GENERALIZED a-WEYL'S THEOREM FOR DIRECT SUMS

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Abstract. If $T$ and $S$ are Hilbert space operators obeying generalized a-Weyl's theorem, then it does not necessarily imply that the direct sum $T \oplus S$ obeys generalized a-Weyl's theorem. In this paper we explore certain conditions on $T$ and $S$ so that the direct sum $T \oplus S$ obeys generalized a-Weyl's theorem.

1. Introduction

Let $H$ be an infinite dimensional separable Hilbert space. Let $B(H)$ be the algebra of all operators on $H$ (bounded linear transformations of $H$ into itself). For an operator $T \in B(H)$, let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, point spectrum and approximate point spectrum of $T$, respectively. Let $\alpha(T)$ and $\beta(T)$ denote the dimension of the kernel $\ker T$ and the codimension of the range $R(T)$, respectively. An operator $T \in B(H)$ is called an upper semi-Fredholm if $\alpha(T) < \infty$ and $T(H)$ is closed, while $T \in B(H)$ is called a lower semi-Fredholm if $\beta(T) < \infty$. However, $T$ is called a semi-Fredholm operator if $T$ is either an upper or a lower semi-Fredholm and $T$ is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in B(H)$ is semi-Fredholm, then the index of $T$ is defined by

$$\text{ind}(T) = \alpha(T) - \beta(T).$$

The ascent of $T$ is defined by the smallest non-negative integer $p := p(T)$ such that $N(T^p) = N(T^{p+1})$. If such an integer does not exist we put $p(T) = \infty$. Analogously, the descent of $T$, is defined by the smallest nonnegative integer $q := q(T)$ such that $R(T^q) = R(T^{q+1})$ and if such an integer does not exist we put $q(T) = \infty$.

An operator $T \in B(H)$ is called a Weyl operator if it is a Fredholm operator of index 0, while $T \in B(H)$ is called a Browder if it is a Fredholm operator of finite ascent and descent. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_W(T)$

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of $T$ are defined by
\[
\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},
\]
\[
\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}.
\]
For $T \in B(H)$, define the set $LD(H)$ by
\[
LD(H) = \{T \in B(H) : p(T) < \infty \text{ and } R(T^{n+1}) \text{ is closed}\}.
\]
An operator $T \in B(H)$ is said to be left Drazin invertible if $T \in LD(H)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of $T$, if $T - \lambda I \in LD(H)$. We denote by $\pi^a(T)$ the set of all left poles of $T$.

We say that Weyl's theorem holds for $T$, if
\[
\sigma(T) \setminus \sigma_W(T) = E_0(T),
\]
where $E_0(T)$ is the set of all isolated point of $\sigma(T)$ which are eigenvalues of finite multiplicity.

For a bounded linear operator $T$ and a nonnegative integer $n$ we define $T_n$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into itself (in particular $T_0 = T$). If for some integer $n$, the range space $R(T^n)$ is closed and $T_n$ is an upper (resp., a lower) semi-Fredholm operator, then $T$ is called an upper (resp., a lower) semi $B$-Fredholm operator. In this situation, $T_m$ is a semi-Fredholm operator and $\text{ind}(T_m) = \text{ind}(T_n)$ for each $m \geq n$ [3, Proposition 2.1]. It permits us to define the index of a semi $B$-Fredholm operator $T$ as the index of the semi-Fredholm operator $T_n$ where $n$ is any integer such that $R(T^n)$ is closed and $T_n$ is a semi-Fredholm operator. Moreover if $T_n$ is a Fredholm operator, then $T$ is called a $B$-Fredholm operator. A semi $B$-Fredholm operator is an upper or a lower semi $B$-Fredholm operator.

An operator $T \in B(H)$ is called a $B$-Weyl operator if it is a $B$-Fredholm operator of index 0. The $B$-Fredholm spectrum $\sigma_{BF}(T)$ and the $B$-Weyl spectrum $\sigma_{BW}(T)$ of $T$ are defined as
\[
\sigma_{BF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a } B\text{-Fredholm operator}\},
\]
\[
\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a } B\text{-Weyl operator}\}.
\]
We say that generalized Weyl's theorem holds for $T$ if
\[
\sigma(T) \setminus \sigma_{BW}(T) = E(T),
\]
where $E(T)$ is the set of isolated eigenvalues of $T$ ([2], Definition 2.13), and that generalized Browder's theorem holds for $T$ if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$, where $\pi(T)$ is the set of all poles of $T$.

