COMPACTNESS AND WEAK COMPACTNESS OF ELEMENTARY OPERATORS ON $B(l^2)$ INDUCED BY COMPOSITION OPERATORS ON $l^2$

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Abstract. In this paper we have given simple proofs of some range inclusion results of elementary operators on $B(l^2)$ induced by composition operators on $l^2$. By using these results we have characterized compact and weakly compact elementary operators on $B(l^2)$ induced by composition operators on $l^2$.

1. Introduction

Definition 1.1. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be $n$-tuples of elements in an algebra $\mathcal{A}$. The elementary operator $E_{a,b}$ on $\mathcal{A}$ into itself associated with $a$ and $b$ is defined by $E_{a,b}(x) = a_1xb_1 + a_2xb_2 + \cdots + a_nxb_n$.

We denote by $M_{a,b}$ the elementary multiplication operator defined by $M_{a,b}(x) = axb$, $x \in \mathcal{A}$, $V_{a,b}(x) = axb - bxa$ for all $x \in \mathcal{A}$. For a fixed $a \in \mathcal{A}$, inner derivation $\delta_a$ is defined by $\delta_a(x) = ax - xa$. For fixed $a, b \in \mathcal{A}$, generalized derivation $\delta_{a,b}$ is defined by $\delta_{a,b}(x) = ax - xb$ for all $x \in \mathcal{A}$.

It is clear that $\delta_a$ and $\delta_{a,b}$ are elementary operators of length 2.

Definition 1.2. Let $X$ and $Y$ be normed linear spaces and $S$ be the closed unit ball in $X$. A linear operator $T : X \to Y$ is

(i) a finite rank operator if dimension of the range of $T$ is finite.

(ii) a compact operator if the closure of $T(S)$ is compact in $Y$.

(iii) a weakly compact operator if $T(S)$ is weakly compact in $Y$.

Definition 1.3. A Banach space $X$ is said to have the approximation property if for every compact subset $C$ of $X$ and for every $\epsilon > 0$ there exists a finite rank operator $T \in B(X)$ such that $\|Tx - x\| < \epsilon$ for each $x \in C$.\medskip

AMS Subject Classification: 47B33, 47B47.

Keywords and phrases: Compactness; composition operators; elementary operators; thin operators.

Research work is supported by CSIR(award no.9/13(951)/2000-EMR-1).

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Since every Banach space with a Schauder basis has the approximation property [1], a separable Hilbert space has approximation property.

**Definition 1.4.** Let $l^2$ be the Hilbert space of all square summable sequences of complex numbers under the standard inner product on it and $\phi$ be a function on $\mathbb{N}$ into itself. We denote by $\chi_n$, characteristic function of $\{n\}$. Let $A_n = \phi^{-1}(n)$ and let $\overline{A}_n$ denote the number of elements in $A_n$. The composition operator $C_\phi$ on $l^2$ is defined by $C_\phi(f) = f \circ \phi$ for all $f \in l^2$.

A necessary and sufficient condition that a function $\phi$ on $\mathbb{N}$ into itself induces a composition operator on $l^2$ is the set $\{A_n : n \in \mathbb{N}\}$ is bounded, see [12].

In the direction of compactness of elementary operators, first study was done by Vala [15] in 1964. He proved that “On $B(X)$ where $X$ is a Banach space the mapping $T \mapsto ATB$ is compact if and only if $A$ and $B$ are compact operators”. Vala defined an element $a$ of a normed algebra $A$ as compact if the mapping $x \mapsto axa$ is compact. By using this notion of compactness K.Ylinen [16] proved that compact elements of $C^*$-algebra $A$ form a closed two sided ideal which is the closure of the finite elements of $A$, i.e. those elements $a$, for which the map $x \mapsto axa$ is a finite rank operator. Akemann and Wright [3] obtained the necessary and sufficient condition for a $C^*$-algebra to admit a nonzero compact or weakly compact derivation. In 1977, Y.Ho [7] proved that derivation induced by non-scalars in $B(H)$ is non-compact. In 1979, Fong and Sourour [5] characterized the compactness of elementary operators on $B(H)$ where $H$ is a separable Hilbert space. Precisely they showed that “An elementary operator on $B(H)$ is compact if and only if it has a representation $E(X) = \sum_{i=1}^{n} A_iXB_i$, where each $A_i$ and each $B_i$ is compact”.

