ON s-CLOSEDNESS AND S-CLOSEDNESS IN TOPOLOGICAL SPACES

Zbigniew Duszyński

Abstract. Some properties of sets s-closed or S-closed relative to a space, and s-closed or S-closed subspaces, are obtained. Relationships between some of them are indicated. New characterizations of Hausdorff spaces in terms of s-closedness and α-compactness relative to a space, are obtained.

1. Preliminaries

Throughout the paper (X, τ) (or (Y, σ)) denotes a topological space. For a subset S of (X, τ), int(S) (or int_X(S)), cl(S) (or cl_X(S), or cl_r(X)) stand for the interior of S and the closure of S, respectively. If X_0 ⊂ X, then (X_0, τ_{X_0}) denotes a subspace of (X, τ), and int_{X_0}(.), cl_{X_0}(.) are interior and closure operators (respectively) in (X_0, τ_{X_0}). CO(X, τ) is the intersection of τ and \{X \setminus S : S ∈ τ\}. A subset S of (X, τ) is said to be regular open (resp. regular closed) if S = int(cl(int(S))) (resp. S = cl(int(S))). A set S is said to be α-open [28] (resp. semi-open [22], semi-closed [8], preopen [25], semi-preopen (or β-open) [2,1]) in (X, τ), if S ⊂ int(cl(int(S))) (resp. S ⊂ cl(int(S)), S ⊂ int(cl(S)), S ⊂ cl(int(cl(S)))). A subset S of (X, τ) is semi-open if and only if there exists a U ∈ τ such that U ⊂ S ⊂ cl(U) [22]. The collection of all regular open (resp. regular closed, α-open, semi-open, semi-closed, preopen, semi-preopen) subsets of (X, τ) is denoted by RO(X, τ) (resp. RC(X, τ), τα, SO(X, τ), SC(X, τ), PO(X, τ), SPO(X, τ)). The family τα forms a topology on X such that α τ. An S is said to be semi-regular [10] (see also [5] and [41]) if it is both semi-closed and semi-open in (X, τ). We denote SO(X, τ) ∩ SC(X, τ) = SR(X, τ). We have in each (X, τ), RO(X, τ) ∪ RC(X, τ) ⊂ SR(X, τ) [41, Lemma 2.3], and RO(X, τ) ∩ RC(X, τ) = CO(X, τ) (see for instance [11, ]). The semi-closure [8] (resp. the semi-interior [8]) of an S ⊂ X is the intersection of all semi-closed subsets of (X, τ) containing S (resp. the union of all semi-open subsets of (X, τ) contained in S), and is denoted

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respectively by scl (S) (or scl\(_X\)(S)) and sint\(_X\)(S). The union of any family of semi-open subsets of (X, \(\tau\)) is semi-open as well [22].

A space (X, \(\tau\)) is said to be extremally disconnected (briefly e.d.) if cl(S) \(\in\) \(\tau\) for any S \(\in\) \(\tau\).

A subset A of a space (X, \(\tau\)) is said to be s-closed [10] (resp. S-closed [32], \(\mathcal{N}\)-closed [7], quasi-\(\mathcal{H}\)-closed [38]) relative to (X, \(\tau\)), if every cover \(\{V_\alpha\}_{\alpha\in\mathcal{V}} \subset\) SO(X, \(\tau\)) (resp. \(\{V_\alpha\}_{\alpha\in\mathcal{V}} \subset\) SO(A, \(\tau\))) of A admits a finite subfamily \(\mathcal{V}_0\) \(\subset\) \(\mathcal{V}\) such that \(A \subset\bigcup_{\alpha\in\mathcal{V}_0}\) scl \((V_\alpha)\) (resp. \(A \subset\bigcup_{\alpha\in\mathcal{V}_0}\) cl \((V_\alpha)\)). In the case \(A = X\), (X, \(\tau\)) is said to be s-closed [10] (resp. S-closed [42]). (\(X_0, \tau_{X_0}\)) is called an s-closed (resp. S-closed) subspace of (X, \(\tau\)) if it is s-closed (resp. S-closed) as a space.

The following results are useful in the sequel:
1. Let S \(\subset\) A \(\in\) SO(X, \(\tau\)). Then S \(\in\) SO(X, \(\tau\)) if and only if S \(\in\) SO(A, \(\tau_A\)) [29, Theorem 5].
2. In any space (X, \(\tau\)),
   
   \[
   \text{scl}(S) = S \cup \text{int}(\text{cl}(S)) \quad [2, \text{Theorem 1.5(a)}],
   \]
   
   \[
   \text{cl}_{\tau^a}(S) = S \cup \text{cl}((\text{int}(\text{cl}(S)))) \quad [2, \text{Theorem 1.5(c)}]
   \]

3. In any space (X, \(\tau\)), cl\(_{\tau}\)(V) = cl\(_{\tau}\)(V) for each V \(\in\) SO(X, \(\tau\)) [17, Lemma 1(i)].
4. In any e.d. space (X, \(\tau\)), \(\tau^a = SO(X, \tau)\) [19, Theorem 2.9].

2. s-closedness

In [4] the following two results have been stated.

**Theorem 1.** [4, Theorem 1] Let A \(\in\) PO(X, \(\tau\)). Then (A, \(\tau_A\)) is s-closed if and only if A is s-closed relative to (X, \(\tau\)).

**Theorem 2.** [4, Theorem 2] Let A \(\subset\) B \(\subset\) X, where B \(\in\) PO(X, \(\tau\)). Then, the set A is s-closed relative to (B, \(\tau_B\)) if and only if it is s-closed relative to (X, \(\tau\)).

Proofs for these theorems are based on [12, Theorem 2.7], which states that SR(A, \(\tau_A\)) = SR(X, \(\tau\)) \(\cap\) A (i.e., SR(A, \(\tau_A\)) = \{S \cap A : S \in SR(X, \tau)\}) for any space (X, \(\tau\)) and any A \(\in\) PO(X, \(\tau\)). Unfortunately, the proof for SR(A, \(\tau_A\)) \(\subset\) SR(X, \(\tau\)) \(\cap\) A given in [12] is far from clear (it is worth to see [20, Lemma 3]). We shall give a proof for [12, Theorem 2.7]. It will make use of the subsequent lemmas.

**Lemma 1.** [37, Teorema 3.2] Let \(X_0\) be an arbitrary subset of a space (X, \(\tau\)). If A \(\in\) SO(\(X_0, \tau_{X_0}\)), then A = \(X_0 \cap B\) for some B \(\in\) SO(X, \(\tau\)).

**Lemma 2.** Let (X, \(\tau\)) be a space and \(X_0\) \(\in\) PO(X, \(\tau\)).
(a) [34, Lemma 2.2] One has B \(\cap\) \(X_0\) \(\in\) SO(\(X_0, \tau_{X_0}\)) for every B \(\in\) SO(X, \(\tau\)).
(b) [34, Lemma 2.3] One has B \(\cap\) \(X_0\) \(\in\) SC(\(X_0, \tau_{X_0}\)) for every B \(\in\) SC(X, \(\tau\)).
Corollary 1. If $A \in \text{PO}(X, \tau)$ then $\text{SR}(X, \tau) \cap A \subset \text{SR}(A, \tau_A)$.

Lemma 3. [34, Theorem 2.4]. If $A \subset X_0 \in \text{PO}(X, \tau)$ then $X_0 \cap \text{scl}_X(A) = \text{scl}_{X_0}(A)$.