Let $SBF(H)$ be the class of all semi $B$-Fredholm operators on $H$, $USBF(H)$ be the class of all upper semi $B$-Fredholm operators on $H$ and $USBF^-(H)$ be the class of all $T \in USBF(H)$ such that $\text{ind}(T) \leq 0$. Also let
\[
\sigma_{usbf^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } USBF^-(H)\}.
We say $T$ obeys generalized a-Weyl’s theorem if

$$\sigma_a(T) \setminus \sigma_{usbf^-}(T) = E^a(T),$$

where $E^a(T)$ is the set of all eigenvalues of $T$ which are isolated in $\sigma_a(T)$ ([2], Definition 2.13). We know that [2] generalized a-Weyl's theorem $\Rightarrow$ generalized Weyl's theorem $\Rightarrow$ Weyl’s theorem. We say $T$ obeys generalized a-Browder’s theorem if

$$\sigma_{usbf^-}(T) = \sigma_a(T) \setminus \sigma^a(T).$$

An operator is called $T$ polaroid if all isolated points of the spectrum of $T$ are poles of the resolvent of $T$ and is called isoloid if each $\lambda \in \sigma_{iso}(T)$ is an eigenvalue of $T$, where $\sigma_{iso}(T)$ is the set of isolated points of $\sigma(T)$. An operator $T$ is called a-isoloid if every $\lambda \in \sigma_{iso}(T)$ is an eigenvalue of $T$, where $\sigma_a^{iso}(T)$ is the set of isolated points of $\sigma_a(T)$. Every a-isoloid operator is isoloid but the converse is generally not true.

2. Generalized a-Weyl’s theorem for direct sums

Let $H$ and $K$ be nonzero complex Hilbert spaces. Although $T \in B(H)$ and $S \in B(K)$ satisfy generalized a-Weyl’s theorem, we do not guarantee that their orthogonal direct sum $T \oplus S$ satisfies generalized a-Weyl’s theorem.

**Example 2.1.** Let us define $S$ for each $x \in (x_i) \in l^1$ by

$$S(x_1, x_2, x_3, \ldots, x_k, \ldots) = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_k x_{k-1}, \ldots)$$

where $(\alpha_i)$ is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$. $\sigma(S) = \sigma_a(S) = \{0\}$. It can be proved that $R(S^n) \neq R(S^n)$ for any $n = 1, 2, \ldots$ Thus, $\sigma_{usbf^-}(S) = \{0\}$. Since $E^a(S) = \phi$, it follows that $S$ satisfies generalized a-Weyl’s theorem. Define $T$ on $X = l^1 \oplus l^1$ by $T = S \oplus 0$. Now $N(T) = \{0\} \oplus l^1, \sigma(T) = \sigma_a(T) = \{0\}, E^a(T) = \{0\}$. As $R(T^n) = R(S^n) \oplus \{0\}, R(T^n)$ is not closed for any $n \in \mathbb{N}$. So $T \notin USBF^-$ and $\sigma_{usbf^-}(T) = \{0\}$. Thus, $\sigma_a(T) \setminus \sigma_{usbf^-}(T) \neq E^a(T)$. Hence $T$ does not satisfy generalized a-Weyl’s theorem.

In this section we discuss certain conditions on $T$ and $S$ to ensure that generalized a-Weyl’s theorem holds for $T \oplus S$. W.Y. Lee [6] proved that if $T \in B(H)$ and $S \in B(K)$ are isoloid and satisfy Weyl’s theorem such that $\sigma_W(T \oplus S) = \sigma_W(T) \cup \sigma_W(S)$ then Weyl’s theorem holds for $T \oplus S$. We now prove the result for generalized a-Weyl’s theorem:

**Theorem 2.2.** Suppose that generalized a-Weyl’s theorem holds for $T \in B(H)$ and $S \in B(K)$. If $T$ and $S$ are a-isoloid and $\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$, then generalized a-Weyl’s theorem holds for $T \oplus S$.

**Proof.** We know $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pair of operators. If $T$ and $S$ are a-isoloid, then

$$E^a(T \oplus S) = [E^a(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap E^a(S)] \cup [E^a(T) \cap E^a(S)]$$

where $\rho_a(.) = \mathbb{C} \setminus \sigma_a(.)$. 


If generalized a-Weyl’s theorem holds for $T$ and $S$, then
\[
[\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_{usbf}-(T) \cup \sigma_{usbf}-(S)]
= [E^a(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap E^a(S)] \cup [E^a(T) \cap E^a(S)].
\]
Thus, $E^a(T \oplus S) = [\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_{usbf}-(T) \cup \sigma_{usbf}-(S)]$.