In the same paper they conjectured that there is no nonzero compact elementary operator on Calkin algebra, which was independently affirmed by Apostol and Fialkow [2], B. Magajna [9] and by M. Mathieu [8]. M. Mathieu generalized above results on $C^*$-algebra. Saksman and Tylli [13] studied compact and weakly compact elementary operators for a large class of Banach spaces. Now we state some known results which are useful in our context.

**Theorem 1.1.** [3, Theorem 3.1] Let $\delta$ be a derivation on $B(H)$. The following are equivalent:

(i) $\delta$ is weakly compact.

(ii) The range of $\delta$ is contained in $K(H)$.

(iii) $\delta = \delta_T$ with $T \in K(H)$.

**Theorem 1.2.** [8, Proposition 3.2] Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be $n$-tuples of elements in $B(H)$ and $E_{A,B}(X) = \sum_{i=1}^{n} A_iXB_i$. If the set $\{B_1, B_2, \ldots, B_n\}$ is linearly independent modulo $K(H)$, then the following are equivalent:

(a) $E_{A,B}$ is weakly compact.

(b) $A_i \in K(H)$ for all $1 \leq i \leq n$. 

Theorem 1.3. [8, Corollary 3.9] A non-zero elementary operator on a prime C*-algebra \( A \) is compact if and only if there are linearly independent subsets \( \{A_1, A_2, \ldots, A_n\} \) and \( \{B_1, B_2, \ldots, B_n\} \) in \( K(A) \) such that \( E(X) = \sum_{i=1}^{n} A_i X B_i \). Here \( K(A) \) is the ideal of all compact elements in \( A \).

Now we state a result due to E. Saksman.

Theorem 1.4. [11, Proposition 5] Let \( X \) be a reflexive Banach space with approximation property. Assume that \( A \) and \( B \) are \( n \)-tuples of operators on \( X \). Then the elementary operator \( E_{A,B} \) on \( B(X) \) is weakly compact if and only if \( E_{A,B}(X) \subseteq K(X) \).

Now we state some results about composition operators on \( l^2 \), which are useful in our context.

Theorem 1.5. [6] Let \( C_\phi \) and \( C_\psi \) be two composition operators on \( l^2 \). Then \( C_\phi - C_\psi \) is a finite rank operator if and only if \( \phi(n) = \psi(n) \) for all but finitely many \( n \in \mathbb{N} \).

Theorem 1.5. [6] The difference of two composition operators \( C_\phi - C_\psi \) is compact if and only if \( C_\phi - C_\psi \) is a finite rank operator.

2. Main Results

In this section we shall characterize compact and weakly compact elementary operators on \( B(l^2) \) induced by composition operators on \( l^2 \).

Theorem 2.1. Let \( C_\phi = (C_{\phi_1}, C_{\phi_2}, \ldots, C_{\phi_n}) \) and \( C_\psi = (C_{\psi_1}, C_{\psi_2}, \ldots, C_{\psi_n}) \) be \( n \)-tuples of composition operators on \( l^2 \). The elementary operator \( E_{C_\phi, C_\psi}(X) = \sum_{i=1}^{n} C_{\phi_i} X C_{\psi_i} \) is never weakly compact, hence never compact.

First we shall prove a lemma.

Lemma 2.1. Sum of a finite number of composition operators on \( l^2 \) is not compact.