Lemma 4. [33, Lemma 3.5] If either $A \in \text{SO}(X, \tau)$ or $B \in \text{SO}(X, \tau)$ then
\[ \text{int}(\text{cl}(A \cap B)) = \text{int}(\text{cl}(A)) \cap \text{int}(\text{cl}(B)). \]

Lemma 5. Let $(X, \tau)$ be any space. The following statements are equivalent:

(a) $S \in \text{SR}(X, \tau)$.

(b) [10, Proposition 2.1(c)] There exists a set $U \in \text{RO}(X, \tau)$ such that $U \subset S \subset \text{cl}_X(U)$.

(c) [41, Lemma 2.2(iii)] $S = \text{scl}_X(\text{sint}_X(S))$.

Lemma 6. (compare [10, Proposition 2.2]) If $S \in \text{SPO}(X, \tau)$ then $\text{scl}(S) \in \text{SR}(X, \tau)$.

Proof. By the use of [2, Theorem 1.5(a)] we obtain
\[ \text{int}(\text{cl}(S)) \subset \text{scl}(S) = S \cup \text{int}(\text{cl}(S)) \subset \text{cl}(\text{int}(\text{cl}(S))). \]

Thus, by Lemma 5(b), $\text{scl}(S) \in \text{SR}(X, \tau)$. $\blacksquare$

Theorem 3. [12, Theorem 2.7] For any space $(X, \tau)$, if $X_0 \in \text{PO}(X, \tau)$ then
\[ \text{SR}(X_0, \tau_{X_0}) = \text{SR}(X, \tau) \cap X_0. \]

Proof. In view of Corollary 1 only the inclusion $\text{SR}(X_0, \tau_{X_0}) \subset \text{SR}(X, \tau) \cap X_0$ requires a proof. Let $S \in \text{SR}(X_0, \tau_{X_0})$ be arbitrarily chosen. By Lemmas 5(c) and 3 we have $\text{scl}_{X_0}(\text{sint}_{X_0}(S)) = X_0 \cap \text{scl}_X(\text{sint}_{X_0}(S))$.

Obviously $\text{sint}_{X_0}(S) \in \text{SO}(X_0, \tau_{X_0})$, so by Lemma 1, $\text{sint}_{X_0}(S) = X_0 \cap B$ for some set $B \in \text{SO}(X, \tau)$. We are to show that $X_0 \cap B \in \text{SPO}(X, \tau)$. Indeed, by Lemma 4 we have the following inclusions:
\[ X_0 \cap B \subset \text{int}(\text{cl}(X_0)) \cap \text{cl}(\text{int}(B)) \subset \text{cl}(\text{int}(\text{cl}(X_0)) \cap \text{int}(\text{cl}(B))) = \text{cl}(\text{int}(X_0 \cap B)). \]

Finally, $\text{scl}_X(X_0 \cap B) \in \text{SR}(X, \tau)$, by Lemma 6, and the proof is complete. $\blacksquare$

Remark 1. Theorems 1 and 2 may be proved independently of Theorem 3 by using Lemmas 1, 2(a), 3, and Lemma 7 below. Details are omitted (it is worth to see for instance [32, Theorems 3.1 and 3.2] and left to the reader.

Lemma 7. Let $B \in \text{PO}(X, \tau)$ and $V \in \text{SO}(X, \tau)$. Then $B \cap \text{scl}(V) \subset \text{scl}(B \cap V)$.

Proof. By [2, Theorem 1.5(a)] and Lemma 4 we have $B \cap \text{scl}(V) = B \cap (V \cup \text{int}(\text{cl}(V))) = (B \cap V) \cup (B \cap \text{int}(\text{cl}(V))) \subset (B \cap V) \cup (\text{int}(B) \cap \text{int}(\text{cl}(V))) = \text{scl}(B \cap V)$. $\blacksquare$
Lemma 6 completes the proof.

\[ \text{regular cover of } S \cup s \]

Statements are equivalent:

\[ \text{PO (} S \text{)} \]

\[ \text{PO (} s \text{)} \]

\[ \nabla \]

\[ \text{PO (} s \text{)} \]

\[ \text{indices} \]

\[ \text{is so since } \text{SO (} X, \tau^a \text{)} = \text{SO (} X, \tau \text{)} \]

\[ \text{[28, Proposition 3], } \text{PO (} X, \tau^a \text{)} = \text{PO (} X, \tau \text{)} \]

\[ \text{[20, Corollary 2.5(a)], } \text{cl}_r(V) = \text{cl}_r(V) \]

\[ \text{[17, Lemma 1(i)], and } \text{cl}_r(B \cap V) \subseteq \text{cl}_r(B \cap V) \]

(to prove this one use Lemma 4 and [2, Theorem 1.5(c)]).

We omit details in the proofs of the next three corollaries.

**Corollary 2.** Let \( A \subset X_0 \subset X_1 \subset X \) and \( X_0, X_1 \in \text{PO (} X, \tau \text{)} \). Then \( A \) is s-closed relative to \( (X_0, \tau_{X_0}) \) if and only if \( A \) is s-closed relative to \( (X_1, \tau_{X_1}) \).

*Proof.* Theorem 2. ■

**Corollary 3.** Let \( A \in \text{PO (} X_0, \tau_{X_0} \text{)} \) and \( X_0 \in \text{PO (} X, \tau \text{)} \). Then \( A \) is an s-closed subspace of \( (X_0, \tau_{X_0}) \) if and only if \( A \) is an s-closed subspace of \( (X, \tau) \).

*Proof.* This follows from Theorems 1–2 and [26, Lemma 2.2]: if \( A \in \text{PO (} X_0, \tau_{X_0} \text{)} \) and \( X_0 \in \text{PO (} X, \tau \text{)} \) then \( A \in \text{PO (} X, \tau \text{)} \). ■

Corollary 3 improves [4, Corollary 1].

**Corollary 4.** Let \( A \in \text{PO (} X_0, \tau_{X_0} \text{)}, X_0 \in \text{PO (} X_1, \tau_{X_1} \text{)}, \) and \( X_1 \in \text{PO (} X, \tau \text{)} \). Then \( A \) is an s-closed subspace of \( (X_0, \tau_{X_0}) \) if and only if it is an s-closed subspace of \( (X_1, \tau_{X_1}) \).

*Proof.* By Corollary 2 and [26, Lemma 2.2]. ■

**Definition 1.** A subset \( S \) of a space \( (X, \tau) \) is said to be sspo-closed relative to \( (X, \tau) \) if, for every cover \( \{V_\alpha : \alpha \in \nabla\} \subset \text{SPO (} X, \tau \text{)} \) of \( S \) there is a finite set of indices \( \nabla_0 \subset \nabla \) such that \( S \subset \bigcup_{\alpha \in \nabla_0} \text{scl}_X(V_\alpha) \). If \( S = X \), then \( (X, \tau) \) is called an sspo-closed space.

**Theorem 4.** In any space \( (X, \tau) \) and for any subset \( S \) of it, the following statements are equivalent:

(a) \( S \) is sspo-closed relative to \( (X, \tau) \),

(b) \( S \) is s-closed relative to \( (X, \tau) \).

*Proof.* (a)⇒(b). Obvious, since \( \text{SO (} X, \tau \text{)} \subset \text{SPO (} X, \tau \text{)} \).