If $\sigma_{usbf}-(T \oplus S) = \sigma_{usbf}-(T) \cup \sigma_{usbf}-(S)$, then $E^a(T \oplus S) = \sigma_a(T \oplus S) \setminus \sigma_{usbf}-(T \oplus S)$. Hence generalized a-Weyl’s theorem holds for $T \oplus S$. ■

**Theorem 2.3.** Suppose $T \in B(H)$ has no isolated point in its approximate spectrum and $S \in B(K)$ satisfies generalized a-Weyl’s theorem. If $\sigma_{usbf}-(T \oplus S) = \sigma_a(T) \cup \sigma_{usbf}-(S)$, then generalized a-Weyl’s theorem holds for $T \oplus S$.

**Proof.** As $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pair of operators, we have
\[
\sigma_a(T \oplus S) \setminus \sigma_{usbf}-(T \oplus S) = [\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_a(T) \cup \sigma_{usbf}-(S)]
= \sigma_a(S) \setminus [\sigma_a(T) \cup \sigma_{usbf}-(S)]
= [\sigma_a(S) \setminus \sigma_{usbf}-(S)] \setminus \sigma_a(T)
= E^a(S) \cap \rho_a(T)
\]
where $\rho_a(T) = C \setminus \sigma_a(T)$.

Let $\sigma_a^{iso}(T)$ be the set of isolated points of $\sigma_a(T)$ and $\sigma_a^{iso}(T \oplus S)$ be the set of isolated points of $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$. If $\sigma_a^{iso}(T) \neq \emptyset$ it implies that $\sigma_a(T) = \sigma_a^{acc}(T)$, where $\sigma_a^{acc}(T) = \sigma_a(T) \setminus \sigma_a^{iso}(T)$ is the set of all accumulation points of $\sigma_a(T)$. Thus we have
\[
\sigma_a^{iso}(T \oplus S) = [\sigma_a^{iso}(T) \cup \sigma_a^{iso}(S)] \setminus [(\sigma_a^{iso}(T) \cap \sigma_a^{acc}(S)) \cup (\sigma_a^{acc}(T) \cap \sigma_a^{iso}(S))]
= (\sigma_a^{iso}(T) \setminus \sigma_a^{acc}(S)) \cup (\sigma_a^{iso}(S) \setminus \sigma_a^{acc}(T))
= \sigma_a^{iso}(S) \setminus \sigma_a(T)
= \sigma_a^{iso}(S) \cap \rho_a(T).
\]
We have that $\sigma_p(T \oplus S) = \sigma_p(T) \cup \sigma_p(S)$ for every pair of operators, therefore
\[
E^a(T \oplus S) = \sigma_a^{iso}(T \oplus S) \cap \sigma_p(T \oplus S)
= \sigma_a^{iso}(S) \cap \rho_a(T) \cap \sigma_p(S)
= E^a(S) \cap \rho_a(T).
\]
Thus, $\sigma_a(T \oplus S) \setminus \sigma_{usbf}-(T \oplus S) = E^a(T \oplus S)$. Hence $T \oplus S$ satisfies generalized a-Weyl’s theorem.

Let $\sigma_1(T)$ denote the compliment of $\sigma_{usbf}-(T)$ in $\sigma_a(T)$ i.e. $\sigma_1(T) = \sigma_a(T) \setminus \sigma_{usbf}-(T)$. A straight forward application of Theorem 2.3 leads to the following corollaries.

**Corollary 2.4.** Suppose $T \in B(H)$ is such that $\sigma_a^{iso}(T) = \emptyset$ and $S \in B(K)$ satisfies generalized a-Weyl’s theorem with $\sigma_a^{iso}(S) \cap \sigma_p(S) = \emptyset$ and $\sigma_1(T \oplus S) = \emptyset$, then $T \oplus S$ satisfies generalized a-Weyl’s theorem.
Thus, where \( \rho_b \) and \( \sigma \) are points of the spectrum of \( \sigma \)
and generalized Weyl’s theorem holds for \( \sigma \). Thus we have the following similar results of generalized Weyl’s theorem for direct sums of operators:

Corollary 2.5. Suppose \( T \in B(H) \) is such that \( \sigma_1(T) \cup \sigma_{a\text{iso}}(T) = \phi \) and \( S \in B(K) \) satisfies generalized a-Weyl’s theorem. If \( \sigma_{usbf}(T \oplus S) = \sigma_{usbf}(T) \cup \sigma_{usbf}(S) \), then generalized a-Weyl’s theorem holds for \( T \oplus S \).

