FIXED POINT THEOREMS FOR SOME GENERALIZED CONTRACTIVE MULTI-VALUED MAPPINGS AND FUZZY MAPPINGS

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Abstract. In this paper, first we give a theorem which generalizes the Banach contraction principle and fixed point theorems given by many authors, and then a fixed point theorem for a multi-valued \((\theta, L)\)-weak contraction. We extend the notion of \((\theta, L)\)-weak contraction to fuzzy mappings and obtain some fixed point theorems. A coincidence point theorem for a hybrid pair of mappings \(f : X \rightarrow X\) and \(T : X \rightarrow W(X)\) is established. Later on we prove a fixed point theorem for a different type of fuzzy mapping.

1. Introduction

Banach contraction principle plays a very important role in nonlinear analysis and has many generalizations (cf. [14] and the references therein). Recently, Suzuki gave a new type of generalization of the Banach contraction principle (cf. [20]). Then Kikkawa and Suzuki gave another generalization, which generalizes the work of Suzuki (cf. [20, Theorem 1]) and the Nadler fixed point theorem (cf. [16]). In [3], M. Berinde and V. Berinde extended the notion of weak contraction from single valued mappings to multi-valued mappings and obtained some convergence theorems for the Picard iteration associated with multi-valued weak contractions. As mentioned by Berinde and Berinde (cf. [3]), a lot of well-known contractive conditions considered in the literature contains \((\theta, L)\)-weak contraction as a special case. But this case, under consideration in this paper, is very general as unlike others the condition that \(\theta + L < 1\) is not required. For details one is referred to [3]. In [12], Kamran further extended the notion of weak contraction and introduced the notion of multi-valued \(f\)-weak contraction and generalized multi-valued \(f\)-weak contraction. In this paper in Theorem 3.1, we generalize the work of Kikkawa and Suzuki (cf. [14, Theorem 2]), Nadler (cf. [16]), Kamran (cf. [12, Theorem 2.9]), and Berinde and Berinde (cf. [3, Theorem 3]). In Theorem 3.4, we proved a fixed point theorem for a multi-valued \((\theta, L)\)-weak contraction defined on a nonempty
closed subset of a complete and convex metric space. In Theorem 4.1, a fixed point theorem for a \((\theta, L)\)-weak contractive fuzzy mapping is obtained which extends the result of Berinde and Berinde (cf. [3, Theorem 3]). In Theorem 4.2, a coincidence point theorem for a hybrid pair of mappings \(f : X \to X\) and \(T : X \to W(X)\); and in Theorem 4.3, a fixed point theorems for a \((\alpha, L)\)-weak contractive fuzzy mapping are obtained (definitions follow). Finally in Theorem 4.5 and in Theorem 4.7, we prove fixed point theorems for a different type of fuzzy mapping \(T : X \to K(X)\).

2. Basic definitions and lemmas

In this section first we give the following basic definitions and lemmas for multi-valued mappings, and then that for the fuzzy mappings. \((X, d)\) always represents a metric space, \(H\) represents the Hausdorff distance induced by the metric \(d\), \(CB(X)\) denotes the family of nonempty closed and bounded subsets of \(X\), and \(C(X)\) the family of nonempty compact subsets of \(X\). Let \(\mathcal{P}(X)\) be the family of all nonempty subsets of \(X\), and let \(T : X \to \mathcal{P}(X)\) be a multi-valued mapping. An element \(x \in X\) such that \(x \in T(x)\) is called a fixed point of \(T\). We denote by \(Fix(T)\) the set of all fixed points of \(T\), i.e.,

\[ Fix(T) = \{ x \in X : x \in T(x) \}. \]

Note that, \(x\) is a fixed point of a multi-valued mapping \(T\) if and only if \(d(x, T(x)) = 0\), whenever \(T(x)\) is a closed subset of \(X\).

**Lemma 2.1.** [16] Let \(A\) and \(B\) be nonempty compact subsets of a metric space \((X, d)\). If \(a \in A\), then there exists \(b \in B\) such that \(d(a, b) \leq H(A, B)\).

**Definition 2.2.** Let \((X, d)\) be a complete metric space. \(X\) is said to be (metrically) convex if \(X\) has the property that for each \(x, y \in X\) with \(x \neq y\) there exists \(z \in X\), \(x \neq z \neq y\) such that

\[ d(x, z) + d(z, y) = d(x, y). \]

**Lemma 2.3.** [5] If \(K\) is a nonempty closed subset of a complete and metrically convex metric space \((X, d)\), then for any \(x \in K\), \(y \notin K\), there exists a point \(z \in \partial K\) (the boundary of \(K\)) such that

\[ d(x, z) + d(z, y) = d(x, y). \]

**Definition 2.4.** A multi-valued mapping \(T : X \to CB(X)\) is said to be a multi-valued weak contraction or a multi-valued \((\theta, L)\)-weak contraction if and only if there exist two constants \(\theta \in [0, 1)\) and \(L \geq 0\) such that

\[ H(T(x), T(y)) \leq \theta d(x, y) + Ld(y, T(x)), \]

for all \(x, y \in X\).
Definition 2.5. Let \( f : X \to X \) and \( T : X \to CB(X) \). The mapping \( T \) is said to be a multi-valued \((f, \theta, L)\)-weak contraction if and only if there exist two constants \( \theta \in [0, 1) \) and \( L \geq 0 \) such that

\[
H(T(x), T(y)) \leq \theta d(f(x), f(y)) + Ld(f(y), T(x)),
\]

for all \( x, y \in X \).

Lemma 2.6. [16] If \( A, B \in CB(X) \) and \( x \in A \), then for each positive number \( \alpha \) there exists \( y \in B \) such that \( d(x, y) \leq H(A, B) + \alpha \), i.e., \( d(x, y) \leq qH(A, B) \) where \( q > 1 \).

Lemma 2.7. [16] Let \( \{A_n\} \) be a sequence of sets in \( CB(X) \), and suppose that \( \lim_{n \to \infty} H(A_n, A) = 0 \), where \( A \in CB(X) \). Then if \( x_n \in A_n \), \( n = 1, 2, \ldots \), and if \( \lim_{n \to \infty} x_n = x_0 \), it follows that \( x_0 \in A \).

Definition 2.8. [21] Let \((X, d)\) be a metric space, \( f : X \to X \) be a self-mapping and \( T : X \to CB(X) \) be a multi-valued mapping. The mappings \( f \) and \( T \) are called \( R \)-weakly commuting if for a given \( x \in X \), \( f(T(x)) \in CB(X) \) and there exists some real number \( R \) such that

\[
H(f(T(x)), T(f(x))) \leq Rd(f(x), T(x)).
\]

Definition 2.9. [11] The mappings \( f : X \to X \) and \( T : X \to CB(X) \) are weakly compatible if they commute at their coincidence points, i.e., if \( f(T(x)) = T(f(x)) \) whenever \( f(x) \in T(x) \).

Definition 2.10. [13] Let \( T : X \to CB(X) \). The mapping \( f : X \to X \) is said to be \( T \)-weakly commuting at \( x \in X \) if \( f(f(x)) \in T(f(x)) \).

Note that \( R \)-weakly commuting mappings commute at their coincidence points.

A real linear space \( X \) with a metric \( d \) is called a metric linear space if \( d(x + z, y + z) = d(x, y) \) and \( \alpha_n \to \alpha \), \( x_n \to x \implies \alpha_n x_n \to \alpha x \). Let \((X, d)\) be a metric linear space. A fuzzy set \( A \) in a metric linear space \( X \) is a function from \( X \) into \([0, 1]\). If \( x \in X \), the function value \( A(x) \) is called the grade of membership of \( x \) in \( A \). The \( \alpha \)-level set (or \( \alpha \)-cut set) of \( A \), denoted by \( A_\alpha \), is defined by

\[
A_\alpha = \{x : A(x) \geq \alpha\} \quad \text{if} \quad \alpha \in (0, 1], \\
A_0 = \{x : A(x) > 0\}.
\]

Here \( \overline{A} \) denotes the closure of the (non-fuzzy) set \( A \).

Definition 2.11. A fuzzy set \( A \) is said to be an approximate quantity if and only if \( A_\alpha \) is compact and convex in \( X \) for each \( \alpha \in [0, 1] \) and \( \sup_{x \in X} A(x) = 1 \).