(a)⇐(b). Let \( \{V_\alpha : \alpha \in \nabla\} \subset \text{SPO (} X, \tau \text{)} \) cover a set \( S \). Then, \( S \subset \bigcup_{\alpha \in \nabla} \text{scl}_X(V_\alpha) \). Since \( S \) is s-closed relative to \( (X, \tau) \) if and only if each semi-regular cover of \( S \) admits a finite subcover [10, Proposition 4.1], application of Lemma 6 completes the proof. ■

**Lemma 8.** Let \( A \) be an arbitrary subset of a space \( (X, \tau) \). If \( U \in \text{SPO (} A, \tau_A \text{)} \) then

\[ \text{int}_X(A) \cap U \subset \text{cl}_X(\text{int}_X(\text{cl}_X(U))) \].
By hypothesis there are finite subfamilies \( s \subset W \) using Lemma 3 we get that closed relative to \( W \in \mathcal{U} \). By Lemma 9, we calculate as follows:

\[
\begin{align*}
\text{int}_X(A \cap U) &\subset \text{int}_X(A) \cap \text{cl}_X(\text{cl}_A(U)) \\
&\subset \text{cl}_X(\text{int}_X(A) \cap \text{int}_A(\text{cl}_A(U))) \\
&= \text{cl}_X(\text{int}_X(A) \cap \text{int}_A(\text{cl}_A(U))) \\
&\subset \text{cl}_X(\text{int}_X(\text{cl}_X(U))).
\end{align*}
\]

**Corollary 5.** If \( A \in \tau \) and \( U \in \text{SPO}(A, \tau_A) \), then \( U \in \text{SPO}(X, \tau) \).

**Corollary 6.** If \( A \in \tau \) and \( U \in \text{SPO}(A, \tau_A) \), then \( \text{cl}_A(U) \in \text{SPO}(X, \tau) \).

**Lemma 9.** If \( A \in \tau \) and \( V \in \text{SPO}(X, \tau) \), then \( A \cap V \in \text{SPO}(A, \tau_A) \).

**Proof.** We have

\[
\begin{align*}
A \cap V &\subset A \cap \text{cl}_X(\text{cl}_X(V)) \\
&\subset \text{cl}_A(\text{int}_X(A \cap \text{cl}_X(V))) \\
&= \text{cl}_A(\text{int}_A(A \cap \text{cl}_X(V))) \\
&\subset \text{cl}_A(\text{int}_A(A \cap V)).
\end{align*}
\]

**Theorem 5.** Let \((X, \tau)\) be a space and \( A \in \tau \). The following are equivalent:

(a) \((A, \tau_A)\) is ssopo-closed,

(b) \((A, \tau_A)\) is s-closed.

**Proof.** (a)\(\Rightarrow\)(b). Making use of Theorems 1 and 4 we will show \( A \) is ssopo-closed relative to \((X, \tau)\). Suppose \( \{ V_\alpha : \alpha \in \nabla \} \subset \text{SPO}(X, \tau) \) is a cover of \( A \). By Lemma 9, \( \{ A \cap V_\alpha : \alpha \in \nabla \} \subset \text{SPO}(A, \tau_A) \) covers \( A \) and hence we get \( A = \bigcup_{\alpha \in \nabla_0} \text{scl}_A(A \cap V_\alpha) \) for some finite \( \nabla_0 \subset \nabla \). It is easy to see that by Lemma 3, \( A \subset \bigcup_{\alpha \in \nabla_0} \text{scl}_X(V_\alpha) \). Thus \((A, \tau_A)\) is s-closed.

(a)\(\Leftarrow\)(b). Suppose \( A \) is s-closed relative to \((X, \tau)\) (utilize Theorem 1). Let \( \{ U_\alpha : \alpha \in \nabla \} \subset \text{SPO}(A, \tau_A) \) be a cover of \( A \). We have \( \{ U_\alpha : \alpha \in \nabla \} \subset \text{SPO}(X, \tau) \) (Corollary 2) and \( A \subset \bigcup_{\alpha \in \nabla} \text{scl}_X(U_\alpha) \), where \( \{ \text{scl}_X(U_\alpha) : \alpha \in \nabla \} \subset \text{SR}(X, \tau) \). By \[10, \text{Proposition 4.1}] \], \( A \subset \bigcup_{\alpha \in \nabla_0} \text{scl}_X(U_\alpha) \) for some finite \( \nabla_0 \subset \nabla \). Hence, using Lemma 3 we get that \( A = \bigcup_{\alpha \in \nabla_0} \text{scl}_A(U_\alpha) \). This completes the proof.

**Lemma 10.** \[12\] (compare also \[24, \text{Example 3.3(ii)}\]). If \( V \in \text{SO}(X, \tau) \) and \( W \subset X \), the following holds:

\[
V \cap \text{scl}(W) \subset \text{cl}(\text{scl}(V \cap W)).
\]

**Theorem 6.** Let \( A, B \in \text{SC}(X, \tau) \) and \( A \cap B \in \text{SO}(X, \tau) \). If \( A \) and \( B \) are both s-closed relative to \((X, \tau)\), then \( A \cap B \) is also s-closed relative to \((X, \tau)\).

**Proof.** Let \( A \cap B \subset \bigcup_{\alpha \in \nabla} V_\alpha \) where \( V_\alpha \in \text{SO}(X, \tau) \) for each \( \alpha \in \nabla \). We have \( A \subset (X \setminus B) \cup \bigcup_{\alpha \in \nabla} V_\alpha \) and \( B \subset (X \setminus A) \cup \bigcup_{\alpha \in \nabla} V_\alpha \), where \( X \setminus A, X \setminus B \in \text{SO}(X, \tau) \). By hypothesis there are finite subfamilies \( \nabla_1, \nabla_2 \subset \nabla \) with

\[
A \subset \text{scl}(X \setminus B) \cup \bigcup_{\alpha \in \nabla_1} \text{scl}(V_\alpha)
\]

and

\[
B \subset \text{scl}(X \setminus A) \cup \bigcup_{\alpha \in \nabla_2} \text{scl}(V_\alpha).
\]
It follows easily from Lemma 10 that
\[ A \cap B = (A \cap B) \cap (A \cup B) \subset \bigcup_{\alpha \in \nabla_1} \text{sel}(V_\alpha) \cup \bigcup_{\alpha \in \nabla_2} \text{sel}(V_\alpha). \]
Thus, \( A \cap B \) is \( s \)-closed relative to \((X, \tau)\).

**Corollary 7.** If \( A, B \in \text{SC}(X, \tau) \), \( A \cap B \in \text{SO}(X, \tau) \), and \( A, B \) are both \( s \)-closed relative to \((X, \tau)\), then \( A \cap B \) is an \( s \)-closed subspace of \((X, \tau)\).

**Proof.** Follows from Theorem 6 and [20, Theorem 4].

It is of worth to compare Corollary 7 with [14, Theorem 2.2].

**Theorem 7.** Let \( A, B \in \text{SO}(X, \tau) \) and \( A \cap B = \emptyset \). If a set \( A \cup B \) is \( s \)-closed relative to \((X, \tau)\), then \( B \) and \( A \) are \( s \)-closed relative to \((X, \tau)\).

**Proof.** Similar to that of Theorem 28 below—one uses Lemma 10.

The notion of \( S \)-connectedness has been introduced by Pipitone and Russo in [37]: \((X, \tau)\) is \( S \)-connected if there are no two nonempty sets \( A_1, A_2 \in \text{SO}(X, \tau) \) such that \( X = A_1 \cup A_2 \) and \( A_1 \cap A_2 = \emptyset \). A space that is not \( S \)-connected is said to be \( S \)-disconnected.

**Corollary 8.** Let \((X, \tau)\) be an \( S \)-disconnected and \( s \)-closed space. Then there exists a nonempty set \( B \in \text{SO}(X, \tau) \) which is \( s \)-closed relative to \((X, \tau)\) and is an \( s \)-closed subspace of \((X, \tau)\).

