ON VARIATION TOPOLOGY

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Abstract. Let \( I \) be a real interval and \( X \) be a Banach space. It is observed that spaces \( \Lambda BV_p([a, b], R) \), \( LBV(I, X) \) (locally bounded variation), \( BV_0(I, X) \) and \( LBV_0(I, X) \) share many properties of the space \( BV([a, b], R) \). Here we have proved that the space \( \Lambda BV_0^{(p)}(I, X) \) is a Banach space with respect to the variation norm and the variation topology makes \( \Lambda BV_0^{(p)}(I, X) \) a complete metrizable locally convex vector space (i.e. a Fréchet space).

Introduction. Looking to the features of the Jordan class the space \( BV([a, b], R) \) of real functions of bounded variation over \([a, b]\) is generalized in many ways and many generalized spaces are obtained [1–5]. Many mathematicians have studied different properties for these generalized classes. Recently we have proved that the class \( \Lambda BV^{(p)}([a, b], R) \) is a Banach space [5]. Also, the concept of bounded variation is extended, from real valued, to function with values in \( R^n \). Many properties of such functions hold for functions in an arbitrary Banach space \( X \). In the present paper we have studied properties of the classes \( \Lambda BV_0^{(p)}(I, X) \) and \( \Lambda BV_0^{(p)}(I, X) \).

Definition. Given a real interval \( I \) (neither empty nor reduced to a singleton), a Banach space \( X \), a non-decreasing sequence of positive real numbers \( \Lambda = \{\lambda_n\} \) \((n = 1, 2, \ldots)\) such that \( \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \) diverges, \( 1 \leq p < \infty \) and a function \( f : I \to X \), we say that \( f \in \Lambda BV^{(p)}(I, X) \) (that is \( f \) is a function of \( p - \Lambda \)-bounded variation over \( I \)) if

\[
V^{(p)}_{\Lambda_p}(f, I) = \sup_S V^{(p)}_{\Lambda_p}(f, S, I) < \infty,
\]

where \( V^{(p)}_{\Lambda_p}(f, S, I) = \left( \sum_{i=1}^{n} \frac{\|f(u_i) - f(u_{i-1})\|_X}{\lambda_{u_i}} \right)^{1/p} \), \( S : u_0 < u_1 < \cdots < u_n \) is a finite ordered set of points of \( I \) and \( \| \cdot \|_X \) denotes the Banach norm in \( X \).

Note that, if \( p = 1 \), one gets the class \( ABV(I, X) \) and the variation \( V^{(p)}_{\Lambda} \) is replaced by \( V_{\Lambda} \); if \( \lambda_m \equiv 1 \) for all \( m \), one gets the class \( BV^{(p)}(I, X) \) and the

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variation $V_{\lambda_p}$ is replaced by $V_p$; if $p = 1$ and $\lambda_m \equiv 1$ for all $m$, one gets the class $BV(I, X)$ and the variation $V_{\lambda_p}$ is replaced by $V$.

$f \in L\Lambda BV^{(p)}(I, X)$ means that $f$ is a function of $I$ to $X$ with locally $p$-$\Lambda$ bounded variation, i.e. it has finite $p$-$\Lambda$ bounded variation on every compact subinterval of $I$. It is observed that the class $LBV(I, X)$ is a vector space and for any $f \in LBV(I, X)$, for any compact subinterval $[a, b]$ of $I$, the mapping $f \mapsto V(f, [a, b])$ is a semi-norm in this space. If $[a, b]$ ranges through the totality of the compact subintervals of $I$, or equivalently through some increasing sequence of such subintervals with union equal to $I$, the collection of the corresponding semi-norms defines on $LBV(I, X)$ a (non Hausdorff) locally convex topology which is called the variation topology.

We shall choose once for all a reference point $t$ in $I$ and consider the space $L\Lambda BV_0^{(p)}(I, X)$ consisting of all those functions in $L\Lambda BV^{(p)}(I, X)$ which are vanishing at the point $t$. Moreau [1] proved that the space $BV_0(I, X)$ is a Banach space in the norm $\|f\|_{var} = V(f, I)$ and the variation topology makes $LBV_0(I, X)$ a Fréchet space. Here we have extended these two results for $\Lambda BV^{(p)}$.

In the first stage, let us consider class of functions whose total $p$-$\Lambda$ variation is finite.

**Theorem 1.** The vector space $\Lambda BV_0^{(p)}(I, X)$ is a Banach space in the norm $\|f\|_{var} = V_{\lambda_p}(f, I)$.

Note that the above mentioned $\|\cdot\|_{var}$ is a semi-norm on the space $\Lambda BV^{(p)}(I, X)$. For $\lambda_n = 1$ for all $n$ and $p = 1$ Theorem 1 gives Moreau’s result [1, Proposition 2.1] as a particular case.

We need the following lemma to prove the theorem.