Let \( \mathcal{F}(X) \) be the collection of all fuzzy sets in \( X \) and \( W(X) \) be a sub-collection of all approximate quantities. When \( A \) is an approximate quantity and \( A(x_0) = 1 \) for some \( x_0 \in X \), \( A \) is identified with an approximation of \( x_0 \). For \( x \in X \), let
\{x\} \in W(X) \text{ with membership function equal to the characteristic function } \chi_x \text{ of the set } \{x\}.

**Definition 2.12.** Let \( A, B \in W(X), \alpha \in [0, 1] \). Then we define

\[
p_\alpha(A, B) = \inf_{x \in A, y \in B} d(x, y), \\
p(A, B) = \sup_\alpha p_\alpha(A, B), \\
D_\alpha(A, B) = H(A_\alpha, B_\alpha), \\
D(A, B) = \sup_\alpha D_\alpha(A, B).
\]

where \( H \) is the Hausdorff distance induced by the metric \( d \).

The function \( D_\alpha(A, B) \) is called an \( \alpha \)-distance between \( A, B \in W(X) \), and \( D \) a metric on \( W(X) \). We note that \( p_\alpha \) is a non-decreasing function of \( \alpha \) and thus \( p(A, B) = p_1(A, B) \). In particular if \( A = \{x\} \), then \( p(\{x\}, B) = p_1(x, B) = d(x, B_1) \). Next we define an order on the family \( W(X) \), which characterizes the accuracy of a given quantity.

**Definition 2.13.** Let \( A, B \in W(X) \). Then \( A \) is said to be more accurate than \( B \), denoted by \( A \subset B \) (or \( B \) includes \( A \)), if and only if \( A(x) \leq B(x) \) for each \( x \in X \).

The relation \( \subset \) induces a partial order on the family \( W(X) \).

**Definition 2.14.** Let \( X \) be an arbitrary set and \( Y \) be any metric linear space. \( F \) is called a fuzzy mapping if and only if \( F \) is a mapping from the set \( X \) into \( W(Y) \).

**Definition 2.15.** For \( F : X \to W(X) \), we say that \( x_0 \in X \) is a fixed point of \( F \) if \( \{x_0\} \subset F(x_0) \), i.e. if \( x_0 \in F(x_0) \).

**Lemma 2.16.** [10] Let \( x \in X \) and \( A \in W(X) \). Then \( \{x\} \subset A \) if and only if \( p_\alpha(x, A) = 0 \) for each \( \alpha \in [0, 1] \).

**Remark 2.17.** Note that from the above lemma it follows that for \( A \in W(X) \), \( \{x\} \subset A \) if and only if \( p(\{x\}, A) = 0 \). If no confusion arises instead of \( p(\{x\}, A) \), we will write \( p(x, A) \).

**Lemma 2.18.** [10] \( p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A) \) for each \( x, y \in X \).

**Lemma 2.19.** [10] If \( \{x_0\} \subset A \), then \( p_\alpha(x_0, B) \leq D_\alpha(A, B) \) for each \( B \in W(X) \).

**Lemma 2.20.** [15] Let \((X, d)\) be a complete metric linear space, \( F : X \to W(X) \) be a fuzzy mapping and \( x_0 \in X \). Then there exists \( x_1 \in X \) such that \( \{x_1\} \subset F(x_0) \).

**Remark 2.21.** Let \( f : X \to X \) be a self map and \( T : X \to W(X) \) be a fuzzy mapping such that \( \cup \{T(X)\}_\alpha \subseteq f(X) \) for each \( \alpha \in [0, 1] \). Then from Lemma 2.20,
it follows that for any chosen point \( x_0 \in X \) there exist points \( x_1, y_1 \in X \) such that \( y_1 = f(x_1) \) and \( \{y_1\} \subset T(x_0) \). Here \( T(x)_\alpha = \{y \in X : T(x)(y) \geq \alpha\} \).

**Definition 2.22.** Let \( f : X \to X \) be a self mapping and \( T : X \to W(X) \) be a fuzzy mapping. Then a point \( x \) is said to be a coincidence point of \( f \) and \( T \) if \( \{f(u)\} \subset T(u) \), i.e., if \( f(u) \in T(u)_1 \).

**Definition 2.23.** A fuzzy mapping \( T : X \to W(X) \) is said to be a weak contraction or a \((\theta, L)\)-weak contraction if and only if there exist two constants \( \theta \in [0, 1) \) and \( L \geq 0 \) such that
\[
D(T(x), T(y)) \leq \theta d(x, y) + Lp(y, T(x)),
\]
for all \( x, y \in X \).

**Definition 2.24.** A fuzzy mapping \( T : X \to F(X) \) is said to be a weak contraction or a \((\theta, L)\)-weak contraction if and only if there exist two constants \( \theta \in [0, 1) \) and \( L \geq 0 \) such that
\[
H(T(x)_{\alpha(x)}, T(y)_{\alpha(y)}) \leq \theta d(x, y) + Ld(y, T(x)_{\alpha(x)}),
\]
for all \( x, y \in X \) where \( T(x)_{\alpha(x)}, T(y)_{\alpha(y)} \) are in \( CB(X) \).

**Definition 2.25.** For a complete metric linear space \( X \), let \( f : X \to X \) be a self mapping and \( F : X \to W(X) \) a fuzzy mapping. \( T \) is said to be a \( f \)-weak contraction or a \((f, \theta, L)\)-weak contraction if and only if there exist two constants \( \theta \in [0, 1) \) and \( L \geq 0 \) such that
\[
D(T(x), T(y)) \leq \theta d(f(x), f(y)) + Lp(f(y), T(x)).
\]

**Definition 2.26.** A fuzzy mapping \( T : X \to W(X) \) is said to be a generalized \((\alpha, L)\)-weak contraction if there exists a function \( \alpha : [0, +\infty) \to [0, 1) \) satisfying \( \limsup_{r \to t+} \alpha(r) < 1 \) for every \( t \in [0, +\infty) \), such that
\[
D(T(x), T(y)) \leq \alpha(d(x, y))d(x, y) + Lp(y, T(x)),
\]
for all \( x, y \in X \) and \( L \geq 0 \).

**Lemma 2.27.** \([17]\) Let \( A \) be a subset of \( X \). Let \( \{A_\alpha : \alpha \in [0, 1]\} \) be a family of subsets of \( A \) such that
\begin{enumerate}
  \item \( A_0 = A \),
  \item \( \alpha \leq \beta \) implies \( A_\beta \subseteq A_\alpha \),
  \item \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \), \( \lim_{n \to \infty} \alpha_n = \alpha \) implies \( A_\alpha = \bigcup_{k=1}^n A_{\alpha_k} \).
\end{enumerate}

Then the function \( \phi : X \to I \) defined by \( \phi(x) = \sup\{\alpha \in I : x \in A_\alpha\} \) has the property that \( A_\alpha = \{x \in X : \phi(x) \geq \alpha\} \).

Conversely, in any fuzzy set \( \mu \) in \( X \) the family of \( \alpha \)-level sets of \( \mu \) satisfies the above conditions from (i) to (iii).

The function \( \phi \) in the above lemma is actually defined on the set \( A \), but we can extend it to \( X \) by defining \( \phi(x) = 0 \) for all \( x \in X - A \). This lemma is known as Negoite-Ralescu representation theorem.
3. Multi-valued mappings

In this section we prove all the main theorems of this paper regarding multi-valued mappings. Theorem 3.1 gives a generalization of Banach contraction principle. In Theorem 3.2 we have stated and proved a further generalization of Theorem 3.1 and Banach contraction theorem, and Theorem 3.4 concerns a multi-valued non-self weak contraction and its fixed point. In proving the existence of a fixed point of such a mapping, we follow the technique of Assad and Kirk (cf. [5]). Our theorems extend the results of several authors.

**Theorem 3.1.** Let \( (X, d) \) be a complete metric space and let \( T : X \to CB(X) \). Suppose that there exists two constants \( \theta \in [0, 1) \) and \( L \geq 0 \) such that

\[
\eta(\theta)d(x, T(x)) \leq d(x, y) \quad \text{implies} \quad H(T(x), T(y)) \leq \theta d(x, y) + Ld(y, T(x))
\]

for all \( x, y \in X \), where \( \eta : [0, 1) \to \left( \frac{1}{2+L}, \frac{1}{1+L} \right] \) defined by \( \eta(\theta) = \frac{1}{1+\theta} \) is a strictly decreasing function. Then

(i) there exists \( z \in X \) such that \( z \in T(z) \), i.e., \( \text{Fix}(T) \neq \emptyset \);

(ii) for any point \( x_0 \in X \), there exists an orbit \( \{x_n\} \) of \( T \) at \( x_0 \) with \( x_{n+1} \in T(x_n) \) such that \( \{x_n\} \) converges to a fixed point \( z \) of \( T \) for which the following estimates hold:

\[
d(x_n, z) \leq \frac{h^n}{1-h}d(x_0, x_1) \quad \text{for} \quad n = 0, 1, 2, \ldots
\]

and

\[
d(x_n, z) \leq \frac{h}{1-h}d(x_{n-1}, x_n) \quad \text{for} \quad n = 1, 2, \ldots
\]

for some \( h < 1 \).