**Proof.** By Theorem 7 and [21, Theorem 4].

**Theorem 8.** Let \((X, \tau)\) be \( s \)-closed and \( A \in \text{SR}(X, \tau) \). Then \( X \setminus A \) is an \( s \)-closed subspace of \((X, \tau)\).

**Proof.** Let \( X \setminus A \subset \bigcup_{\alpha \in \nabla} V_\alpha \) where \( \{ V_\alpha : \alpha \in \nabla \} \subset \text{SR}(X, \tau) \). Then \( X = A \cup \bigcup_{\alpha \in \nabla} V_\alpha \), and by [10, Proposition 3.1] there exists some finite \( \nabla_0 \subset \nabla \) with \( X = A \cup \bigcup_{\alpha \in \nabla_0} V_\alpha \). So, \( X \setminus A \) is \( s \)-closed relative to \((X, \tau)\) and by [21, Theorem 4] it is an \( s \)-closed subspace.

**Theorem 9.** Let \( A \in \text{CO}(X, \tau) \) be a set \( s \)-closed relative to \((X, \tau)\). Then \((X, \tau)\) is \( s \)-closed if and only if \( X \setminus A \) is an \( s \)-closed subspace of it.

**Proof.** Necessity. Theorem 8. Sufficiency. By Theorem 1, \( X \setminus A \) is \( s \)-closed relative to \((X, \tau)\). Hence \( X = A \cup (X \setminus A) \) is \( s \)-closed relative to \((X, \tau)\) [4, Theorem 4]; i.e., \((X, \tau)\) is \( s \)-closed.

**Lemma 11.** Let \( B \in \text{SR}(X, \tau) \), \( A \subset X \), and \( A \cup B \) be \( s \)-closed relative to \((X, \tau)\). Then, \( A \setminus B \) is \( s \)-closed relative to \((X, \tau)\).

**Proof.** Follows easily from [10, Proposition 4.1] and the identity \( A \setminus B = (A \cup B) \cap (X \setminus B) \).
THEOREM 10. Let, in a space \((X, \tau), (A, \tau_A)\) and \((B, \tau_B)\) be s-closed subspaces. If \(A \in \tau^o\) and \(B \in CO(X, \tau)\), then \((A \setminus B, \tau_{A \setminus B})\) is an s-closed subspace of \((X, \tau)\).

Proof. By Theorem 1, \(A\) and \(B\) are s-closed relative to \((X, \tau)\). Using [4, Theorem 4] and Lemma 11 we get that \(A \setminus B\) is s-closed relative to \((X, \tau)\). It is enough now to recall that \(CO(X, \tau) = CO(X, \tau^o)\).

REMARK 3. The above Theorems 7 to 10 should be compared with respective Theorems 28 to 31 in the sequel (Section 4).

Recall the following notions [10, p.227]: a point \(x\) of a space \((X, \tau)\) is said to be a \emph{semi-\(\theta\)-adherent point} of a subset \(S \subset X\) if \(S \cap scl_X(U) \neq \emptyset\) for every set \(U \in SO(X, \tau)\) with \(x \in U\). The set of all semi-\(\theta\)-adherent points of an \(S\) is called the \emph{semi-\(\theta\)-closure} of \(S\) in \((X, \tau)\). A set \(S \subset X\) is called \emph{semi-\(\theta\)-closed} if the semi-\(\theta\)-closure of \(S\) is \(S\).

THEOREM 11. Let \(A \in SPO(X, \tau)\). If \(A \cup (X \setminus scl_X(A))\) is s-closed relative to \((X, \tau)\), then \(A\) is s-closed relative to \((X, \tau)\).

Proof. Let \(A \subset \bigcup_{\alpha \in \nabla} V_{\alpha}\) where \(\{V_{\alpha} : \alpha \in \nabla\} \subset SR(X, \tau)\). By Lemma 6, \(scl_X(A) \subset SR(X, \tau)\) and hence \(scl_X(A)\) is semi-\(\theta\)-closed [12, Proposition 2.3(b)]. Thus, for each \(x \in X \setminus scl_X(A)\) there exists \(V_x \in SO(X, \tau)\) with \(x \in V_x\), such that \(scl_X(V_x) \subset X \setminus scl_X(A)\). The family \(\{scl_X(V_x) : x \in X \setminus scl_X(A)\} \cup \{V_{\alpha} : \alpha \in \nabla\}\) covers the set \(A \cup (X \setminus scl_X(A))\). Thus, by hypothesis, there exists a finite \(\nabla_0 \subset \nabla\) with \(A \subset \bigcup_{\alpha \in \nabla_0} V_{\alpha}\).

COROLLARY 9. Let \((X, \tau)\) be an s-closed space and \(A \in SPO(X, \tau)\). If \(scl_X(A) \setminus A \in SR(X, \tau)\) then \(A\) is s-closed relative to \((X, \tau)\).

Proof. By the proof of Theorem 8 the set \(X \setminus (scl_X(A) \setminus A)\) is s-closed relative to \((X, \tau)\). Apply now Theorem 11.

A space \((X, \tau)\) is said to be \emph{weakly-T\(_2\)} [40], if each point of \(X\) can be expressed as an intersection of regular closed subsets of \((X, \tau)\). In [10, Proposition 4.3] the following is proved: if \(K\) is s-closed relative to a weakly-T\(_2\) space, then \(K\) is semi-\(\theta\)-closed in \((X, \tau)\).

THEOREM 12. Let \(A \subset X\) be a set s-closed relative to \((X, \tau)\). Assume that

for each \(x \in X \setminus A\) and \(y \in A\), there exist sets

\[
V_x \in \tau^o, \quad V_y \in SO(X, \tau), \quad V_x \ni x, \quad V_y \ni y, \quad \text{with} \quad V_x \cap V_y = \emptyset.
\]

Then, \(A\) is semi-\(\theta\)-closed in \((X, \tau)\).

Proof. Pick an arbitrary \(x_0 \in X \setminus A\). For each \(y \in A\), there exist sets \(V_{x_0, y} \in \tau^o\), \(V_{x_0, y} \ni x_0\), and \(V_y \in SO(X, \tau)\), \(V_y \ni y\), with \(V_{x_0, y} \cap V_y = \emptyset\). Thus, \(\{V_y : y \in A\}\) covers \(A\) and, as \(A\) is s-closed relative to \((X, \tau)\), we have \(A \subset \bigcup_{i=1}^n scl(V_{y_i})\) for some \(y_1, \ldots, y_n \in A\). Making use of Lemma 7 (or Lemma 10) we get \(V_{x_0, y_i} \cap
scl $(V_{y_i}) = \emptyset$, $i = 1, \ldots, n$. We have also $A \subset \bigcup_{i=1}^n \text{scl} (V_{y_i}) = V \in \text{SO} (X, \tau)$ and $x_0 \in \bigcap_{i=1}^n V_{x_0, y_i} = B \in \tau^n$. So, by [17, Lemma 1(i)],

$$B \cap \text{cl}_r (V) = B \cap \text{cl}_{r^n} (V) \subset \text{cl}_{r^n} (B \cap V) = \emptyset,$$

where $\text{cl}_r (V) \in \text{SR} (X, \tau)$. This implies that $x_0 \in X \setminus \text{cl}_r (V) \in \text{SR} (X, \tau)$; i.e., there is a $U \in \text{SO} (X, \tau)$ containing $x_0$ such that $\text{scl}_X (U) \cap A = \emptyset$. Thus, $x_0$ is not a semi $\theta$-adherent point of $A$ and hence $A$ is semi $\theta$-closed.

Example 1. There exist a space $(X, \tau)$ which is not weakly-$T_2$, and a subset $A \subseteq X$ such that (1) of Theorem T12 holds. Indeed, if $X = \{a, b, c, d, e\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}, \{e\}\}$, then consider $A = \{c, d, e\}$.

