ON $L^1$-CONVERGENCE OF CERTAIN GENERALIZED MODIFIED TRIGONOMETRIC SUMS

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Abstract. In this paper we define new modified generalized sine sums $K_{nr}(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} (\triangle^r a_{k-1} - \triangle^r a_{k+1}) \tilde{S}^{r-1}_k(x)$ and study their $L^1$-convergence under a newly defined class $K^\alpha$. Our results generalize the corresponding results of Kaur, Bhatia and Ram [6] and Kaur [7].

1. Introduction

Let
\[ K(x) = \sum_{k=1}^{\infty} a_k \cos kx \] (1.1)

and
\[ K_n(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\triangle a_{j-1} - \triangle a_{j+1}) \sin kx. \] (1.2)

Using modified cosine sums
\[ g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\triangle a_j) \cos kx \]

of Garett and Stanojević [4], Kaur and Bhatia [5] proved the following theorem under the class of generalized semi-convex coefficients.

THEOREM 1. If \( \{a_n\} \) is a generalized semi-convex null sequence, then \( g_n(x) \) converges to \( K(x) \) in the $L^1$-metric if and only if \( \triangle a_n \log n = o(1) \), as \( n \to \infty \).

The above mentioned result motivated the authors [6] to define a new modified sums (1.2) and to study these sums under a different class $K$ of coefficients in the following way.

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Theorem 2. Let $k$ be a positive real number. If
\[ a_k = o(1), \quad k \to \infty \quad (1.3) \]
and
\[ \sum_{k=1}^{\infty} k|\Delta^2 a_{k-1} - \Delta^2 a_{k+1}| < \infty, \quad (1.4) \]
then $K_n(x)$ converges to $K(x)$ in the $L^1$-norm.

Any sequence satisfying (1.3) and (1.4) is said to belong to the class $K_0$.

In particular in [6] the following corollary to Theorem 2 is proved.

Theorem 3. If $\{a_n\}$ belongs to the class $K_0$, then the necessary and sufficient condition for $L^1$-convergence of the cosine series (1.1) is
\[ \lim_{n \to \infty} a_n \log n = 0. \]

Definition. Let $\alpha$ be a positive real number. If (1.3) holds and
\[ \sum_{k=1}^{\infty} k^\alpha |\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}| < \infty, \quad (a_0 = 0), \]
then we say that $\{a_n\}$ belongs to the class $K_\alpha$.

For $\alpha = 1$, the class $K_\alpha$ reduces to the class $K_0$.

Applying Abel's transformation to $K_n(x)$, we can easily see that
\[ K_n(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} (\Delta a_{k-1} - \Delta a_{k+1}) \tilde{S}_k^0(x), \]
where $\tilde{S}_k^0(x) = \tilde{D}_k(x) = \sin x + \sin 2x + \sin 3x + \cdots + \sin nx$. So in [7] the author studied the $L^1$-convergence of $K_n(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} (\Delta a_{k-1} - \Delta a_{k+1}) \tilde{S}_k^0(x)$ to $K(x)$ by proving the following result.

Theorem 4. Let the sequence $\{a_k\}$ belong to class $K_\alpha$. Then $K_n(x)$ converges to $K(x)$ in the $L^1$-norm.

If we take $\alpha = 1$, then this theorem reduces to Theorem 2.

It is natural to seek ways to prove the $L^1$-convergence of the generalized sums of the form
\[ K_{nr}(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} (\Delta^r a_{k-1} - \Delta^r a_{k+1}) \tilde{S}_k^{-r}(x), \quad (1.5) \]
where $r$ is any real number greater than or equal to 1. It is obvious that for $r = 1$, $K_{nr}(x)$ reduces to $K_n(x)$.

The purpose of this paper is to prove the $L^1$-convergence of $K_{nr}(x)$ to $K(x)$ under the class $K_\alpha$. 
2. Notation and Formulae

We use the following notations [10].