**Proof.** (i) Suppose \( q > 1 \). We select a sequence \( \{x_n\} \) in \( X \) in the following way. Let \( x_0 \in X \) and \( x_1 \in T(x_0) \). Then we have

\[
\eta(\theta)d(x_0, T(x_0)) \leq \eta(\theta)d(x_0, x_1) \leq d(x_0, x_1).
\]

Hence from the given hypothesis we have,

\[
H(T(x_0), T(x_1)) \leq \theta d(x_0, x_1) + Ld(x_1, T(x_0)) = \theta d(x_0, x_1).
\]

There exists a point \( x_2 \in T(x_1) \) such that

\[
d(x_1, x_2) \leq qH(T(x_0), T(x_1)) \leq q[\theta d(x_0, x_1) + Ld(x_1, T(x_0))] \leq q\theta d(x_0, x_1).
\]

Since the above inequality is valid for any \( q \geq 1 \), we choose \( q > 1 \) such that \( h = q\theta < 1 \) for any \( \theta \in [0, 1) \). Thus, \( d(x_1, x_2) \leq hd(x_0, x_1) \).

Let \( x_3 \in T(x_2) \) be such that \( d(x_2, x_3) \leq qH(T(x_1), T(x_2)) \). Note that \( \eta(\theta)d(x_1, T(x_1)) \leq \eta(\theta)d(x_1, x_2) \leq d(x_1, x_2) \), and so by the given hypothesis,

\[
H(T(x_1), T(x_2)) \leq \theta d(x_1, x_2) + Ld(x_2, T(x_1)) = \theta d(x_1, x_2).
\]

Hence we have,

\[
d(x_2, x_3) \leq hd(x_1, x_2).
\]

Proceeding in this way we can obtain a sequence \( \{x_n\} \) in \( X \) such that \( d(x_{n-1}, x_n) \leq hd(x_{n-2}, x_{n-1}) \). It can easily be shown that \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete let \( x_n \to z \in X \).
Next we show that $d(z, T(x)) \leq \theta d(z, x) + Ld(x, z)$ for all $x \in X \setminus \{z\}$. Since $x_n \to z$, for $x \in X \setminus \{z\}$ there exists $\nu \in \mathbb{N}$ such that $d(z, x_n) \leq \frac{1}{3} d(z, x)$ for all $n \in \mathbb{N}$ with $n \geq \nu$. Then we have,

$$\eta(\theta)d(x_n, T(x_n)) \leq d(x_n, T(x_n)) \leq d(x_n, x_{n+1}) \leq d(x_n, z) + d(z, x_{n+1}) \leq \frac{1}{3} d(z, x) + \frac{1}{3} d(x, z) = \frac{2}{3} d(x, z) = d(x, z) - \frac{1}{3} d(x, z) \leq d(x, z) - d(x, z) \leq d(x_n, z) \leq d(x, x_n) + d(x_n, z) - d(x_n, z) = d(x, x_n),$$

i.e., $\eta(\theta)d(x_n, T(x_n)) \leq d(x_n, x)$ for $n \geq \nu$, which implies $H(T(x_n), T(x)) \leq \theta d(x_n, x) + Ld(x, T(x_n))$ for $n \geq \nu$. For $n \geq \nu$, this implies

$$d(x_{n+1}, T(x)) \leq \theta d(x_n, x) + Ld(x, T(x_n)) \leq \theta d(x_n, x) + Ld(x, x_{n+1}).$$

Taking $n \to \infty$ we have, $d(z, T(x)) \leq \theta d(z, x) + Ld(x, z)$ for all $x \in X \setminus \{z\}$. Next we show that

$$H(T(x), T(z)) \leq \theta d(x, z) + Ld(z, T(x)) \text{ for all } x \in X. \tag{1}$$

Equation (1) is satisfied when $x = z$. Now we take $x \neq z$. For every $n \in \mathbb{N}$ there exists $y_n \in T(x)$ such that $d(z, y_n) \leq d(z, T(x)) + \frac{1}{n} d(x, z)$ as $d(z, T(x)) = \inf_{y \in T(x)} d(z, y)$. Consider the following

$$d(x, T(x)) \leq d(x, y_n) \leq d(x, z) + d(z, y_n) \leq d(x, z) + d(z, T(x)) + \frac{1}{n} d(x, z) \leq d(x, z) + (\theta + L)d(x, z) + \frac{1}{n} d(x, z) = (1 + \theta + L + \frac{1}{n})d(x, z).$$

Dividing both sides by $1 + \theta + L$ we have,

$$\frac{1}{1 + \theta + L}d(x, T(x)) \leq (1 + \frac{1}{n(1 + \theta + L)})d(x, z),$$

for any $n$, and hence $\eta(\theta)d(x, T(x)) \leq d(x, z)$. Then by the given hypothesis, $H(T(x), T(z)) \leq \theta d(x, z) + Ld(z, T(x))$ is satisfied for all $x \in X$. Now we have,

$$d(z, T(z)) = \lim_{n \to \infty} d(x_{n+1}, T(z)) \leq \lim_{n \to \infty} H(T(x_n), T(z)) \leq \lim_{n \to \infty} \{\theta d(x_n, z) + Ld(z, T(x_n))\} \leq \lim_{n \to \infty} \{\theta d(x_n, z) + L[d(z, x_{n+1}) + d(x_{n+1}, T(x_n))]\} = 0,$$

which implies $d(z, T(z)) = 0$, and hence $z \in T(z)$, i.e., $FixT \neq \emptyset$ as $T(z)$ is closed.
To prove (ii) let us proceed as follows: The sequence \( \{x_n\} \) obtained in the proof of (i) are such that \( x_{n+1} \in T(x_n) \) for \( n \geq 0 \) and satisfies
\[
d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \leq h^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq h^n d(x_0, x_1).
\]
Also we have
\[
d(x_{n+k}, x_{n+k+1}) \leq h^{k+1} d(x_{n-1}, x_n) \text{ for any } k \geq 0.
\]
Using the above inequalities we have
\[
d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})
\]
\[
\leq h^n d(x_0, x_1) + h^{n+1} d(x_0, x_1) + \cdots + h^{n+p-1} d(x_0, x_1)
\]
\[
= h^n \frac{(1 - h^p)}{1 - h} d(x_0, x_1), \tag{2}
\]
and
\[
d(x_n, x_{n+p}) \leq h d(x_{n-1}, x_n) + h^2 d(x_{n-1}, x_n) + \cdots + h^p d(x_{n-1}, x_n)
\]
\[
= h \frac{(1 - h^p)}{1 - h} d(x_{n-1}, x_n). \tag{3}
\]
Taking \( p \to \infty \), and noting the fact that \( \lim_{n \to \infty} d(x_n, x_{n+p}) = d(x_n, z) \) and \( \lim_{p \to \infty} h^p = 0 \), from (2) and (3) we obtain the assertion (ii) of the Theorem. \( \blacksquare \)

The above theorem is a generalization of Theorem 2 of Kikkawa and Suzuki (cf. [14]) which is obtained when \( L = 0 \). It is also a generalization of Theorem 3 of Berinde and Berinde (cf. [3]).

**Corollary 3.1.1.** [14, Theorem 2] Define a strictly decreasing function \( \eta \) from \([0, 1]\) onto \([\frac{1}{2}, 1]\) by \( \eta(r) = \frac{1}{1 + r} \). Let \((X, d)\) be a complete metric space and let \(T\) be a mapping from \(X\) into \(CB(X)\). Assume that there exists \( r \in [0, 1) \) such that
\[
\eta(r)d(x, T(x)) \leq d(x, y) \text{ implies } H(T(x), T(y)) \leq rd(x, y)
\]
for all \( x, y \in X \). Then there exists \( z \in X \) such that \( z \in T(z) \).