**Lemma.** If $f \in \Lambda BV_0^{(p)}(I, X)$ then $f$ is bounded.

**Proof.** For any $u \in I$, observe that

$$\|f(u)\|_X = \lambda_1 \left( \frac{\|f(u) - f(t)\|_X}{\lambda_1} \right) \leq (\lambda_1)^{(1/p)} V_{\lambda_p}(f, I).$$

Hence

$$\|f\|_\infty = \sup_{u \in I} \|f(u)\|_X \leq (\lambda_1)^{(1/p)} V_{\lambda_p}(f, I).$$

Similarly, for any $f \in L\Lambda BV_0^{(p)}(I, X)$ and for any $[a, b] \subset I$ containing the point $t$, we get

$$\sup_{u \in [a, b]} \|f(u)\|_X \leq (\lambda_1)^{(1/p)} V_{\lambda_p}(f, [a, b]).$$

Therefore, in the space $L\Lambda BV_0^{(p)}(I, X)$ the variation topology is stronger than the topology of uniform convergence on compact subsets of $I$. \[ \black nabla \]

**Proof of Theorem 1.** Consider a Cauchy sequence $\{f_n\}$ in the given normed linear space. Then there exists a constant $C > 0$ such that

$$\|f_n\| \leq C, \quad \forall n \in N.$$

(1.1)
In view of the Lemma, \(\{f_n\}\) is also Cauchy sequence in the sup norm \(\|\cdot\|_\infty\) so it converges in the latter norm to some function \(f_\infty : I \to X\), with \(f_\infty(t) = 0\).

For every finite ordered set of points of \(I\), say \(u_0 < u_1 < \cdots < u_m\), and every \(f : I \to X\), let us denote

\[
V_{\Lambda_p}(f, S) = \left( \sum_{i=1}^{m} \frac{\|f(u_i) - f(u_{i-1})\|_X^p \lambda_i}{\lambda_i} \right)^{1/p}.
\]

Since, at every point \(u_i\) of \(S\), the element \(f_\infty(u_i)\) of \(X\) equals the limit of \(f_n(u_i)\) in the \(\|\cdot\|_X\) norm, one has

\[
V_{\Lambda_p}(f_\infty, S) = \lim_{n \to \infty} \left( \sum_{i=1}^{m} \frac{\|f(u_i) - f(u_{i-1})\|_X^p \lambda_i}{\lambda_i} \right)^{1/p}.
\]

Due to (1.1), this is majorized by \(C\) whatever is \(S\), hence \(f_\infty \in \Lambda BV_{0}^p(I, X)\).

Now, let us prove that \(f_n\) converges to \(f_\infty\) in the norm \(\|\cdot\|_{\text{var}}\). In view of Cauchy property, for any \(\epsilon > 0\) there exists \(n \in N\) such that

\[l \geq n \Rightarrow \|f_l - f_n\|_{\text{var}} \leq \epsilon.\]

Hence, for every \(l \geq n\),

\[V_{\Lambda_p}(f_l - f_n, S) \leq V_{\Lambda_p}(f_l - f_n, I) \leq \epsilon.\]

Thus

\[V_{\Lambda_p}(f_\infty - f_n, S) \leq V_{\Lambda_p}(f_\infty - f_l, S) + V_{\Lambda_p}(f_l - f_n, S) \leq \epsilon + V_{\Lambda_p}(f_l - f_\infty, S).\]

By letting \(l\) tending to \(+\infty\), one concludes that \(V(f_n - f_\infty, S) \leq \epsilon\) for every finite sequence \(S\), hence \(\|f_n - f_\infty\|_{\text{var}} \leq \epsilon\) for every finite sequence \(S\). Hence the result follows.

Let us drop the assumption of finite total variation on \(I\). The variation topology on \(\Lambda BV_{0}^p(I, X)\) is defined by the collection of norms \(f \mapsto N_k(f) = \|f\|_{\text{var}, K_k} = V_{\Lambda_p}(f, K_k)\), where \(\{K_k\}\) denotes a nondecreasing sequence of compact subintervals whose union equals \(I\). Additionally assume that all intervals \(K_k\) are large enough to contain \(t\). Therefore the resulting topology is metrizable and Hausdorff.

**Theorem 2.** The variation topology makes \(\Lambda BV_{0}^p(I, X)\) a complete metrizable locally convex vector space (i.e. Fréchet space).

Note that for \(\lambda_n = 1\) for all \(n\) and \(p = 1\) Theorem 2 gives Moreau’s result [1, Proposition 2.2] as a particular case.

**Proof.** Let \(\{f_n\}\) be a Cauchy sequence in \(\Lambda BV_{0}^p(I, X)\). By definition, for every neighborhood \(U\) of the origin in this space there exists \(n \in N\) such that

\[l \geq n \text{ and } q \geq n \Rightarrow f_l - f_q \in U.\]
For \( k \in N \) and for any \( \epsilon > 0 \) define the semi-ball,
\[
U_{k, \epsilon} = \{ u \in \Lambda BV_0^{(p)}(I, X) : N_k(u) < \epsilon \}.
\]
Thus
\[
l \geq n \text{ and } q \geq n \Rightarrow N_k(f_l - f_q) < \epsilon.
\]
Therefore the restriction of the functions \( \{ f_n \} \) to \( K_k \) make a Cauchy sequence in \( \Lambda BV_0^{(p)}(K_k, X) \). In view of Theorem 1, this sequence converges to some element \( f^k \) in the latter space. If the same construction is effected for another compact subinterval \( K_{k'} \), with \( k' > k \), the resulting function \( f^{k'} : K_{k'} \rightarrow X \) is an extension of \( f^k \). Inductively, a function \( f \) is constructed on the whole \( I \), which constitutes the limit of the sequence \( \{ f_n \} \) in the variation topology. Hence the result follows. ■

REFERENCES


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