Given a sequence $S_0, S_1, S_2, \ldots$, we define the sequence $S_0^\alpha, S_1^\alpha, S_2^\alpha, \ldots$ for every $\alpha = 0, 1, 2, \ldots$ by the conditions $S_0^\alpha = S_\alpha, S_\alpha = S_0^\alpha + S_{\alpha-1}^\alpha + S_{\alpha-2}^\alpha + \cdots + S_0^\alpha$ $(\alpha = 1, 2, \ldots; n = 0, 1, 2, \ldots)$. Similarly we define the sequence of numbers $A_0^\alpha, A_1^\alpha, A_2^\alpha, \ldots$ for $\alpha = 0, 1, 2, \ldots$ by the conditions $A_0^\alpha = 1, A_\alpha^\alpha = A_{\alpha-1}^\alpha + A_{\alpha-2}^\alpha + \cdots + A_0^\alpha$ $(\alpha = 1, 2, \ldots; n = 0, 1, 2, \ldots)$. Let $\sum a_n$ be a given infinite series. The conjugate Cesàro sums of order $\alpha$ of $\sum a_n$ for any real number $\alpha$ are defined by

$$\tilde{S}_n^\alpha(a_p) = \tilde{S}_n^\alpha = \sum_{p=0}^{\infty} A_\alpha^\alpha = \sum_{p=0}^{\infty} A_{\alpha-p}^\alpha \tilde{S}_p,$$

where $\tilde{S}_n = \tilde{S}_n^0 = \tilde{D}_n$, and $A_p^\alpha$ denotes the binomial coefficients. The conjugate Cesàro means $\tilde{T}_n^\alpha$ of order $r$ of $\sum a_n$ will be defined by

$$\tilde{T}_n^\alpha = \frac{\tilde{S}_n^\alpha}{A_\alpha^\alpha}.$$

The following formulae will also be needed:

$$\tilde{S}_n^\alpha(\tilde{S}_p^\alpha) = \tilde{S}_{n+r}^{\alpha+1},$$

$$\tilde{S}_n^{\alpha+1} - \tilde{S}_{n-1}^{\alpha+1} = \tilde{S}_n^\alpha, \quad \sum_{p=0}^{\infty} A_{\alpha-p}^\beta = A_{\alpha+\beta}^{\alpha+1}.$$ 

The differences of order $\alpha$ of the sequence $\{a_n\}$ for any positive integer $\alpha$ are defined by the equations $\Delta^\alpha a_n = \Delta(\Delta^{\alpha-1} a_n), n = 0, 1, 2, \ldots \Delta^1 a_n = a_n - a_{n+1}$. Since $A_m^{-\alpha-1} = 0$ for $m \geq \alpha + 1$, we have

$$\Delta^\alpha a_n = \sum_{m=0}^{\alpha} A_m^{-\alpha-1} a_{n+m} = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m}.$$

(2.1)

If the series (2.1) is convergent for some $\alpha$ which is not a positive integer, then we denote the differences

$$\Delta^\alpha a_n = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m} \quad n = 0, 1, 2, 3, \ldots.$$

The broken differences $\Delta^\alpha a_p$ are defined by

$$\Delta^\alpha a_p = \sum_{m=0}^{n-p} A_m^{-\alpha-1} a_{p+m}.$$

By repeated partial summation of order $r$, we have

$$\sum_{p=0}^{n} a_p b_p = \sum_{p=0}^{n} \tilde{S}_p^{\alpha-1}(a_p) \Delta^\alpha b_p.$$
3. Lemmas

We need the following lemmas for the proof of our result.