**Corollary 3.1.2.** (Nadler [16]) Let \((X, d)\) be a complete metric space and let \(T\) be a mapping from \(X\) into \(CB(X)\). If there exists \( r \in [0, 1) \) such that
\[
H(Tx, Ty) \leq rd(x, y) \text{ for all } x, y \in X,
\]
then there exists \( z \in X \) such that \( z \in Tz \).

**Proof.** Given that \( T \) satisfies the condition of Nadler’s theorem, i.e.,
\[
H(Tx, Ty) \leq rd(x, y) \text{ for all } x, y \in X \text{ and } r \in [0, 1), \tag{4}
\]
we have to prove that there exists \( z \in X \) such that \( z \in T(z) \).
For any \( x \in X, \ y \in T(x) \) we have \( d(y, T(y)) \leq H(T(x), T(y)) \). Hence by (4) we have,
\[
d(y, T(y)) \leq H(T(x), T(y)) \leq rd(x, y),
\]
i.e., \( \eta(r)d(y, T(y)) \leq rd(x, y) \), i.e.
\[
\eta(r)d(x, T(x)) \leq d(y, x) = d(x, y) \tag{5}
\]
as \( r < 1 \) and \( \eta(r) < 1 \). Hence by (4), (5) and Theorem 3.1 for \( L = 0 \), it follows that there exists \( z \in X \) such that \( z \in T(z) \). □

**Theorem 3.2.** Let \((X, d)\) be a metric space, \( T : X \to CB(X) \) and \( f : X \to X \). Suppose that there exist two constants \( \theta \in [0, 1) \) and \( L \geq 0 \) such that
\[
\eta(\theta)d(f(x), T(x)) \leq d(f(x), f(y)) \quad \text{implies} \quad H(T(x), T(y)) \leq \theta d(f(x), f(y)) + Ld(f(y), T(x))
\]
for all \( x, y \in X \), where \( \eta : [0, 1) \to \left[ \frac{1}{2+L}, \frac{1}{1+L} \right] \) defined by \( \eta(\theta) = \frac{1}{1+\theta+L} \) is a strictly decreasing function, \( T(X) \subset f(X) \) and \( f(X) \) is complete. Then
(i) the set of coincidence point of \( f \) and \( T \), \( C(f, T) \) is nonempty.
(ii) for any \( x_0 \in X \), there exists an \( f \)-orbit \( O_f(x_0) = \{ f(x_n) : n = 1, 2, 3 \ldots \} \)
of \( T \) at the point \( x_0 \) such that \( f(x_n) \to f(u) \), where \( u \) is a coincidence point of \( f \) and \( T \), for which the following estimates hold:
\[
d(f(x_n), f(u)) \leq \frac{h^n}{1-h}d(f(x_0), f(x_1)), \quad n = 0, 1, 2, \ldots,
\]
\[
d(f(x_n), f(u)) \leq \frac{h^n}{1-h}d(f(x_{n-1}), f(x_n)), \quad n = 1, 2, \ldots
\]
for a certain constant \( h < 1 \). Further, if \( f \) is \( R \)-weakly commuting at \( u \) and \( f(f(u)) = f(u) \), then \( f \) and \( T \) have a common fixed point.

**Proof.** Let \( x_0 \in X \), and \( x_1 \in X \) such that \( f(x_1) \in T(x_0) \). Then
\[
\eta(\theta)d(f(x_0), T(x_0)) \leq d(f(x_0), f(x_1)), \quad \text{and so by the given hypothesis we have}
\]
\[
H(T(x_0), T(x_1)) \leq \theta d(f(x_0), f(x_1)) + Ld(f(x_1), T(x_0)) = \theta d(f(x_0), f(x_1))
\]
Let \( x_2 \in X \) be such that \( f(x_2) \in T(x_1) \) and then
\[
d(f(x_1), f(x_2)) \leq qH(T(x_0), T(x_1)) \leq \theta d(f(x_0), f(x_1)), \quad \text{where} \ q > 1 \text{ and } q \text{ is chosen in such a way that} \ h = q\theta < 1.
\]
Now
\[
\eta(\theta)d(f(x_1), T(x_1)) \leq \eta(\theta)d(f(x_1), f(x_2)) \leq d(f(x_1), f(x_2)),
\]
which implies
\[
H(T(x_1), T(x_2)) \leq \theta d(f(x_1), f(x_2)) + Ld(f(x_2), T(x_1)) = \theta d(f(x_1), f(x_2))
\]
Let \( x_3 \in X \) be such that \( f(x_3) \in T(x_2) \) and then
\[
d(f(x_2), f(x_3)) \leq qH(T(x_1), T(x_2)) \leq \theta d(f(x_1), f(x_2)) \leq h^2d(f(x_0), f(x_1)). \quad \text{Proceeding in this way we obtain a sequence} \ \{ f(x_n) \} \in X. \text{ It can easily be shown that the sequence} \ \{ f(x_n) \} \text{ is a Cauchy sequence in} \ X. \text{ Since} \ f(X) \text{ is complete, the sequence converges to some
point \( f(u) \in f(X) \). So there exists a positive integer \( \nu \) such that for all \( x \in X \setminus \{u\} \) we have \( d(f(x_n), f(u)) \leq \frac{1}{3} d(f(x), f(u)) \) for \( n \geq \nu \). Then for \( n \geq \nu \) we can write

\[
\eta(\theta)d(f(x_n), T(x_n)) \leq d(f(x_n), T(x_n)) \leq d(f(x_n), f(x_{n+1})) \\
\leq d(f(x_n), f(u)) + d(f(u), f(x_{n+1})) \\
\leq \frac{1}{3} d(f(x), f(u)) + \frac{1}{3} d(f(x), f(u)) \\
= d(f(x), f(u)) - \frac{1}{3} d(f(x), f(u)) \\
\leq d(f(x), f(u)) - d(f(x_n), f(u)) \\
\leq d(f(x), f(x_n)) + d(f(x_n), f(u)) - d(f(x_n), f(u)) \\
= d(f(x_n), f(x)).
\]

Hence from the given hypothesis it follows that

\[
H(T(x_n), T(x)) \leq \theta d(d(x_n), f(x)) + Ld(f(x), T(x_n)).
\]

This implies for any \( n \geq \nu \),

\[
d(f(x_{n+1}), T(x)) \leq H(T(x_n), T(x)) \\
\leq \theta d(f(x_n), f(x)) + Ld(f(x), T(x_n)) \\
\leq \theta d(f(x_n), f(x)) + Ld(f(x), f(x_{n+1})) + Ld(f(x_{n+1}), T(x_n)) \\
= \theta d(f(x_n), f(x)) + Ld(f(x), f(x_{n+1})).
\]

Hence taking \( n \to \infty \) we have \( d(f(u), T(x)) \leq \theta d(d(u), f(x)) + Ld(f(x), f(u)) = (\theta + L)d(f(x), f(u)) \) for each \( x \in X \setminus \{u\} \). Next we show

\[
H(T(x), T(u)) \leq \theta d(f(x), f(u)) + Ld(f(u), T(x)) \tag{6}
\]

for all \( x \in X \). It is true if \( x = u \). Suppose \( x \neq u \). Since \( d(f(u), T(x)) = \inf_{v \in T(x)} d(f(u), v) \) for each \( n \in \mathbb{N} \) we can obtain a sequence \( \{v_n\} \) in \( T(x) \) such that \( d(f(u), v_n) \leq d(f(u), T(x)) + \frac{1}{n} d(f(x), f(u)) \) for each \( n \in \mathbb{N} \). Hence for \( x \neq u \) we have

\[
d(f(x), T(x)) \leq d(f(x), v_n) \leq d(f(x), f(u)) + d(f(u), v_n) \\
\leq d(f(x), f(u)) + d(f(u), T(x)) + \frac{1}{n} d(f(x), f(u)) \\
\leq d(f(x), f(u)) + (\theta + L)d(f(x), f(u)) + \frac{1}{n} d(f(x), f(u)) \\
= (1 + \theta + L + \frac{1}{n})d(f(x), f(u)).
\]

and so \( \frac{1}{1 + \theta + L} d(f(x), T(x)) \leq (1 + \frac{1}{1 + \theta + L}) d(f(x), f(u)) \) for any \( n \), and hence

\[
\eta(\theta)d(f(x), T(x)) \leq d(f(x), f(u)),
\]
which implies \( H(T(x), T(u)) \leq \theta d(f(x), f(u)) + Ld(f(u), T(x)) \), and hence (6) is proved. Now

\[
d(f(u), T(u)) = \lim_{n \to \infty} d(f(x_{n+1}), T(u)) \leq \lim_{n \to \infty} H(T(x_n), T(u))
\]

\[
\leq \lim_{n \to \infty} [\theta d(f(x_n), f(u)) + Ld(f(u), T(x_n))]
\]

\[
\leq \lim_{n \to \infty} [\theta d(f(x_n), f(u)) + Ld(f(u), f(x_{n+1}) + d(f(x_{n+1}), T(x_n))]
\]

\[= 0,
\]

and hence \( f(u) \in T(u) \) which completes the proof of (i).