Remark 4. Recall that $(X, \tau)$ is called a semi-$T_2$-space [23], if for any distinct points $x_1, x_2 \in X$ there exist disjoint $V_1, V_2 \in \text{SO} (X, \tau)$ with $V_1 \ni x_1$ and $V_2 \ni x_2$. Using [19, Theorem 2.9] and the fact that $(X, \tau)$ is $T_2$ if and only if $(X, \tau^o)$ is $T_2$ [11, Theorem 3], we obtain that every e.d. semi-$T_2$ space is $T_2$. So, directly from [10, Proposition 4.3] we infer what follows: in any e.d. semi-$T_2$ space $(X, \tau)$, every subset $s$-closed relative to $(X, \tau)$ is semi $\theta$-closed in $(X, \tau)$.

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be semi-continuous [22] (resp. s-open [6]) if $f^{-1}(V) \in \text{SO} (X, \tau)$ (resp. $f(U) \in \sigma$) for every $V \in \sigma$ (resp. $U \in \text{SO} (X, \tau)$). An $f$ is semi-continuous if and only if for every $S \subset X$, $f(\text{scl}_X (S)) \subset \text{cl}_Y (f(S))$ [9, Theorem 1.16].

Theorem 13. Consider a function $f : (X, \tau) \to (Y, \sigma)$ and a subset $G$ $s$-closed relative to $(X, \tau)$.

(a) If $f$ is semi-continuous and s-open then $f(G)$ is $\mathcal{N}$-closed relative to $(Y, \sigma)$.

(b) If $f$ is semi-continuous then $f(G)$ is quasi $\mathcal{H}$-closed relative to $(Y, \sigma)$.

Proof. (a) Let $\{V_\alpha : \alpha \in \nabla\} \subset \sigma$ be a cover of $f(G)$. Then $\{f^{-1}(V_\alpha) : \alpha \in \nabla\} \subset \text{SO} (X, \tau)$ is a cover of $G$. There is a finite $\nabla_0 \subset \nabla$ such that $G \subset \bigcup_{\alpha \in \nabla_0} \text{scl}_X (f^{-1}(V_\alpha))$. As $f$ is semi-continuous and s-open, we obtain

\[ f(G) \subset \bigcup_{\alpha \in \nabla_0} f(\text{scl}_X (f^{-1}(V_\alpha))) \subset \bigcup_{\alpha \in \nabla_0} \text{int}_Y (\text{cl}_Y (f(f^{-1}(V_\alpha)))) \]

Thus, $f(G)$ is $\mathcal{N}$-closed relative to $(Y, \sigma)$.

(b) Similar to the case (a).

Example 2. (a). Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}, \{a\}\}$, $Y = \{a, b, c, d\}$, and $\sigma = \{\emptyset, Y, \{a, b\}, \{a, d\}, \{a, b, d\}\}$. Define $f : (X, \tau) \to (Y, \sigma)$ as the identity on $X$. One checks that $f$ is semi-continuous. But, $f$ is not s-open since $f(\{a, b\}) \notin \sigma$. 
(b). Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a, b\}\} \), and \( \sigma = \{\emptyset, X, \{c, \{a, b\}\}\} \). Let again \( f : (X, \tau) \to (Y, \sigma) \) be the identity on \( X \). Then \( f \) is \( s \)-open not being semi-continuous as \( f^{-1}(\{c\}) \notin \text{SO}(X, \tau) \).

**Definition 2.** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( \text{SR-open} \) (resp. \( \text{R-open} \)), if \( f(U) \in \text{SR}(Y, \sigma) \) (resp. \( f(U) \in \text{RO}(Y, \sigma) \)) for every \( U \in \text{SR}(X, \tau) \) (resp. \( U \in \text{RO}(X, \tau) \)).

**Theorem 14.** Let a set \( B \) be \( s \)-closed relative to \((Y, \sigma)\). If a bijection \( f : (X, \tau) \to (Y, \sigma) \) is \( \text{SR-open} \) then \( f^{-1}(B) \) is \( s \)-closed relative to \((X, \tau)\).

**Proof.** Use [10, Proposition 4.1]. ■

A function \( f : (X, \tau) \to (Y, \sigma) \) is called \( \text{a.c.H.} \) ([18, 25] and [39, Theorem 4]) if \( f^{-1}(V) \in \text{PO}(X, \tau) \) for every \( V \in \sigma \).

**Theorem 15.** If a function \( f : (X, \tau) \to (Y, \sigma) \) is \( \text{a.c.H.} \) and \( \text{R-open} \), then it is \( \text{SR-open} \).

**Proof.** Let \( A \in \text{SR}(X, \tau) \). There exists a set \( U \in \text{RO}(X, \tau) \) such that \( U \subset A \subset \text{cl}(U) \) [10, Proposition 2.1]. Since \( f \) is \( \text{a.c.H.} \), \( f(\text{cl}(S)) \subset \text{cl}(f(S)) \) for every \( S \in \tau \) [39, Theorem 6]. Thus, by \( \text{R-openness} \) of \( f \) and, again, by [10, Proposition 2.1] we obtain that \( f \) is \( \text{SR-open} \). ■

### 3. Hausdorffness of spaces

In this section we offer some characterizations of \( T_2 \) and semi-\( T_2 \) spaces.

**Theorem 16.** A space \((X, \tau)\) is \( T_2 \) if and only if, for each \( A \subseteq X \) \( s \)-closed relative to \((X, \tau)\) and each point \( x \in X \setminus A \) there exist disjoint sets \( U_1, U_2 \in \text{RO}(X, \tau) \) with \( U_1 \ni x \) and \( U_2 \supset A \).

**Proof.** Necessity. Let \( x_0 \in X \setminus A \) be arbitrary. By Hausdorffness of \((X, \tau)\), for each \( y \in A \) there are disjoint \( V_{x_0, y_1}, V_{y_1} \in \tau^a \) with \( V_{x_0, y_1} \ni x_0 \) and \( V_{y_1} \ni y \) [17, Theorem 3]. Since \( A \) is \( s \)-closed relative to \((X, \tau)\), \( A \subseteq \bigcup_{i=1}^{n} \text{cl}(V_{y_i}) \) for certain \( y_1, \ldots, y_n \in A \). It is enough to show that

\[
\text{scl} \left( \bigcap_{i=1}^{n} V_{x_0, y_i} \right) \cap \text{scl} \left( \bigcup_{i=1}^{n} \text{scl}(V_{y_i}) \right) = \emptyset,
\]

because \( \text{scl}(S) = \text{int}(\text{cl}(S)) \) for any \( S \in \tau^a \subset \text{PO}(X, \tau) \) [20, Proposition 2.7(a)]. Indeed, we get by Lemma 7 (for instance), [8, Theorem 1.7(4)], and Lemma 4:

\[
\text{scl} \left( \bigcap_{i=1}^{n} V_{x_0, y_i} \right) \cap \text{scl} \left( \bigcup_{i=1}^{n} \text{scl}(V_{y_i}) \right) \subset \text{scl} \left( \text{scl} \left( \bigcap_{i=1}^{n} V_{x_0, y_i} \right) \cap \bigcup_{i=1}^{n} \text{scl}(V_{y_i}) \right)
\]

\[
\subset \text{scl} \left( \text{int} \left( \text{cl} \left( \bigcap_{i=1}^{n} V_{x_0, y_i} \cap \bigcup_{i=1}^{n} V_{y_i} \right) \right) \right) = \text{scl} \left( \text{int} \left( \text{cl}(\emptyset) \right) \right) = \emptyset.
\]
Thus, if we put
\[ U_1 = \text{scl} \left( \bigcap_{i=1}^{n} V_{x_0,y_i} \right) \in \text{RO}(X,\tau), \quad U_2 = \text{scl} \left( \bigcup_{i=1}^{n} \text{scl}(V_{y_i}) \right) \in \text{RO}(X,\tau), \]
then
\[ x_0 \in U_1, \quad A \subset U_2, \quad \text{and } U_1 \cap U_2 = \emptyset. \]

**Sufficiency.** This is clear as every singleton is \(s\)-closed relative to \((X,\tau)\) (compare [10, Proposition 4.1]).