Proof. Let \( C_{\phi_1}, C_{\phi_2}, \ldots, C_{\phi_n} \) be the composition operators on \( l^2 \) and let \( M = \{n_i : \phi_i^{-1}(n_i) \text{ is nonempty}\} \). Clearly \( M \) is an infinite subset of \( \mathbb{N} \) and \( \{\chi_{n_i}\}_{n_i \in M} \) is a weakly convergent sequence of orthonormal vectors in \( l^2 \). We have

\[
(C_{\phi_1} + C_{\phi_2} + \cdots + C_{\phi_k})(\chi_{n_i}) = \chi_{\phi_1^{-1}(n_i)} + \cdots + \chi_{\phi_k^{-1}(n_i)}.
\]

It follows that

\[
\| (C_{\phi_1} + \cdots + C_{\phi_k})(\chi_{n_i}) \|^2 = \| \chi_{\phi_1^{-1}(n_i)} + \cdots + \chi_{\phi_k^{-1}(n_i)} \|^2 \geq \frac{1}{\phi^{-1}(n_i)} \geq 1
\]

for \( n_i \in M \). Therefore \( \{(C_{\phi_1} + C_{\phi_2} + \cdots + C_{\phi_k})(\chi_{n_i})\}_{n_i \in M} \) does not converge strongly to zero in \( l^2 \). Hence \( (C_{\phi_1} + C_{\phi_2} + \cdots + C_{\phi_k}) \) is not compact. \( \blacksquare \)
Proof of Theorem 2.1. We have \( EC_{\phi}C_{\psi}(I) = C_{\phi_1}C_{\psi_1} + \cdots + C_{\phi_n}C_{\psi_n} \). Due to the fact that composition of two composition operators is a composition operator, by above lemma we get \( EC_{\phi}C_{\psi}(I) \notin K(l^2) \). Since \( l^2 \) has approximation property, \( EC_{\phi}C_{\psi} \) is not weakly compact by Theorem 1.4. Hence \( EC_{\phi}C_{\psi} \) is not compact. \(\blacksquare\)

Now we give simple proofs of some range inclusion results on elementary operators induced by composition operators on \( l^2 \). Here recall that an operator \( T \in B(H) \) of the form scalar plus compact is called thin.

**Theorem 2.2.** Let \( \delta_{C_{\phi}} \) be an inner derivation on \( B(l^2) \) defined by \( \delta_{C_{\phi}}(X) = C_{\phi}X - XC_{\phi} \). Then

(i) If \( C_{\phi} \) is a thin composition operator then \( R(\delta_{C_{\phi}}) \subseteq F(l^2) \).

(ii) Suppose \( C_{\phi} \) is not a thin composition operator on \( l^2 \) then \( R(\delta_{C_{\phi}}) \nsubseteq K(l^2) \).

**Proof.** (i) Let \( C_{\phi} \) be a thin composition operator on \( l^2 \). From Theorem 1.5 it follows that \( C_{\phi} = I + F_{\phi} \), where \( F_{\phi} \) is a finite rank operator on \( l^2 \). Now

\[
\delta_{C_{\phi}}(X) = C_{\phi}X - XC_{\phi} = (I + F_{\phi})X - X(I + F_{\phi})
\]

\[
= F_{\phi}X - XF_{\phi} \in F(l^2), \text{ for each } X \in B(l^2).
\]

Thus \( R(\delta_{C_{\phi}}) \subseteq F(l^2) \).

(ii) Suppose \( C_{\phi} \) is not a thin operator. Let \( M_w \) be a multiplication operator on \( l^2 \) defined by \( M_w(f) = \sum_{j=1}^{\infty} w_j f(j) \chi_j \) for each \( f \in l^2 \), where \( w \) is a weight function with \( w_j \{0, 1\} \), and we will define the sequence \( w_j \) later. We shall show that \( C_{\phi}M_w^* - M_w^*C_{\phi} \notin K(l^2) \).

Now \( (C_{\phi}M_w^* - M_w^*C_{\phi})^* = -(C_{\phi}^*M_w - M_wC_{\phi}^*) \). We have

\[
(C_{\phi}^*M_w - M_wC_{\phi}^*)(\chi_j) = C_{\phi}^*M_w(\chi_j) - M_wC_{\phi}^*(\chi_j) = C_{\phi}^*(w_j \chi_j) - M_w(\chi_{\phi(j)})
\]

\[
= w_j \chi_{\phi(j)} - w_{\phi(j)} \chi_{\phi(j)} = (w_j - w_{\phi(j)}) \chi_{\phi(j)}
\]

Since \( C_{\phi} \) is not thin, \( M = \{n \in N : \phi(j) \neq j\} \) is an infinite subset of \( N \) by Theorem (1.5).