To prove (ii) let us proceed as follows: The sequence \( \{f(x_n)\} \) obtained in the proof of (i) are such that \( f(x_{n+1}) \in T(x_n) \) for \( n \geq 0 \) and satisfies

\[
d(f(x_n), f(x_{n+1})) \leq h d(f(x_{n-1}), f(x_n)) \leq h^2 d(f(x_{n-2}), f(x_{n-1}))
\]

\[
\leq \cdots \leq h^n d(f(x_0), f(x_1)).
\]

Also we have

\[
d(f(x_{n+k}), f(x_{n+k+1})) \leq h^{k+1} d(f(x_{n-1}), f(x_n)) \text{ for any } k \geq 0 \text{ and } n \geq 1.
\]

Using the above inequalities we have

\[
d(f(x_n), f(x_{n+p}))
\]

\[
\leq d(f(x_n), f(x_{n+1})) + d(f(x_{n+1}), f(x_{n+2})) + \cdots + d(f(x_{n+p-1}), f(x_{n+p}))
\]

\[
\leq h^n d(f(x_0), f(x_1)) + h^{n+1} d(f(x_0), f(x_1)) + \cdots + h^{n+p-1} d(f(x_0), f(x_1))
\]

\[
= h^n \frac{(1 - h^p)}{1 - h} d(f(x_0), f(x_1)), \tag{7}
\]

and

\[
d(f(x_n), f(x_{n+p}))
\]

\[
\leq d(f(x_n), f(x_{n+1})) + d(f(x_{n+1}), f(x_{n+2})) + \cdots + d(f(x_{n+p-1}), f(x_{n+p}))
\]

\[
\leq h d(f(x_{n-1}), f(x_n)) + h^2 d(f(x_{n-1}), f(x_n)) + \cdots + h^p d(f(x_{n-1}), f(x_n))
\]

\[
= h(1 - h^p) \frac{1}{1 - h} d(f(x_{n-1}), f(x_n)). \tag{8}
\]

Taking \( p \to \infty \), and noting the fact that \( \lim_{n \to \infty} d(f(x_n), f(x_{n+p})) = d(f(x_n), f(u)) \) and \( \lim_{p \to \infty} h^p = 0 \), from (7) and (8) we obtain the assertion (ii) of the Theorem.

If \( f \) is \( R \)-weakly commuting at \( u \) we have \( H(f(T(u)), T(f(u))) \leq Rd(f(u), T(u))) \). As \( f(u) \in T(u) \) this implies \( f(T(u)) = T(f(u)) \). Again \( f(f(u)) = f(u) \), and so \( f(u) \in T(u) \) implies \( f(f(u)) \in f(T(u)) = T(f(u)) \), i.e., \( f(u) \in T(f(u)) \). Hence, \( f(u) \) is a fixed point of both \( f \) and \( T \), i.e., \( f \) and \( T \) have a common fixed point.

**Remark 3.3.** In Definition 2.9 we need \( f(T(x)) \in CB(X) \). If \( f \) is continuous and \( T(x) \in C(X) \), then \( f(T(x)) \) also belongs to \( C(X) \). The above theorem is
a generalization of Theorem 3.1, since by taking \( f \) as the identity mappings in Theorem 3.2 we obtain Theorem 3.1. It is easy to see that the map \( f : X \to X \) is \( T \)-weakly commuting at a coincidence point of \( f \) and \( T \). Hence Theorem 3.2 is generalization of Theorem 2.9 of Kamran (cf. [12]). In some sense the above theorem is also a generalization of Theorem 3 of Kikkawa and Suzuki (cf. [14]) in two directions: The mapping \( T \) is multi-valued and we have an additional term in the second inequality. If we take \( L = 0 \) and \( T : X \to X \) (single-valued), then we get Theorem 3 of [14] without the continuity condition on the mapping \( f \), but with an additional condition that \( f(X) \) is complete. If \( X \) is assumed to be compact then \( f(X) \) is compact when \( f \) is continuous.

**Theorem 3.4.** Let \( K \) be a nonempty closed subset of a complete and convex metric space \((X, d)\) and \( T : K \to CB(X) \) be a multi-valued \((\theta, L)\)-weak contraction (see Definition 2.4). If \( T(x) \subset K \) for each \( x \in \partial K \) (the boundary of \( K \)), then \( T \) has a fixed point.

**Proof.** We select a sequence \( \{x_n\} \) in the following way. Let \( x_0 \in K \) and \( x'_1 \in T(x_0) \). If \( x'_1 \in K \) let \( x_1 = x'_1 \); otherwise select a point \( x_1 \in \partial K \) s.t. \( d(x_0, x_1) + d(x_1, x'_1) = d(x_0, x_1') \). Thus \( x_1 \in K \) and by Lemma 2.6 we can choose a point \( x'_2 \in T(x_1) \) so that \( d(x'_1, x'_2) \leq H(T(x_0), T(x_1)) + \theta \), where \( \theta < 1 \). Now put \( x'_2 = x_2 \) if \( x'_2 \in K \), otherwise let \( x_2 \) be a point of \( \partial K \) such that \( d(x_1, x_2) + d(x_2, x'_2) = d(x_1, x'_2) \).

By induction we can obtain a sequence \( \{x_n\} \), \( \{x'_n\} \) such that for \( n = 1, 2, 3, \ldots \)

(i) \( x'_{n+1} \in T(x_n) \)
(ii) \( d(x'_n, x'_{n+1}) \leq H(T(x_{n-1}), T(x_n)) \) where
(iii) \( x'_{n+1} = x_{n+1} \) if \( x'_{n+1} \in K \), or
(iv) \( d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1}) \) if \( x_{n+1} \notin K \). Now let
\[
\begin{align*}
P &= \{x_i \in \{x_n\} : x_i = x'_i, i = 1, 2, \ldots \} \\
Q &= \{x_i \in \{x_n\} : x_i \neq x'_i, i = 1, 2, \ldots \}.
\end{align*}
\]
Observe that if \( x_n \in Q \) for some \( n \), then \( x_{n+1} \in P \). Now for \( n \geq 2 \) we estimate the distance \( d(x_n, x_{n+1}) \). There arises three cases:

Case 1. The case that \( x_n \in P \) and \( x_{n+1} \in P \). In this case we have,
\[
\begin{align*}
d(x_n, x_{n+1}) &= d(x'_n, x'_{n+1}) \\
&\leq \theta d(x_{n-1}, x_n) + Ld(x_{n-1}, T(x_{n-1})) + \theta^n \\
&\leq \theta d(x_{n-1}, x_n) + \theta^n.
\end{align*}
\]

Case 2. The case that \( x_n \in P \) and \( x_{n+1} \in Q \). In this case we use (iv) and proceeding in the same way as Case 1 we obtain,
\[
d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) = d(x'_n, x'_{n+1}) \leq H(T(x_{n-1}), T(x_n)) \leq \theta d(x_{n-1}, x_n) + \theta^n.
\]

Case 3. The case that \( x_n \in Q \) and \( x_{n+1} \in P \). From the construction of the sequence \( \{x_n\} \) it is clear that two consecutive terms of \( \{x_n\} \) can not be in \( Q \), and hence \( x_{n-1} \in P \) and \( x'_{n-1} = x_{n-1} \). Using this below we obtain,
\[
d(x_n, x_{n+1}) \leq d(x_n, x_n) + d(x'_n, x_{n+1})
\]
\[ \begin{align*}
= d(x_n, x_n') + d(x_n', x_{n+1}') \\
\leq d(x_n, x_n') + H(T(x_{n-1}), T(x_n)) + \theta^n \\
\leq d(x_n, x_n') + \theta d(x_{n-1}, x_n) + \theta^n \\
\leq d(x_n, x_n') + d(x_{n-1}, x_n) + \theta^n \\
= d(x_{n-1}, x_n') + \theta^n = d(x_{n-1}', x_n') + \theta^n \\
\leq H(T(x_{n-2}), T(x_{n-1})) + \theta^{n-1} + \theta^n \\
\leq \theta d(x_{n-2}, x_{n-1}) + \theta^{n-1} + \theta^n.
\end{align*} \]