Recall that a subset \(A\) of a space \((X,\tau)\) is said to be \(\alpha\)-compact relative to \((X,\tau)\) [3], if every \(\tau^\alpha\)-cover of \(A\) admits a finite subcover.

**Theorem 17.** A space \((X,\tau)\) is \(T_2\) if and only if, for each \(A \subseteq X\), \(\alpha\)-compact relative to \((X,\tau)\) and each point \(x \in X \setminus A\), there exist disjoint sets \(U_1,U_2 \in \text{RO}(X,\tau)\) with \(U_1 \ni x\) and \(U_2 \supset A\).

**Proof.** Very similar to that of Theorem 16 (after few modifications—details left to the reader).

In [15] the author has proved that a space \((X,\tau)\) is semi-\(T_2\) if and only if, for any distinct \(x,y \in X\), there are sets \(U_x,U_y \in \text{SR}(X,\tau)\) such that \(x \in U_x, y \in U_y, U_x \cap U_y = \emptyset\). So, since every singleton is \(s\)-closed relative to \((X,\tau)\) [10, Proposition 4.1], we get as a corollary

**Theorem 18.** Assume that for each subset \(A \subseteq X\), \(s\)-closed relative to \((X,\tau)\), and for each point \(x \in X \setminus A\), there exist disjoint \(U_1,U_2 \in \text{SR}(X,\tau)\) with \(U_1 \ni x\) and \(U_2 \supset A\). Then \((X,\tau)\) is semi-\(T_2\).

Combining Theorem 18 with [21, Theorem 6] we obtain the following characterization of e.d. semi-\(T_2\) spaces.

**Theorem 19.** An e.d. space \((X,\tau)\) is semi-\(T_2\) if and only if, for any \(A \subseteq X\), \(s\)-closed relative to \((X,\tau)\), and each \(x \in X \setminus A\), there exist disjoint semi-regular subsets \(U\) and \(V\) with \(U \ni x\) and \(V \supset A\).

### 4. \(S\)-closedness

The following result has been stated by Khan, Ahmad, and Noiri [21, Theorem 5]: if every semi-regular subset of an e.d. space \((X,\tau)\) is an \(s\)-closed subspace of \((X,\tau)\), then \((X,\tau)\) is \(s\)-closed. In this theorem \(\text{"(X,\tau) is } s\text{-closed" may be replaced by \"(X,\tau) is } S\text{-closed\" since in e.d. spaces these two notions coincide} [27, Theorem 14]. Moreover, the next result we state shows that after this replacement, the assumption \(\text{"(X,\tau) is } e.d.\text{" becomes superfluous.

**Theorem 20.** If every semi-regular subset of \((X,\tau)\) is an \(s\)-closed subspace of \((X,\tau)\), then \((X,\tau)\) is \(S\)-closed.

**Proof.** Suppose \(\{V_\alpha : \alpha \in \nabla\} \subset \text{SO}(X,\tau)\) is a cover of \((X,\tau)\). Take into consideration a set \(\text{cl}_X(V_\beta) \neq X\) with \(V_\beta \neq \emptyset\). Obviously, \(\text{cl}_X(V_\beta) \in \text{SR}(X,\tau)\) and
hence \( X \setminus \text{cl}_X(V_β) \in \text{SR}(X, τ) \) as well. By hypothesis \( X \setminus \text{cl}_X(V_β) \) is an \( s \)-closed subspace of \((X, τ)\), and since it is open in \((X, τ)\), we infer from Theorem 1 that \( X \setminus \text{cl}_X(V_β) \) is \( s \)-closed relative to \((X, τ)\). We have \( X \setminus \text{cl}_X(V_β) \subset \bigcup_{α \in ∇} V_α \) and there is a finite \( ∇_0 \subset ∇ \) such that

\[
X \setminus \text{cl}_X(V_β) \subset \bigcup_{α \in ∇_0} \text{scl}_X(V_α).
\]

Thus, one gets \( X = \bigcup_{α \in ∇ \cup β} \text{cl}_X(V_α) \). This shows that \((X, τ)\) is \( S \)-closed. ■

In [32, Theorem 3.1] Noiri proved that if \( A \in τ^α \), then the subspace \((A, τ_A)\) is \( S \)-closed if and only if it is \( S \)-closed relative to \((X, τ)\). Combining this result with Theorem 1, it is easy to show that for \( A \in τ^α \), if \((A, τ_A)\) is \( s \)-closed then it is \( S \)-closed. The theorem below is a strong improvement of this corollary.

**Theorem 21.** Let \( A \) be an arbitrary subset of \((X, τ)\). If \((A, τ_A)\) is \( s \)-closed then it is \( S \)-closed.

**Proof.** Let \( \{U_α : α \in ∇\} \subset \text{SO}(A, τ_A) \) be a cover of \( A \). By assumption, there is a finite \( ∇_0 \subset ∇ \) such that \( A = \bigcup_{α \in ∇_0} \text{scl}_A(U_α) \). So, \( A = \bigcup_{α \in ∇_0} \text{cl}_A(U_α) \). ■

**Theorem 22.** Let \( A \in τ^α \) be a subset of an e.d. space \((X, τ)\). Then, \((A, τ_A)\) is \( S \)-closed if and only if it is \( s \)-closed.

**Proof.** Let \((A, τ_A)\) be \( S \)-closed. By [32, Theorem 3.1] it is equivalent \( A \) being \( S \)-closed relative to \((X, τ)\). By means of [27, Theorem 14] and Theorem 1, the latter is equivalent \((A, τ_A)\) being \( s \)-closed. ■

**Remark 5.** The following is an interesting consequence of [27, Theorem 14]: for any subset \( A \) of \((X, τ)\) such that \((A, τ_A)\) is e.d., \((A, τ_A)\) is \( S \)-closed if and only if \( A \) is \( s \)-closed.

In [14, Theorem 2.7] the author proved that if \( A \in τ^α \) is an \( S \)-closed subspace of \((X, τ)\), then \((\text{scl}_X(A), τ_{\text{scl}_X(A)})\) is also \( S \)-closed. Since \( \text{scl}_X(A) = \text{int}_X(\text{cl}_X(A)) \) for any \( A \in \text{PO}(X, τ) \) [20, Proposition 2.7(a)], by the use of Theorem 22 it follows that if \( A \in τ^α \) is an \( s \)-closed subspace of an e.d. \((X, τ)\), then \((\text{scl}_X(A), τ_{\text{scl}_X(A)})\) is \( s \)-closed too. This result shall be extended to \( A \in \text{PO}(X, τ) \) (in e.d. spaces) in Theorem 23 below.