For some \( n_1 \in M \), define \( w_{n_1} = 1 \) and \( w_{\phi(n_1)} = 0 \), suppose \( \phi(n_1) = m_1 \). Now there is \( n_2 \in M - (\{n_1\} \cup \phi^{-1}(m_1)) \). Define \( w_{n_2} = 1 \) and \( w_{\phi(n_2)} = 0 \), suppose \( \phi(n_2) = m_2 \). Similarly there is an \( n_3 \in M - (\{n_1, n_2\} \cup (\bigcup_{i=1}^{k-1} \phi^{-1}(n_i))) \).

Define \( w_{n_3} = 1 \) and \( w_{\phi(n_3)} = 0 \); suppose \( \phi(n_3) = m_3 \). In this way inductively we can get \( n_k \in M - (\{n_1, n_2, \ldots, n_k\} \cup (\bigcup_{i=1}^{k-1} \phi^{-1}(n_i))) \).

Define \( w_{n_k} = 1 \) and \( w_{\phi(n_k)} = 0 \); suppose \( \phi(n_k) = m_k \). Define \( w_j = 0 \) for \( j \in N - (\{m_1, m_2, \ldots, \} \cup (\bigcup_{i=1}^{k-1} \phi^{-1}(n_i))) \). Thus \( w_j - w_{\phi(j)} = 1 \) for infinitely many \( j \in N \). Let \( M_1 = \{j \in M : w_j - w_{\phi(j)} = 1\} \). Clearly \( M_1 \) is an infinite subset of \( N \). Now we have \( \| (C_{\phi}M_w - M_wC_{\phi}^*)(\chi_j) \| \geq 1 \) for all \( j \in M_1 \). It follows that \( C_{\phi}M_w - M_wC_{\phi} \) is not compact and so \( C_{\phi}M_w^* - M_w^*C_{\phi} \) is not compact. Hence \( R(\delta_{C_{\phi}}) \nsubseteq K(l^2) \). \(\blacksquare\)

**Corollary 2.1.** \( R(\delta_{C_{\phi}}) \subseteq K(l^2) \) if and only if \( R(\delta_{C_{\phi}}) \subseteq F(l^2) \) if and only if \( C_{\phi} \) is thin.
THEOREM 2.3. Let $C_\phi$ and $C_\psi$ be two composition operators on $l^2$ and $\delta_{C_\phi,C_\psi}$ be the generalized derivation on $B(l^2)$ defined by $\delta_{C_\phi,C_\psi} = C_\phi X - XC_\psi$. Then $R(\delta_{C_\phi,C_\psi}) \subseteq F(l^2)$ if and only if $C_\phi$ and $C_\psi$ are thin operators.

Proof. Let $C_\phi$ and $C_\psi$ be two thin composition operators on $l^2$. Then $C_\phi = I + F_\phi$ and $C_\psi = I + F_\psi$ for some finite rank operator $F_\phi$ and $F_\psi$. We get $\delta_{C_\phi,C_\psi} = C_\phi X - XC_\psi \in F(l^2)$, for all $X \in B(l^2)$. Thus $R(\delta_{C_\phi,C_\psi}) \subseteq F(l^2)$.

Conversely, suppose $R(\delta_{C_\phi,C_\psi}) \subseteq F(l^2)$ i.e. $C_\phi X - XC_\psi \in F(l^2)$ for all $X \in B(l^2)$. In particular $\delta_{C_\phi,C_\psi}(I) = C_\phi - C_\psi \in F(l^2)$ i.e. $C_\phi - C_\psi = F, F \in F(l^2)$. It follows that $\delta_{C_\phi}(X) \in F(l^2)$ for all $X \in B(l^2)$ which implies that $C_\phi$ is thin by Corollary 2.1. Therefore $C_\psi = C_\phi - F$ is also thin. Thus both $C_\phi$ and $C_\psi$ are thin operators on $l^2$.