The only other possibility, \( x_n \in Q, x_{n+1} \in Q \) can not occur. Thus for \( n \geq 2 \) we have

\[ d(x_n, x_{n+1}) \leq \begin{cases} 
\theta d(x_{n-1}, x_n) + \theta^n, & \text{or} \\
\theta d(x_{n-2}, x_{n-1}) + \theta^{n-1} + \theta^n
\end{cases} \quad (9) \]

Let \( \delta = \theta^{-1/2} \max\{d(x_0, x_1), d(x_1, x_2)\} \). Now as in [5], it can be we proved that for
\( n \geq 1, \)

\[ d(x_n, x_{n+1}) \leq \theta^{n/2}(\delta + n). \quad (10) \]

From (10) it follows that

\[ d(x_k, x_N) \leq \delta \sum_{i=N}^{\infty} (\theta^{1/2})^i + \sum_{i=N}^{\infty} i(\theta^{1/2})^i, \quad k > N \geq 1. \]

This implies \( \{x_n\} \) is a Cauchy sequence in \( K \), and since \( X \) is complete and \( K \) is closed, \( \{x_n\} \) converges to a point in \( K \). Let \( u = \lim_{n \to \infty} x_n \). Hence there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) each of whose terms is in the set \( P \) (i.e., \( x_{n_k} = x_{n_k}' \) for \( k = 1, 2, \ldots \)). Thus by (i), \( x_{n_k}' \in T(x_{n_k}-1) \) for \( k = 1, 2, \ldots \), and since \( x_{n_k-1} \to u \) as \( k \to \infty \) we have \( T(x_{n_k-1}) \to T(u) \) as \( k \to \infty \) in the Hausdorff metric. Hence it follows from Lemma 2.7 that \( u \in T(u) \), i.e., \( T \) has a fixed point, which completes the proof. \( \blacksquare \)

4. Fuzzy mappings

Many authors considered the class of fuzzy sets with nonempty compact \( \alpha \)-cut sets in a metric space or nonempty compact convex \( \alpha \)-cut sets in a metric linear space, but some have given attention to class of fuzzy sets with nonempty closed and bounded \( \alpha \)-cut sets in a metric space. Theorems 4.1–4.3 deal with fuzzy mappings with \( \alpha \)-cut sets as nonempty, compact and convex subsets of \( X \). Next following the work in [2, 7, 22], we present Theorem 4.5 and Theorem 4.7 concerning a different kind of fuzzy mappings with special \( \alpha \)-cut sets as nonempty, closed and bounded subsets of \( X \).

**Theorem 4.1.** Let \( (X, d) \) be a complete metric linear space and \( T : X \to W(X) \) be a \((\theta, L)\)-weak contractive fuzzy mapping (see Definition 2.24). Then

(i) \( \text{Fix}(T) \neq \emptyset; \)
(ii) For any \( x_0 \in X \), there exists an orbit \( \{x_n\}_{n=0}^{\infty} \) of \( T \) at the point \( x_0 \) that converges to a fixed point \( u \) of \( T \), for which the following estimates hold:

\[
d(x_n, u) \leq \frac{\theta^n}{1-\theta} d(x_0, x_1), \quad n = 0, 1, 2, \ldots,
\]

\[
d(x_n, u) \leq \frac{\theta}{1-\theta} d(x_{n-1}, x_n), \quad n = 1, 2, \ldots.
\]

Proof. Let \( x_0 \in X \). Then there exists \( x_1 \in X \) such that \( \{x_1\} \subset T(x_0) \). If \( D(T(x_0), T(x_1)) = 0 \), then \( T(x_0) = T(x_1) \), i.e., \( \{x_1\} \subset T(x_1) \), which actually means that \( \text{Fix}(T) \neq \emptyset \). Let \( D(T(x_0), T(x_1)) \neq 0 \). Then by Lemmas 2.20 and 2.21, we can find \( x_2 \in X \) such that \( \{x_2\} \subset T(x_1) \) and

\[
d(x_1, x_2) \leq H(T(x_0)_{11}, T(x_1)_{11}) = D_1(T(x_0), T(x_1))
\]

\[
\leq D(T(x_0), T(x_1)) \leq \theta d(x_0, x_1) + L \rho(x_1, T(x_0))
\]

\[
\leq \theta d(x_0, x_1).
\]

If \( D(T(x_1), T(x_2)) = 0 \) then \( T(x_1) = T(x_2) \), i.e., \( \{x_2\} \subset T(x_2) \). Otherwise, we assume \( D(T(x_1), T(x_2)) \neq 0 \) and \( x_3 \in X \) such that \( \{x_3\} \subset T(x_2) \) and

\[
d(x_2, x_3) \leq H(T(x_1)_{11}, T(x_2)_{11}) = D_1(T(x_1), T(x_2))
\]

\[
\leq D(T(x_1), T(x_2)) \leq \theta d(x_1, x_2) + L \rho(x_2, T(x_1))
\]

\[
\leq \theta d(x_1, x_2).
\]

In this manner, we obtain an orbit \( \{x_n\}_{n=0}^{\infty} \) at \( x_0 \) for \( T \) satisfying

\[
d(x_n, x_{n+1}) \leq \theta d(x_{n-1}, x_n), \quad n = 1, 2, \ldots
\]

(11)

From (11) we obtain inductively,

\[
d(x_n, x_{n+1}) \leq \theta^n d(x_0, x_1) \quad \text{and} \quad d(x_{n+k}, x_{n+k+1}) \leq \theta^{k+1} d(x_{n-1}, x_n)
\]

(12)

for \( k \in \mathbb{N}, n \geq 1 \). Now from (12) we have,

\[
d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})
\]

\[
= (\theta^n + \theta^{n+1} + \cdots + \theta^{n+p-1})d(x_0, x_1)
\]

\[
= \frac{\theta^n (1 - \theta^p)}{1-\theta}d(x_0, x_1),
\]

(13)

which in view of \( 0 < \theta < 1 \) shows that \( \{x_n\} \) is a Cauchy sequence. Since \( (X, d) \) is complete, it follows that \( \{x_n\}_{n=0}^{\infty} \) converges to some point in \( X \). Let \( u = \lim_{n \to \infty} x_n \). Then we have,

\[
p(u, T(u)) \leq d(u, x_{n+1}) + p(x_{n+1}, T(u))
\]

\[
\leq d(u, x_{n+1}) + D(T(x_n), T(u))
\]

\[
\leq d(u, x_{n+1}) + \theta d(x_n, u) + L \rho(u, T(x_n))
\]

\[
\leq d(u, x_{n+1}) + \theta d(x_n, u) + L \rho(u, x_{n+1}) + L \rho(u, T(x_n)).
\]
Noting that \( p(x_{n+1}, T(x_n)) = 0 \) and taking \( n \to \infty \) we have, \( p(u, T(u)) \leq 0 \implies p(u, T(u)) = 0 \implies \{u\} \subset T(u) \).

From (13) taking \( p \to \infty \) we have
\[
d(x_n, u) \leq \frac{\theta^n}{1 - \theta}d(x_0, x_1), n = 0, 1, 2, \ldots
\]

Again by (12) we have
\[
d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})
= (\theta + \theta^2 + \cdots + \theta^p)d(x_{n-1}, x_n)
= \theta(1 - \theta^p) \frac{1}{1 - \theta}d(x_{n-1}, x_n).
\]

Taking \( p \to \infty \) we have,
\[
d(x_n, u) \leq \frac{\theta}{1 - \theta}d(x_{n-1}, x_n).
\]

Hence the proof is complete. \( \blacksquare \)

**Theorem 4.2.** Let \( (X, d) \) be a complete metric linear space, \( f : X \to X \) be a self mapping, and \( T : X \to W(X) \) be a \((f, \theta, L)\)-weak contractive fuzzy mapping (see Definition 2.25). Suppose \( \cup\{T(X)\}_\alpha \subseteq f(X) \) for \( \alpha \in [0, 1] \) and \( f(X) \) is complete. Then there exists \( u \in X \) such that \( u \) is a coincidence point of \( f \) and \( T \), that is \( \{f(u)\} \subset T(u) \). Here \( T(x)_\alpha = \{y \in X : (T(x))(y) \geq \alpha\} \).