**Lemma 12.** For any \((X, τ)\) and \( S_1, S_2 \subset X \),

\[
\text{int}(\text{cl}(S_1 \cup S_2)) = \text{int}(\text{cl}(\text{int}(S_1)) \cup \text{int}(\text{cl}(S_2))).
\]

**Proof.** Clearly, \( \text{int}(\text{cl}(S_1)) \cup \text{int}(\text{cl}(S_2)) \subset \text{int}(\text{cl}(S_1 \cup S_2)) \). Next, we calculate as follows: \( \text{int}(\text{cl}(S_1 \cup S_2)) \subset \text{cl}(\text{int}(S_1 \cup S_2)) = \text{cl}(\text{int}(\text{cl}(S_1)) \cup \text{cl}(S_2)) \) by the dual to Lemma 4. So,

\[
\text{int}(\text{cl}(S_1 \cup S_2)) \subset \text{cl}(\text{int}(\text{cl}(S_1)) \cup \text{int}(\text{cl}(S_2))) \subset \text{cl}(\text{int}(\text{cl}(S_1 \cup S_2))),
\]

and this concludes the proof. ■
**Lemma 13.** Let \((X, \tau)\) be e.d. Then for every \(S_1, S_2 \subset X\),
\[
\text{int}(\text{cl}(S_1 \cup S_2)) = \text{int}(\text{cl}(S_1)) \cup \text{int}(\text{cl}(S_2)).
\]

**Proof.** Follows easily from Lemma 12. \(\blacksquare\)

**Lemma 14.** In any \((X, \tau)\), if \(A \subset X\) and \(U \in \text{SO}(\text{scl}_X(A), \tau_{\text{scl}_X(A)})\) then \(U \cap A \in \text{SO}(A, \tau_A)\).

**Proof.** For a certain \(O \in \tau\), \(V = O \cap \text{scl}_X(A) \subset U \subset \text{cl}_{\text{scl}_X(A)}(V)\). Then \(V \subset U \subset \text{cl}_X(V) \cap \text{scl}_X(A) \subset \text{cl}_X(O \cap \text{cl}_X(A)) \cap \text{scl}_X(A) \\
\subset \text{cl}_X(O \cap A) \subset \text{cl}_X(O \cap A) \cap \text{scl}_X(A)\). Therefore we obtain
\[
O \cap A \subset U \cap A \subset \text{cl}_X(O \cap A) \cap A = \text{cl}_A(O \cap A).
\]

**Theorem 23.** Let \((A, \tau_A)\) be an s-closed subspace of e.d. \((X, \tau)\), where \(A \in \text{PO}(X, \tau)\). Then the subspace \((\text{scl}_X(A), \tau_{\text{scl}_X(A)})\) is s-closed.

**Proof.** Let \(\{U_\alpha : \alpha \in \nabla\} \subset \text{SO}(\text{scl}_X(A), \tau_{\text{scl}_X(A)})\) cover \(\text{scl}_X(A)\). By Lemma 14 the family \(\{U_\alpha \cap A : \alpha \in \nabla\} \subset \text{SO}(A, \tau_A)\) forms a cover of \(A\). Since \((A, \tau_A)\) is s-closed, \(A = \bigcup_{\alpha \in \nabla_0} \text{scl}_A(U_\alpha \cap A)\) for some finite \(\nabla_0 \subset \nabla\). Hence by Lemma 3 and by [20, Proposition 2.7(a)] we get \(A \subset \bigcup_{\alpha \in \nabla_0} (\text{int}_X(\text{cl}_X(A)) \cap \text{scl}_X(U_\alpha))\), and since \((X, \tau)\) is e.d. we have by Lemmas 13 and 4
\[
\text{scl}_X(A) \subset \text{int}_X \left( \text{cl}_X \left( \bigcup_{\alpha \in \nabla_0} (\text{int}_X(\text{cl}_X(A)) \cap \text{scl}_X(U_\alpha)) \right) \right).
\]
\[
= \bigcup_{\alpha \in \nabla_0} (\text{int}_X(\text{cl}_X(A)) \cap \text{int}_X(\text{scl}_X(U_\alpha))).
\]

So, as \(\text{scl}_X(U_\alpha) \in \text{SC}(X, \tau), \alpha \in \nabla_0\), we obtain \(\text{scl}_X(A) = \bigcup_{\alpha \in \nabla_0} \text{scl}_{\text{scl}_X(A)}(U_\alpha)\). Thus \(\text{scl}_X(A)\) is s-closed. \(\blacksquare\)

**Lemma 15.** Let \(A \in \text{SO}(X, \tau)\). If \((\text{int}_X(A), \tau_{\text{int}_X(A)})\) is s-closed, then for any cover \(\{V_i : i \in \nabla\} \subset \text{SPO}(X, \tau)\) of \(A\) there is some finite \(\nabla_0 \subset \nabla\) such that \(A \subset \bigcup_{i \in \nabla_0} \text{cl}_{\text{int}_X(A)}(V_i)\).

**Proof.** Let \(\emptyset \neq \text{int}_X(A) \subset C \subset \bigcup_{i \in \nabla} V_i\), where \(V_i \in \text{SPO}(X, \tau)\) for each \(i \in \nabla\). Then \(\text{int}_X(A) = \bigcup_{i \in \nabla} (\text{int}_X(A) \cap V_i)\) and by Lemma 9 we have
\[
\text{int}_X(A) \cap V_i \subset \text{SPO}(\text{int}_X(A), \tau_{\text{int}_X(A)})
\]
for \(i \in \nabla\). By hypothesis there exists a finite \(\nabla_0 \subset \nabla\) with
\[
\text{int}_X(A) = \bigcup_{i \in \nabla_0} \text{scl}_{\text{int}_X(A)}(\text{int}_X(A) \cap V_i)
\]
(see Theorem 5). Making use of Lemmas 3 and 8 we get
\[
\text{int}_X(A) \subset \bigcup_{i \in \nabla_0} \text{scl}_X(\text{int}_X(A) \cap (\text{int}_X(A) \cap V_i)) \subset \bigcup_{i \in \nabla_0} \text{cl}_X(\text{int}_X(\text{cl}_X(V_i))).
\]
On the other hand, by [2, Theorem 1.5(c)], \( \text{cl}_{\tau^*}(V) = \text{cl}_X(\text{int}_X(\text{cl}_X(V))) \) for each \( V \in \text{SPO}(X, \tau) \). Therefore, since \( A \in \text{SO}(X, \tau) \),
\[
A \subset \bigcup_{i \in \nabla_0} \text{cl}_{\tau^*}(V_i). 
\]

**Theorem 24.** Let \( A \in \text{SO}(X, \tau) \). If the subspace \( \left( \text{int}_X(A), \tau_{\text{int}_X(A)} \right) \) is \( s \)-closed then \( (A, \tau_A) \) is \( S \)-closed.

**Proof.** Let \( A = \bigcup_{i \in \nabla} U_i \) where \( U_i \in \text{SO}(A, \tau_A) \) for each \( i \in \nabla \). By [29, Theorem 5], \( U_i \in \text{SO}(X, \tau) \). Since \( \text{SO}(X, \tau) \subset \text{SPO}(X, \tau) \), from Lemma 15 we infer that for some finite \( \nabla_0 \subset \nabla \), \( A \subset \bigcup_{i \in \nabla_0} \text{cl}_{\tau^*}(U_i) = \bigcup_{i \in \nabla_0} \text{cl}_{\tau}(U_i) \) [17, Lemma 1(i)]. Consequently, \( A = \bigcup_{i \in \nabla_0} \text{cl}_A(U_i) \).

By [19, Theorem 2.9] we have for each subset \( S \) of \( X \) that \( \text{cl}_{\tau^*}(S) = \text{scl}_X(S) \). Thus, by [17, Lemma 1(i)], it leads to the following theorem.