**Lemma 3.1** [3] If $r \geq 0$, $p \geq 0$,

(i) $\epsilon_n = o(n)^{-p}$, and

(ii) $\sum_{n=0}^{\infty} A_n^{r+p} |\Delta^{r+1} \epsilon_n| < \infty$,

then

(iii) $\sum_{n=0}^{\infty} A_n^{\lambda+p} |\Delta^{\lambda+1} \epsilon_n| < \infty$, for $-1 \leq \lambda \leq r$ and

(iv) $A_n^{\lambda+p} \Delta^{\lambda} \epsilon_n$ is of bounded variation for $0 \leq \lambda \leq r$ and tends to zero as $n \to \infty$.

**Lemma 3.2** [1] Let $r$ be a non-negative real number. If the sequence $\{\epsilon_n\}$ satisfies the conditions

(i) $\epsilon_n = O(1)$, and

(ii) $\sum_{n=1}^{\infty} n^r |\Delta^{r+1} \epsilon_n| < \infty$,

then $\Delta^{\beta} \epsilon_n = \sum_{m=0}^{\infty} A_m^{r-\beta} \Delta^{r+1} \epsilon_{n+m}$, for $\beta > 0$.

**Lemma 3.3** [2] If $0 < \delta < 1$ and $0 \leq n < m$, then

$$\left| \sum_{i=0}^{m} A_{n-i}^{\delta-1} S_i \right| \leq \max_{0 \leq p \leq m} |S_p|.$$

**Lemma 3.4** [10] Let $S_n(x)$ and $T_n^{\alpha}$ be the $n$th partial sum and Cesàro mean of order $\alpha > 0$, respectively, of the series $\sin x + \sin 2x + \sin 3x + \cdots + \sin nx + \cdots$.

Then

(i) $\int_{0}^{\pi} |S_n(x)| \, dx \sim \log n$,

(ii) $\int_{0}^{\pi} |T_n^{\alpha}| \, dx$ remains bounded for all $n$.

4. Main result

The main result of this paper is the following theorem.

**Theorem 4.1.** Let $\alpha$ be a positive real number. If a sequence $\{a_k\}$ belongs to the class $K^\alpha$, then for $\alpha \leq r \leq \alpha + 1$

(i) $K_{nr}(x)$ converges to $K(x)$ pointwise for $0 < \delta \leq x \leq \pi$, and

(ii) $K_{nr}(x) \to K(x)$ in the $L^1$-norm.

If we take $\alpha = 1$ and $r = 1$, then Theorem 2 is obtained as a particular case and also Theorem 4 can be deduced as a special case of Theorem 4.1 if we take $r = 1$ in (1.5).
Proof. We have

\[ K(x) = \frac{1}{2 \sin x} \sum_{k=1}^{\infty} (\triangle^r a_{k-1} - \triangle^r a_{k+1}) \tilde{S}_k^{r-1}(x) \]

and

\[ K_{nr}(x) = \frac{1}{2 \sin x} \sum_{k=1}^{n} (\triangle^r a_{k-1} - \triangle^r a_{k+1}) \tilde{S}_k^{r-1}(x). \]

Case 1. Let \( r = \alpha + 1 \). Then

\[ K_{nr}(x) = \frac{1}{2 \sin x} \sum_{k=1}^{n} (\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}) \tilde{S}_k^{\alpha}(x). \]

So \( K_{nr}(x) \) converges to \( K(x) \) pointwise for \( 0 < \delta \leq x \leq \pi \).

Now

\[
\int_0^\pi |K(x) - K_{nr}(x)| \, dx \\
= \int_0^\pi \left| \frac{1}{2 \sin x} \sum_{k=n+1}^{\infty} (\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}) \tilde{S}_k^{\alpha}(x) \right| \, dx \\
\leq C \sum_{k=n+1}^{\infty} \left| (\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}) \right| \int_0^\pi |\tilde{S}_k^{\alpha}(x)| \, dx \\
= C \sum_{k=n+1}^{\infty} A_k^{\alpha} \left| (\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}) \right| \int_0^\pi |\tilde{T}_k^{\alpha}(x)| \, dx \\
\leq C_1 \sum_{k=n+1}^{\infty} A_k^{\alpha} \left| (\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}) \right| = o(1), \text{ by hypothesis of Theorem 4.1.}
\]

Therefore, \( K_{nr}(x) \) converges to \( K(x) \) as \( n \to \infty \) in the \( L^1 \)-norm.