**Proof.** Let \( x_0 \in X \) and \( y_0 = f(x_0) \). Since \( \cup\{T(X)\}_\alpha \subset f(X) \) for each \( \alpha \in [0, 1] \), by Remark 2.21 for \( x_0 \in X \) there exist points \( x_1, y_1 \in X \) such that \( y_1 = f(x_1) \) and \( \{y_1\} \subset T(x_0) \). By Remark 2.21 and Lemma 2.1, for \( x_1 \in X \) there exist points \( x_2, y_2 \in X \) such that \( y_2 = f(x_2) \) and \( \{y_2\} \subset T(x_1) \), and
\[
d(y_1, y_2) \leq H(T(x_0)_1, T(x_1)_1) \leq D(T(x_0), T(x_1))
\leq \theta d(f(x_0), f(x_1)) + Lp(f(x_1), T(x_0)) = \theta d(y_0, y_1).
\]

By repeating this process we can select points \( x_k, y_k \in X \) such that \( y_k = f(x_k) \) and \( \{y_k\} \subset T(x_{k-1}) \), and hence
\[
d(y_k, y_{k+1}) \leq H(T(x_{k-1})_1, T(x_k)_1) \leq D(T(x_{k-1}), T(x_k))
\leq \theta d(f(x_{k-1}), f(x_k)) + Lp(f(x_k), T(x_{k-1})) = \theta d(y_{k-1}, y_k).
\]

(14)

From (14) we obtain inductively,
\[
d(y_n, y_{n+1}) \leq \theta^n d(y_0, y_1) \quad \text{and} \quad d(y_{n+k}, y_{n+k+1}) \leq \theta^{k+1} d(y_{k-1}, y_k)
\]

(15)

for all \( k \in \mathbb{N}, n \geq 1 \).
Now from (15), we have for any $p \geq 1$,
\[
d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}) \\
\leq (\theta^n + \theta^{n+1} + \cdots + \theta^{n+p-1})d(y_0, y_1) \\
= \frac{\theta^n(1 - \theta^p)}{1 - \theta}d(y_0, y_1).
\]

In view of $0 < \theta < 1$, we see that $\{y_n\}$ is a Cauchy sequence. Since $f(X)$ is complete, $\{y_n\}$ converges to some point in $f(X)$. Let $y = \lim_{n \to \infty} y_n$ and $u \in X$ be such that $y = f(u)$. Now
\[
p(f(u), T(u)) = p(y, T(u)) \leq d(y, y_{k+1}) + \theta d(f(x_k), f(u)) + L_{p}(f(u), T(x_k)) \\
\leq d(y, y_{k+1}) + \theta d(y, y) + L_{p}[d(y, y_{k+1}) + p(y_{k+1}, T(x_k))].
\]

Noting that $p(y_{k+1}, T(x_k)) = 0$ and the fact that $y_k \to y$ as $k \to \infty$ we have, $p(f(u), T(u)) = 0$, i.e., $\{f(u)\} \subset T(u)$. ■

**Theorem 4.3.** Let $(X, d)$ be a complete metric linear space and $T : X \to W(X)$ be a generalized $(\alpha, L)$-weak contraction (see Definition 2.26). Then $T$ has a fixed point.

**Proof.** Let $x_0 \in X$. Then by Lemma 2.20, there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$. Now by Lemma 2.20 and Lemma 2.1, there exists a point $x_2 \in X$ such that $\{x_2\} \subset T(x_1)$ and
\[
d(x_1, x_2) \leq H(T(x_0)_1, T(x_1)) \leq D(T(x_0), T(x_1)) \\
\leq \alpha d(x_0, x_1) + L_{p}(x_1, T(x_0)) \leq d(x_0, x_1).
\]
By repeating this process we can select a point $x_{k+1} \in X$ such that $\{x_{k+1}\} \subset T(x_k)$ and
\[
d(x_k, x_{k+1}) \leq H(T(x_{k-1}), T(x_{k+1})) \leq D(T(x_{k-1}), T(x_{k})) \\
\leq \alpha d(x_{k-1}, x_k) + L_{p}(x_{k-1}, T(x_k)) \leq d(x_{k-1}, x_k).
\]

Let $d_k = d(x_{k-1}, x_k)$. Since $d_k$ is a non-increasing sequence of nonnegative real numbers, therefore \(\lim_{k \to \infty} d_k = 0\). By hypothesis we get \(\limsup_{t \to c+} \alpha(t) < 1\). Therefore there exists $k_0$ such that $k \geq k_0$ implies that $\alpha(d_k) < h$, where $\limsup_{t \to c+} \alpha(t) < h < 1$. Now by (16) we deduce that the sequence $\{d_k\}$ satisfies the following recurrence inequality:
\[
d_{k+1} \leq \alpha(d_k) d_k \leq \alpha(d_k) \alpha(d_{k-1}) d_{k-1} \cdots \\
\leq \prod_{i=1}^{k} \alpha(d_i) d_1 \leq \prod_{i=1}^{k} \alpha(d_i) \prod_{i=k_0}^{k} \alpha(d_i) d_1 \leq \prod_{i=1}^{k_0-1} \alpha(d_i) h^{k-k_0+1} d_1 \\
= C h^k, \quad \text{(where } C \text{ is a generic positive constant).}
Hence for \( p \geq 1 \) we have,

\[
d(x_k, x_{k+p}) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{k+p-1}, x_{k+p}) \\
= d_{k+1} + d_{k+2} + \cdots + d_{k+p-1} \leq C(h^k + h^{k+1} + \cdots + h^{k+p-1}) \\
= C\frac{h^k(1 - h^p)}{1 - h}.
\]

which in view of \( 0 < h < 1 \) shows that \( \{x_k\} \) is a Cauchy sequence. Since \( X \) is complete, the sequence \( \{x_k\} \) converges to some point in \( X \). Let \( u = \lim_{k \to \infty} x_k \).

Now we have,

\[
p(u, T(u)) \leq d(u, x_{k+1}) + p(x_{k+1}, T(u)) \\
\leq d(u, x_{k+1}) + D(T(x_k), T(u)) \\
\leq d(u, x_{k+1}) + \alpha(d(x_k, u))d(x_k, u) + Lp(u, T(x_k)) \\
\leq d(u, x_{k+1}) + \alpha(d(x_k, u))d(x_k, u) + L[d(u, x_{k+1}) + p(x_{k+1}, T(x_k))].
\]

Using the fact that \( x_{k+1} \in T(x_k) \) and the fact that \( x_k \to u \) we have, \( p(u, T(u)) \leq 0 \), i.e., \( p(u, T(u)) = 0 \), i.e., \( \{u\} \subset T(u) \). Hence the proof is complete. ■

**Note.** Du first introduced the concept of Reich-functions as follows (cf. [8]):

**Definition 4.4.** A function \( \phi : [0, \infty) \to [0, 1) \) is called to be a Reich-function (R-function for short) if for each \( t \in [0, \infty) \) there exists \( r_t \in [0, 1) \) and \( \epsilon_t > 0 \) such that \( \phi(s) \leq r_t \) for all \( s \in [t, t + \epsilon_t) \).

**Examples.** Let \( \phi : [0, \infty) \to [0, 1) \) be a function.

(i) Obviously, if \( \phi \) is defined by \( \phi(t) = c \), where \( c \in [0, 1) \), then \( \phi \) is a R-function;

(ii) If \( \phi \) is nondecreasing function, then \( \phi \) is a R-function;

(iii) It is easy to see that if \( \phi \) satisfies \( \limsup_{s \to t+} \phi(s) < 1 \) for all \( t \in [0, \infty) \), then \( \phi \) is a R-function.