**Theorem 25.** Let \( (X, \tau) \) be an e.d. space. Any of the two conditions: ‘for every semi-open (or open) cover \( \mathcal{U} \) of \( A \subset X \) there is a finite subfamily \( \mathcal{U}_0 \) with \( A \subset \text{scl}_X(\bigcup \mathcal{U}_0) \)’, coincides with any of the properties: ‘\( A \) is \( S \)-closed relative to \( (X, \tau) \)’, ‘\( A \) is \( S \)-closed relative to \( (X, \tau) \)’, ‘\( A \) is \( \mathcal{N} \)-closed relative to \( (X, \tau) \)’, ‘\( A \) is quasi \( \mathcal{H} \)-closed relative to \( (X, \tau) \)

**Proof.** We use [27, Theorem 14] (the reader is advised to compare [27, Theorem 2]).

The following result has been stated in [5, Theorem 2]: a space \( (X, \tau) \) is \( S \)-closed if and only if every cover \( \{V_\alpha : \alpha \in \nabla\} \subset \text{RC}(X, \tau) \) of \( X \) admits a finite subcover. This fact is a particular case of our next theorem.

**Theorem 26.** A subset \( A \) of \( (X, \tau) \) is \( S \)-closed relative to \( (X, \tau) \) if and only if every cover \( \{V_\alpha : \alpha \in \nabla\} \subset \text{RC}(X, \tau) \) of \( A \) admits a finite subcover.

**Proof.** Necessity. Let \( A \subset \bigcup_{\alpha \in \nabla} V_\alpha \) where \( V_\alpha \in \text{RC}(X, \tau) \subset \text{SO}(X, \tau) \) for each \( \alpha \in \nabla \). So, by our assumption, \( A \subset \bigcup_{\alpha \in \nabla_0} \text{cl}(V_\alpha) = \bigcup_{\alpha \in \nabla_0} V_\alpha \) for some finite \( \nabla_0 \subset \nabla \).

Sufficiency. Let \( A \subset \bigcup_{\alpha \in \nabla} V_\alpha \) where \( V_\alpha \in \text{SO}(X, \tau) \) for each \( \alpha \in \nabla \). Obviously \( A \subset \bigcup_{\alpha \in \nabla} \text{cl}(V_\alpha) \) and since \( \text{cl}(S) = \text{cl}(\text{int}(S)) \) for every \( S \in \text{SO}(X, \tau) \) [30, Lemma 2], we get by hypothesis that there exists a finite \( \nabla_0 \subset \nabla \) with \( A \subset \bigcup_{\alpha \in \nabla_0} \text{cl}(V_\alpha) \).

**Lemma 16.** Let \( A \in \text{RO}(X, \tau) \). Then for each \( G \subset A, G \in \text{RO}(X, \tau) \) if and only if \( G \in \text{RO}(A, \tau_A) \).

**Proof.** Strong necessity. Let \( A \in \tau \). We have \( G = A \cap \text{int}_X(\text{cl}_X(G)) = \text{int}_X(A \cap \text{cl}_X(G)) = \text{int}_X(\text{cl}_A(G)) = \text{int}_A(\text{cl}_A(G)) \).

Sufficiency. This has been shown in the proof of [4, Theorem 6].
In [31, Theorem 1.3] the following was proved: \((X, \tau)\) is \(S\)-closed if and only if its every proper subset \(S \in \text{RO}(X, \tau)\) is \(S\)-closed.

**Theorem 27.** Let \(A \in \text{RO}(X, \tau)\). Then, the subspace \((A, \tau_A)\) is \(S\)-closed if and only if every proper subset \(G \subset A\) with \(G \in \text{RO}(X, \tau)\) is \(S\)-closed.

**Theorem 28.** Let \(A \in \text{SO}(X, \tau)\), \(B \in \text{PO}(X, \tau)\), \(A \cap B = \emptyset\). If the union \(A \cup B\) is \(S\)-closed relative to \((X, \tau)\), then \(B\) is \(S\)-closed relative to \((X, \tau)\).

**Proof.** Let a family \(\mathcal{F} \subset \text{SO}(X, \tau)\) be a cover of \(B\). Then, the family \(\mathcal{F} \cup \{A\}\) covers \(A \cup B\). There exist \(V_1, \ldots, V_n \in \mathcal{F}\) such that \(A \cup B \subset \text{cl}(A) \cup \bigcup_{i=1}^n \text{cl}(V_i)\). So, by [35, Lemma 2.1] (see Remark 2) we obtain \(B \subset \bigcup_{i=1}^n \text{cl}(V_i)\). This completes the proof. ■

By [13, Theorem 1] the author has proved that a space \((X, \tau)\) is \(S\)-disconnected if and only if there exists nonempty \(U_1 \in \text{SO}(X, \tau)\), \(U_2 \in \tau^\alpha\) such that \(X = U_1 \cup U_2\) and \(\emptyset = U_1 \cap U_2\). Directly from this result together with Theorem 28, follows

**Corollary 10.** Let \((X, \tau)\) be an \(S\)-disconnected and \(S\)-closed space. Then there exists a nonempty set \(B \in \tau^\alpha\) which is \(S\)-closed relative to \((X, \tau)\) (hence it is also such a subspace of \((X, \tau)\) [32, Theorem 3.1]).

**Theorem 29.** Let \((X, \tau)\) be \(S\)-closed and \(A \in \text{CO}(X, \tau)\). Then \(X \setminus A\) is an \(S\)-closed subspace of \((X, \tau)\).

**Proof.** Let \(X \setminus A \subset \bigcup_{\alpha \in \nabla} V_\alpha\) where \(\{V_\alpha : \alpha \in \nabla\} \subset \text{RC}(X, \tau)\). By [5, Theorem 2] there is a finite \(\nabla_0 \subset \nabla\) such that \(X \subset A \cup \bigcup_{\alpha \in \nabla_0} V_\alpha\). From Theorem 26 we infer that \(X \setminus A\) is \(S\)-closed relative to \((X, \tau)\). Therefore, in view of [32, Theorem 3.1], \(X \setminus A\) is \(S\)-closed as a subspace. ■

**Theorem 30.** Let \(A \in \text{CO}(X, \tau)\) be an \(S\)-closed subspace of \((X, \tau)\). Then, \((X, \tau)\) is \(S\)-closed if and only if \(X \setminus A\) is an \(S\)-closed subspace of \((X, \tau)\).

**Proof.** Necessity. Theorem 29.

**Sufficiency.** By [32, Theorem 3.1], the set \(X \setminus A\) is \(S\)-closed relative to \((X, \tau)\). Thus, by [32, Theorem 3.6], \(X = A \cup (X \setminus A)\) is \(S\)-closed relative to \((X, \tau)\); i.e., \((X, \tau)\) is \(S\)-closed. ■

**Lemma 17.** Let \(A \subset X\) be arbitrary, \(B \in \text{RC}(X, \tau)\), an let \(A \cup B\) be \(S\)-closed relative to \((X, \tau)\). Then \(A \setminus B\) is \(S\)-closed relative to \((X, \tau)\).

**Proof.** This follows from Theorem 26. ■

**Theorem 31.** Let \((A, \tau_A)\) and \((B, \tau_B)\) be \(S\)-closed subspaces of \((X, \tau)\). If \(A, B \in \text{CO}(X, \tau)\) then \((A \setminus B, \tau_{A \setminus B})\) is \(S\)-closed too.

**Proof.** Use [32, Theorems 3.1 and 3.6] and Lemma 17. ■
On $s$-closedness and $S$-closedness


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Institute of Mathematics, Casimir the Great University, Pl. Weyssenhoffa 11, 85-072 Bydgoszcz, Poland

E-mail: imath@ukw.edu.pl