By Corollary 2.1 and the above Theorem, we have the following corollary.

COROLLARY 2.2. $R(\delta_{C_\phi,C_\psi}) \subseteq K(l^2)$ if and only if $C_\phi$ and $C_\psi$ are thin.

EXAMPLE 2.1. Let $A = 2I + K$ and $B = I + K$, $K \in K(l^2)$ be two thin operators. $\delta_{A,B}(I) = (2I + K)I - (I + K) = I \notin K(l^2)$.

This shows that Theorem 2.3 may not be true for general thin operators.

THEOREM 2.4. Let $C_\phi$ and $C_\psi$ be two composition operators on $l^2$ and $V_{C_\phi,C_\psi}$ be an elementary operator on $B(l^2)$ defined by $V_{C_\phi,C_\psi}(X) = C_\phi XC_\psi - C_\psi XC_\phi$. Then $R(V_{C_\phi,C_\psi}) \subseteq F(l^2)$ if and only if $C_\phi - C_\psi$ is a finite rank operator.

Proof. We have $V_{C_\phi,C_\psi}(X) = C_\phi XC_\psi - C_\psi XC_\phi$. Suppose $C_\phi - C_\psi = F$, where $F$ is a finite rank operator on $l^2$. Then $V_{C_\phi,C_\psi}(X) = FXC_\psi - C_\phi XF \in F(l^2)$ for all $X \in B(l^2)$. Thus $R(V_{C_\phi,C_\psi}) \subseteq F(l^2)$.

Conversely, suppose $C_\phi - C_\psi$ is not a finite rank operator, i.e. $\phi(n) \neq \psi(n)$ for infinitely many $n \in \mathbb{N}$, by Theorem 1.5. Let $M_w$ be a multiplication operator on $l^2$ defined by $M_w(f) = \sum_{j=1}^{\infty} w_j f(j) \chi_j$, where $w$ is a weight function with $w_j \{0,1\}$, and we will define the sequence $w_j$ later. We shall show that $C_\phi^* M_w C_\psi - C_\psi^* M_w C_\phi \notin K(l^2)$.

\[
(C_\phi^* M_w C_\psi - C_\psi^* M_w C_\phi)(\chi_k) = (C_\phi^* M_w C_\psi)(\chi_k) - (C_\psi^* M_w C_\phi)(\chi_k) \\
= C_\phi^* M_w(\chi_{\phi(k)}) - C_\psi^* M_w(\chi_{\psi(k)}) = C_\phi^* (w_{\phi(k)} \chi_{\phi(k)}) - C_\psi^* (w_{\psi(k)} \chi_{\psi(k)}) \\
= w_{\psi(k)} \chi_{(\phi \circ \psi)(k)} - w_{\phi(k)} \chi_{(\psi \circ \phi)(k)}.
\]

Now

\[
\|(C_\phi^* M_w C_\psi - C_\psi^* M_w C_\phi)(\chi_k)\|^2 \\
= |w_{\psi(k)}|^2 + |w_{\phi(k)}|^2 - (w_{\psi(k)} w_{\phi(k)} + w_{\phi(k)} w_{\phi(k)}) \langle \chi_{(\phi \circ \psi)(k)}, \chi_{(\psi \circ \phi)(k)} \rangle.
\]

If $\phi \circ \psi(k) \neq \psi \circ \phi(k)$, then

\[
\|(C_\phi^* M_w C_\psi - C_\psi^* M_w C_\phi)(\chi_k)\|^2 = |w_{\psi(k)}|^2 + |w_{\phi(k)}|^2.
\]
If \( \phi \circ \psi(k) = \psi \circ \phi(k) \), then
\[
\|(C^*_\phi M_w C^*_\psi - C^*_\psi M_w C^*_\phi)(\chi_k)\| = |w(\phi(k)) - w(\psi(k))|^2.
\]
(2)