Note that in Theorem 4.3, \( \alpha \) is a R-function and so the proof of showing the sequence \( \{x_k\} \) is Cauchy can be done in another way as follows. Since \( \alpha \) is a R-function, there exists \( r_c \in [0, 1) \) and \( \epsilon_c > 0 \) such that \( \phi(s) \leq r_c \) for all \( s \in [c, c + \epsilon_c) \). Again \( \{d_k\} \) being non-increasing and \( d_k \to c \) as \( k \to \infty \), there exists \( k_0 \) such that for all \( k \geq k_0 \) we have \( d_k \in [c, c + \epsilon_c) \). Hence, by (16) we have

\[
d_{k+1} \leq \alpha(d_k)d_k \leq r_c d_k \leq r_c^2 d_{k-1} \leq \cdots \leq r_c^{k-k_0+1}d_{k_0} \leq r_c^k \frac{d_{k_0}}{r_c^{k_0-1}}.
\]

Hence, for \( p \geq 1 \) we have

\[
d(x_k, x_{k+p}) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{k+p-1}, x_{k+p}) \\
\leq (r_c^k + r_c^{k+1} + \cdots + r_c^{k+p-1}) \frac{d_{k_0}}{r_c^{k_0-1}} \\
= \frac{r_c^k(1 - r_c^p)}{1 - r_c} \frac{d_{k_0}}{r_c^{k_0-1}},
\]

which in view of \( r_c \in [0, 1) \) shows that \( \{x_k\} \) is a Cauchy sequence.
Theorem 4.5. Let $(X,d)$ be a complete metric space and $T : X \to \mathcal{F}(X)$ be a $(\theta,L)$-weak contractive fuzzy mapping satisfying the condition that for each $x \in X$ there is $\alpha(x) \in (0,1]$ such that $T(x)_{\alpha(x)}$ is a nonempty closed bounded subset of $X$. Then

(i) There exists a point $u \in X$ such that $u \in T(u)_{\alpha(u)}$;

(ii) For any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of $T$ at the point $x_0$ that converges to a point $u \in X$, for which the following estimates hold:

\[
d(x_n,u) \leq \frac{h^n}{1-h}d(x_0,x_1), \quad n = 0,1,2,\ldots,
\]

\[
d(x_n,u) \leq \frac{h}{1-h}d(x_{n-1},x_n), \quad n = 1,2,\ldots
\]

for a certain constant $h < 1$, such that $u \in T(u)_{\alpha(u)}$.

Proof. Let $F : X \to \mathcal{F}(X)$ be a fuzzy mapping. By assumption, there exists $\alpha(x) \in (0,1]$ such that $F(x)_{\alpha(x)}$ is a nonempty closed bounded subset of $X$. Let us now construct a sequence $\{x_n\}$ $(n \geq 0)$ as follows. By $\alpha_{n+1}$ we denote $\alpha_{n+1} = \alpha(x_n)$ for $n \geq 0$. Let $x_0 \in X$. Let $x_1 \in T(x_0)_{\alpha_1}$ and $q > 1$. Then there exists a point $x_2 \in T(x_1)_{\alpha_2}$ such that

\[
d(x_1,x_2) \leq qH(T(x_0)_{\alpha_1},T(x_1)_{\alpha_2})
\]

\[
\leq q\theta d(x_0,x_1) + qLd(x_1,T(x_0)_{\alpha_1}) \leq q\theta d(x_0,x_1).
\]

Let us choose $q > 1$ in such a way that $h = q\theta < 1$ for any $\theta \in [0,1)$, and then $d(x_1,x_2) \leq hd(x_0,x_1)$. Now for $x_2 \in T(x_1)_{\alpha_2}$, there exists a point $x_3 \in T(x_2)_{\alpha_3}$ such that

\[
d(x_2,x_3) \leq qH(T(x_1)_{\alpha_2},T(x_2)_{\alpha_3})
\]

\[
\leq q\theta d(x_1,x_2) + qLd(x_2,T(x_1)_{\alpha_2}) \leq hd(x_1,x_2).
\]

In this manner, we obtain an orbit $\{x_n\}_{n=0}^{\infty}$ at $x_0$ for $T$ satisfying

\[
d(x_n,x_{n+1}) \leq hd(x_{n-1},x_n), \quad n = 1,2,\ldots \tag{17}
\]

From (17) we obtain inductively,

\[
d(x_n,x_{n+1}) \leq h^n d(x_0,x_1) \quad \text{and} \quad d(x_{n+k},x_{n+k+1}) \leq h^{k+1}d(x_{n-1},x_n) \tag{18}
\]

for $k \in \mathbb{N}$, $n \geq 1$. Now from (18) we have

\[
d(x_n,x_{n+p}) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+p-1},x_{n+p})
\]

\[
= (h^n + h^{n+1} + \cdots + h^{n+p-1})d(x_0,x_1)
\]

\[
= \frac{h^n(1-h^p)}{1-h}d(x_0,x_1), \tag{19}
\]

which in view of $0 < h < 1$ shows that $\{x_n\}$ is a Cauchy sequence. Since $(X,d)$ is complete, it follows that $\{x_n\}_{n=0}^{\infty}$ converges to some point in $X$. Let $u = \lim_{n \to \infty} x_n$. Then we have,

\[
d(u,T(u)_{\alpha(u)}) \leq d(u,x_{n+1}) + d(x_{n+1},T(u)_{\alpha(u)})
\]
Noting that and Lemma 4.6, we get the following result.

\[ d(u, x_{n+1}) + d(T(x_n)_{\alpha_{n+1}}, T(u)_{\alpha(u)}) \leq \left(1 - \frac{h}{2}\right) d(u, T(u)_{\alpha(u)}) \]

Taking \( h \) and \( 1 - \frac{h}{2} \) we have, \( d(u, T(u)_{\alpha(u)}) \leq 0 \implies d(u, T(u)_{\alpha(u)}) = 0 \implies u \in T(u)_{\alpha(u)} \).

From (19) taking \( p \to \infty \) we have

\[ d(x_n, u) \leq \frac{h^n}{1 - h} d(x_0, x_1), \quad n = 0, 1, 2, \ldots \]

Again by (18) we have,

\[ d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \]

\[ = (h + h^2 + \cdots + h^p) d(x_{n-1}, x_n) \]

\[ = \frac{h(1 - h^p)}{1 - h} d(x_{n-1}, x_n). \]

Taking \( p \to \infty \) we have, \( d(x_n, u) \leq \frac{h}{1 - h} d(x_{n-1}, x_n) \). This completes the proof. \( \blacksquare \)

Now we discuss a different type of fuzzy mapping. As defined earlier we know a fuzzy set in \( X \) is a function with domain \( X \) and range in \([0, 1])\), \( \mathcal{F}(X) \) denotes the collection of all fuzzy set in \( X \) and \( CB(X) \) represents the nonempty closed and bounded subsets of \( X \). Let \( K(X) \) be the set of all fuzzy sets \( \mu : X \to [0, 1] \) such that \( \hat{\mu} \in CB(X) \) where \( \hat{\mu} = \{ x \in X : \mu(x) = \max_{y \in X} \mu(y) \} \), i.e., \( K(X) = \{ \mu \in \mathcal{F}(X) : \hat{\mu} \in CB(X) \} \).

A fuzzy mapping \( T : X \to K(X) \) is a mapping from \( X \) into \( K(X) \). For a fuzzy mapping \( T : X \to K(X) \) and a mapping \( \Lambda : K(X) \to CB(X) \), the composition \( \hat{T} = \Lambda \circ T : X \to CB(X) \) is defined as \( (\Lambda \circ T)(x) = T(x) = \{ y \in X : T(x)(y) = \max_{z \in X} T(x)(z) \} \).

A point \( x^* \in X \) is called a fixed point of a fuzzy mapping \( T : X \to K(X) \) if \( T(x^*)(x^*) \geq T(x^*)(x) \) for all \( x \in X \), i.e., \( T(x^*)(x^*) = \max_{y \in X} T(x^*)(y) \).

**Lemma 4.6.** [2] A point \( x^* \in X \) is a fixed point of a fuzzy mapping \( T : X \to K(X) \) if and only if \( x^* \) is a fixed point of the induced mapping \( \hat{T} : X \to CB(X) \).

Now we define \( \alpha(x) = \max_{y \in X} T(x)(y) \), and then \( T(x)_{\alpha(x)} = \{ y \in X : T(x)(y) = \max_{z \in X} T(x)(z) \} = \{ y \in X : T(x)(y) \geq \alpha \} \). Then using Theorem 4.5 and Lemma 4.6, we get the following result.

**Theorem 4.7.** Let the fuzzy mapping \( T : X \to K(X) \) be a \((\theta,L)\)-weak contractive fuzzy mapping. Then \( T \) has a fixed point.

**Proof.** By Theorem 4.5, there exists \( u \in X \) such that \( u \in T(u)_{\alpha(u)} \). But here \( \alpha(u) \) by definition is \( \max_{y \in X} T(u)(y) \), i.e. \( u \in \hat{T}(u) \), i.e., \( u \) is a fixed point of the induced mapping \( \hat{T} \). Then by Lemma 4.6, \( u \) is a fixed point of the fuzzy mapping \( T : X \to K(X) \). \( \blacksquare \)
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