F)$ is a sequentially-closed subset of $f(S)$, so $g(S) = f|_S(S)$ is a closed subset of $\sigma f(S)$.

(b) $g$ is a compact mapping: Let $y \in \sigma f(S)$. Then $f^{-1}(y)$ is a compact subset of $X$. Note that $f^{-1}(y)$ has a $G_{\delta}$-diagonal. So $f^{-1}(y)$ is compact metrizable from [17, Theorem 1.4.10]. Thus the topology on $f^{-1}(y) \cap S$ as a subspace of $\sigma S$ is equivalent to the topology on $f^{-1}(y) \cap S$ as a subspace of $X$. Consequently, $g^{-1}(y) = f^{-1}(y) \cap S$ is compact.

By the above (a) and (b), $g$ is a perfect mapping. Note that $\sigma S = T$ is an $\aleph$-space and perfect mappings preserve $\aleph$-spaces [14, Theorem 2.2]. So $\sigma f(S)$ is an $\aleph$-space. It is easy to see that $f(S)$, as a subspace of $Y$, is sof-countable. By [16, Corollary 2.8], $\sigma f(S)$ is first countable. Thus $\sigma f(S)$ is metrizable from Theorem 1.2, so $\sigma S$ is a perfect pre-image of a metrizable space. By [11, Corollary 3.8], $\sigma S$ is metrizable. This contradicts that $\sigma S = T$ is not metrizable.$\blacksquare$

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