Definition 2.6. An operator \( T \in B(H) \) is called left-polaroid if every isolated point of the spectrum of \( \sigma_a(T) \) is left pole of \( T \).

Theorem 2.7. Suppose generalized a-Browder’s theorem holds for \( T \in B(H) \) and \( S \in B(K) \). Suppose \( T \) and \( S \) are left-polaroid and \( \sigma_{usbf}(T \oplus S) = \sigma_{usbf}(T) \cup \sigma_{usbf}(S) \), then generalized a-Browder’s theorem holds for \( T \oplus S \).

Proof. If \( T \) and \( S \) are left-polaroid, then

\[
\pi^n(T \oplus S) = [\pi^n(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap \pi^n(S)] \cup [\pi^n(T) \cap \pi^n(S)]
\]

where \( \rho_a(.) = \mathbb{C} \setminus \sigma_a(.) \).

Since generalized a-Browder’s theorem holds for \( T \) and \( S \), we have

\[
[\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_{usbf}(T) \cup \sigma_{usbf}(S)]
= [\pi^n(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap \pi^n(S)] \cup [\pi^n(T) \cap \pi^n(S)].
\]

Thus, \( \pi^n(T \oplus S) = [\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_{usbf}(T) \cup \sigma_{usbf}(S)] \).

If \( \sigma_{usbf}(T \oplus S) = \sigma_{usbf}(T) \cup \sigma_{usbf}(S) \), then \( \pi^n(T \oplus S) = \sigma_a(T) \cup \sigma_a(S) \setminus \sigma_{usbf}(T \oplus S) \). Hence, generalized a-Browder’s theorem holds for \( T \oplus S \).

3. Generalized Weyl’s theorem for direct sums

We know that generalized a-Weyl’s theorem \( \Rightarrow \) generalized Weyl’s theorem [2]. Thus we have the following similar results of generalized Weyl’s theorem for direct sum of operators:

Theorem 3.1. Suppose that generalized Weyl’s theorem holds for \( T \in B(H) \) and \( S \in B(K) \). If \( T \) and \( S \) are isoloid and \( \sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S) \), then generalized Weyl’s theorem holds for \( T \oplus S \).

A straightforward application of Theorem 3.1 leads to the following corollary:

Corollary 3.2. Suppose \( T \in B(H) \) is an isoloid operator that satisfies generalized Weyl’s theorem, then \( T \oplus S \) satisfies generalized Weyl’s theorem whenever \( S \in B(K) \) is a normal operator.

Proof. It is shown in [4] that if \( K \) is a Hilbert space and an operator \( S \in B(K) \) satisfies \( \sigma_{BF}(S) = \sigma_{BW}(S) \), then \( \sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S) \) for every
Hilbert space $H$ and $T \in B(H)$. As $S \in B(K)$ is a normal operator we have $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$. Since every normal operator is isoloid and satisfies generalized Weyl’s theorem [1], therefore $S$ is isoloid and satisfies generalized Weyl’s theorem. Hence the required result follows from the Theorem 3.1.

**Theorem 3.3.** Suppose $T \in B(H)$ has no isolated point in its spectrum and $S \in B(K)$ satisfies generalized Weyl’s theorem. Suppose $\sigma_{BW}(T \oplus S) = \sigma(T) \cup \sigma_{BW}(S)$, then generalized Weyl’s theorem holds for $T \oplus S$.

We denote by $\sigma_0(T)$ the complement of $\sigma_{BW}(T)$ in $\sigma(T)$. We have the following consequences of the above result.

**Corollary 3.4.** Suppose $T \in B(H)$ is such that $\sigma_{iso}(T) = \phi$ and $S \in B(K)$ satisfies generalized Weyl’s theorem with $\sigma_{iso}(S) \cap \sigma_p(S) = \phi$ and $\sigma_0(T \oplus S) = \phi$, then $T \oplus S$ satisfies generalized Weyl’s theorem.

**Corollary 3.5.** Suppose $T \in B(H)$ is such that $\sigma_0(T) \cup \sigma_{iso}(T) = \phi$ and $S \in B(K)$ satisfies generalized Weyl’s theorem. If $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$, then generalized Weyl’s theorem holds for $T \oplus S$.

**Theorem 3.6.** Suppose generalized Browder’s theorem holds for $T \in B(H)$ and $S \in B(K)$. Suppose $T$ and $S$ are polaroid and $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$, then generalized Browder’s theorem holds for $T \oplus S$.

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