Case 2. Let \( \alpha < r < \alpha + 1 \). Take \( r = \alpha + 1 - \delta \) and \( 0 < \delta < 1 \). Then

\[ K_{nr}(x) = \frac{1}{2 \sin x} \sum_{k=1}^{n} (\triangle^r a_{k-1} - \triangle^r a_{k+1}) \tilde{S}_k^{r-1}(x) \\
= \frac{1}{2 \sin x} \sum_{k=1}^{n} (\triangle^{\alpha+1-\delta} a_{k-1} - \triangle^{\alpha+1-\delta} a_{k+1}) \tilde{S}_k^{\alpha-\delta}(x). \]

Applying Abel’s transformation of order \( -\delta \) and using Lemma 3.2, we have

\[
\frac{1}{2 \sin x} \sum_{k=1}^{n} (\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}) \tilde{S}_k^{\alpha}(x) \\
= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n} \tilde{S}_k^{\alpha-\delta}(x) \sum_{m=1}^{n-k} A_m^{\delta-1} (\triangle^{\alpha+1} a_{m+k-1} - \triangle^{\alpha+1} a_{m+k+1}) \right] \\
= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n} \tilde{S}_k^{\alpha-\delta}(x) \left\{ (\triangle^{\alpha-\delta+1} a_{k-1} - \triangle^{\alpha-\delta+1} a_{k+1}) \\
- \sum_{m=n-k+1}^{\infty} A_m^{\delta-1} (\triangle^{\delta+1} a_{m+k-1} - \triangle^{\delta+1} a_{m+k+1}) \right\} \right].
\]
This implies that

\[ R_n(x) = \sum_{k=1}^{n} S_k(x) \left( A_{n-k+1}^{\delta-1}(\Delta^{\delta+1}a_n - \Delta^{\delta+1}a_{n+2}) + A_{n-k+2}^{\delta-1}(\Delta^{\delta+1}a_{n+1} - \Delta^{\delta+1}a_{n+3}) + \cdots \right) \]

This implies that

\[ K_{nr}(x) = \frac{1}{2\sin x} \left[ \sum_{k=1}^{n} S_k(x)(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1}) + R_n(x) \right]. \]

Hence

\[
\int_0^{\pi} |K(x) - K_{nr}(x)| \, dx \\
\leq C \int_0^{\pi} \left| \frac{1}{2\sin x} \left\{ \sum_{k=n+1}^{\infty} S_k(x)(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1}) - R_n(x) \right\} \right| \, dx \\
\leq C \sum_{k=n+1}^{\infty} |(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1})| \int_0^{\pi} |S_k(x)| \, dx + \int_0^{\pi} |R_n(x)| \, dx \\
= C \sum_{k=n+1}^{\infty} A_k(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1}) |(x) \int_0^{\pi} |S_k(x)| \, dx + \int_0^{\pi} |R_n(x)| \, dx \\
\leq \sum_{k=n+1}^{\infty} A_k(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1}) + \int_0^{\pi} |R_n(x)| \, dx \\
= o(1) + \int_0^{\pi} |R_n(x)| \, dx, \text{ by hypotheses of Theorem } 4.1. \tag{4.1} \]