Now \( M = \{ n \in \mathbb{N} : \phi(n) \neq \psi(n) \} \) is an infinite subset of \( \mathbb{N} \). For some \( n_1 \in M \), define \( w_{\phi(n_1)} = 1 \) and \( w_{\psi(n_1)} = 0 \), suppose \( \phi(n_1) = l_1 \) and \( \psi(n_1) = m_1 \). Now there is some \( n_2 \in M - (\phi^{-1}(l_1) \cup \psi^{-1}(l_1) \cup \psi^{-1}(m_1)) \). Define \( w_{\phi(n_2)} = 1 \) and \( w_{\psi(n_2)} = 0 \), suppose \( \phi(n_2) = l_2 \) and \( \psi(n_2) = m_2 \). Now there is some
\[
n_3 \in M - (\bigcup_{i=1}^{k-1} \phi^{-1}(l_i)) \cup (\bigcup_{i=1}^{k-1} \psi^{-1}(m_i)) \cup (\bigcup_{i=1}^{k-1} \psi^{-1}(l_i)) \cup (\bigcup_{i=1}^{k-1} \psi^{-1}(m_i)).
\]

Define \( w_{\phi(n_3)} = 1 \) and \( w_{\psi(n_3)} = 0 \), suppose \( \phi(n_3) = l_3 \) and \( \psi(n_3) = m_3 \).

In this way inductively we can find
\[
n_k \in M - (\bigcup_{i=1}^{k-1} \phi^{-1}(l_i)) \cup (\bigcup_{i=1}^{k-1} \psi^{-1}(m_i)) \cup (\bigcup_{i=1}^{k-1} \psi^{-1}(l_i)) \cup (\bigcup_{i=1}^{k-1} \psi^{-1}(m_i)).
\]

Define \( w_n = 0 \) for \( n \in \mathbb{N} - (\{ l_i : i \in \mathbb{N} \}) \cup \{ m_i : i \in \mathbb{N} \} \). Clearly \( w_{\phi(n)} - w_{\psi(n)} = 1 \) for infinitely many \( n \in \mathbb{N} \), and so \( M_1 = \{ n \in M : w_{\phi(n)} - w_{\psi(n)} = 1 \} \) is an infinite subset of \( M \).

Now for \( n \in M_1 \), by equations (1) and (2), we have
\[
\|(C^*_\phi M_w C^*_\psi - C^*_\psi M_w C^*_\phi)(\chi_n)\|^2 \geq 1,
\]
which implies that \( C^*_\phi M_w C^*_\psi - C^*_\psi M_w C^*_\phi \) and so \( C^*_\phi M_w C^*_\psi - C^*_\psi M_w C^*_\phi \) is not compact on \( l^2 \).

Thus \( R(V_{C^*_\phi, C^*_\psi}) \not\subseteq F(l^2) \). Hence the proof. ■

As a consequence of the proof of Theorem 2.4, we have the following corollary.

**Corollary 2.3.** \( R(V_{C^*_\phi, C^*_\psi}) \subseteq K(l^2) \) if and only if \( C_{\phi} - C_{\psi} \) is compact.

In view of Theorem 1.4 and Corollaries 2.1, 2.2 and 2.3 we have the following characterization of weakly compact elementary operators on \( l^2 \).

**Theorem 2.5.** Let \( C_{\phi} \) and \( C_{\psi} \) be two composition operators on \( l^2 \). Then
(i) \( C_{\phi} \) is weakly compact if and only if \( C_{\phi} \) is a thin operator on \( l^2 \).
(ii) \( \delta_{C_{\phi}, C_{\psi}} \) is weakly compact if and only if \( C_{\phi} \) and \( C_{\psi} \) are thin operators on \( l^2 \).
(iii) \( V_{C_{\phi}, C_{\psi}} \) is weakly compact if and only if \( C_{\phi} - C_{\psi} \) is a compact operator on \( l^2 \).

**Acknowledgements.** 1. The author is grateful to Prof. Nand Lal for his helpful suggestions and discussions.

2. The author is grateful to the referee for his helpful suggestions.

**References**

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(received 02.07.2008, in revised form 14.04.2009)

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