Now

\[
\int_0^{\pi} |R_n(x)| \, dx \\
= \int_0^{\pi} \left| \left( \sum_{k=1}^{n} S_k(x) A_{n-k+1}^{\delta-1}(\Delta^{\delta+1}a_n - \Delta^{\delta+1}a_{n+2}) + A_{n-k+2}^{\delta-1}(\Delta^{\delta+1}a_{n+1} - \Delta^{\delta+1}a_{n+3}) + \cdots \right) \right| \, dx \\
\leq \int_0^{\pi} |(\Delta^{\delta+1}a_n - \Delta^{\delta+1}a_{n+2})| \left| \sum_{k=1}^{n} S_k(x) A_{n-k+1}^{\delta-1} \right| \, dx \\
+ \int_0^{\pi} |(\Delta^{\delta+1}a_{n+1} - \Delta^{\delta+1}a_{n+3})| \left| \sum_{k=1}^{n} S_k(x) A_{n-k+2}^{\delta-1} \right| \, dx + \cdots
\]
Thus
\[ \int_0^\pi |R_n(x)| \, dx = o(1), \quad n \to \infty. \]

Hence by (4.1)
\[
\lim_{n \to \infty} \int_0^\pi |K(x) - K_{nr}(x)| \, dx = o(1).
\]

**Case 3.** Let \( \alpha = r. \) In this case

\[ K_{nr}(x) = \frac{1}{2 \sin x} \sum_{k=1}^n (\Delta^\alpha a_{k-1} - \Delta^\alpha a_{k+1}) \tilde{S}_k^{\alpha-1}(x). \]

Applying Abel's transformation, we have
\[
K_{nr}(x) = \left[ \sum_{k=1}^n \left( \Delta^\alpha a_{k-1} - \Delta^\alpha a_{k+1} \right) \tilde{S}_k^{\alpha}(x) + \left( \Delta^\alpha a_n - \Delta^\alpha a_{n+2} \right) \tilde{S}_n^{\alpha}(x) \right].
\]

Since \( \tilde{S}_k^{\alpha}(x) \) are bounded for \( 0 < \delta \leq x \leq \pi, \)
\[
K_{nr}(x) \to K(x) = \frac{1}{2 \sin x} \sum_{k=1}^\infty (\Delta^\alpha a_{k-1} - \Delta^\alpha a_{k+1}) \tilde{S}_k^{\alpha}(x)
\]
pointwise for \( 0 < \delta \leq x \leq \pi. \) So
\[
\int_0^\pi |K(x) - K_{nr}(x)| \, dx
\]
\[
= \int_0^\pi \left| \frac{1}{2 \sin x} \sum_{k=n+1}^\infty \left( \Delta^\alpha a_{k-1} - \Delta^\alpha a_{k+1} \right) \tilde{S}_k^{\alpha}(x) - (\Delta^\alpha a_n - \Delta^\alpha a_{n+2}) \tilde{S}_n^{\alpha}(x) \right| \, dx
\]
\[
\leq C \sum_{k=n+1}^\infty \int_0^\pi \left| (\Delta^\alpha a_{k-1} - \Delta^\alpha a_{k+1}) \tilde{S}_k^{\alpha}(x) \right| \, dx
\]
\[
+ \int_0^\pi \left| (\Delta^\alpha a_n - \Delta^\alpha a_{n+2}) \tilde{S}_n^{\alpha}(x) \right| \, dx
\]
\[
= \sum_{k=n+1}^\infty A_k \int_0^\pi \left| (\Delta^\alpha a_{k-1} - \Delta^\alpha a_{k+1}) \tilde{S}_k^{\alpha}(x) \right| \, dx.
\]
\[ A_n^\alpha \left( \Delta^\alpha a_n - \Delta^\alpha a_{n+2} \right) \int_0^\pi |\tilde{T}_n^\alpha(x)| \, dx \]
\[ \leq C \sum_{k=n+1}^\infty A_k^\alpha \left( \Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1} \right) + C_1 \left| \Delta^\alpha a_n - \Delta^\alpha a_{n+2} \right| \]
\[ = o(1) + o(1) = o(1), \text{ by the hypotheses of Theorem 4.1 and Lemma 3.1.} \]

Thus the proof is complete.

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