SZÁSZ-MIRAKJAN TYPE OPERATORS OF TWO VARIABLES
PROVIDING A BETTER ESTIMATION ON $[0,1] \times [0,1]$

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Abstract. This paper deals with a modification of the classical Szász-Mirakjan type operators of two variables. It introduces a new sequence of non-polynomial linear operators which hold fixed the polynomials $x^2 + \alpha x$ and $y^2 + \beta y$ with $\alpha, \beta \in [0, \infty)$ and we study the convergence properties of the new approximation process. Also, we compare it with Szász-Mirakjan type operators and show an improvement of the error of convergence in $[0,1] \times [0,1]$. Finally, we study statistical convergence of this modification.

1. Introduction

Most of the approximating operators, $L_n$, preserve $e_i(x) = x^i$, ($i = 0, 1$), i.e., $L_n(e_i; x) = e_i(x)$, $n \in \mathbb{N}$, $i = 0, 1$, but $L_n(e_2; x) \neq e_2(x) = x^2$. Especially, these conditions hold for the operators given by Agratini [1], the Bernstein polynomials [4, 5] and the Szász-Mirakjan type operators [3, 14]. Agratini [2] has investigated a general technique to construct operators which preserve $e_2$. Recently, King [13] presented a non-trivial sequence of positive linear operators defined on the space of all real-valued continuous functions on $[0,1]$ while preserving the functions $e_0$ and $e_2$. Duman and Orhan [7] have studied King’s results using the concept of statistical convergence. Recently, Duman and Özarslan [8] have investigated some approximation results on the Szász-Mirakjan type operators preserving $e_2(x) = x^2$.

The functions $f_0(x, y) = 1$, $f_1(x, y) = x$ and $f_2(x, y) = y$ are preserved by most of approximating operators of two variables, $L_{m,n}$, i.e., $L_{m,n}(f_0; x, y) = f_0(x, y)$, $L_{m,n}(f_1; x, y) = f_1(x, y)$ and $L_{m,n}(f_2; x, y) = f_2(x, y)$, $m, n \in \mathbb{N}$, but $L_{m,n}(f_3; x, y) \neq f_3(x, y) = x^2 + y^2$. These conditions hold, specifically, for the Bernstein polynomials of two variables, the Szász-Mirakjan type operators of two variables. In this paper, we give a modification of the well-known Szász-Mirakjan type operators of two variables and show that this modification holds fixed some polynomials different from $f_i(x, y)$. The resulting approximation processes turn out to have an order of approximation at least as good as the one of Szász-Mirakjan.

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type operators of two variables in certain subsets of \([0, \infty) \times [0, \infty)\). Finally, we study \(A\)-statistical convergence of this modification.

We first recall the concept of \(A\)-statistical convergence for double sequences. Let \(A = (a_{j,k,m,n})\) be a four-dimensional summability matrix. For a given double sequence \(x = (x_{m,n})\), the \(A\)-transform of \(x\), denoted by \(Ax := ((Ax)_{j,k})\), is given by
\[
(Ax)_{j,k} = \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n} x_{m,n}
\]
provided the double series converges in Pringsheim’s sense for every \((j, k) \in \mathbb{N}^2\).

A two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two-dimensional matrix transformations is known as Silverman-Toeplitz conditions (see, for instance, \([12]\)). In 1926, Robison \([18]\) presented a four-dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double Pringsheim convergent (\(P\)-convergent) sequence is not necessarily bounded. The definition and the characterization of regularity for four-dimensional matrices is known as Robison-Hamilton conditions, or briefly, \(RH\)-regularity (see \([11, 18]\)).

Recall that a four-dimensional matrix \(A = (a_{j,k,m,n})\) is said to be \(RH\)-regular if it maps every bounded \(P\)-convergent sequence into a \(P\)-convergent sequence with the same \(P\)-limit. The Robison-Hamilton conditions state that a four-dimensional matrix \(A = (a_{j,k,m,n})\) is \(RH\)-regular if and only if
(i) \( P - \lim_{j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n} = 0 \) for each \((m, n) \in \mathbb{N}^2\);
(ii) \( P - \lim_{j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n} = 1 \);
(iii) \( P - \lim_{j,k} \sum_{m \in \mathbb{N}} |a_{j,k,m,n}| = 0 \) for each \(n \in \mathbb{N}\);
(iv) \( P - \lim_{j,k} \sum_{n \in \mathbb{N}} |a_{j,k,m,n}| = 0 \) for each \(m \in \mathbb{N}\);
(v) \( \sum_{(m,n) \in \mathbb{N}^2} |a_{j,k,m,n}| \) is \(P\)-convergent for each \(j, k \in \mathbb{N}\);
(vi) there exist finite positive integers \(A\) and \(B\) such that \( \sum_{m,n > B} |a_{j,k,m,n}| < A \) holds for every \((j, k) \in \mathbb{N}^2\).

Now let \(A = (a_{j,k,m,n})\) be a non-negative \(RH\)-regular summability matrix, and let \(K \subset \mathbb{N}^2\). Then \(A\)-density of \(K\) is given by
\[
\delta_A^{(2)}\{K\} := P - \lim_{j,k} \sum_{(m,n) \in K} a_{j,k,m,n}
\]
provided that the limit on the right-hand side exists in Pringsheim’s sense. A real double sequence \(x = (x_{m,n})\) is said to be \(A\)-statistically convergent to a number \(L\) if, for every \(\varepsilon > 0\),
\[
\delta_A^{(2)}\{(m, n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\} = 0.
\]
In this case, we write \( st^2_{(A)} - \lim_{m,n} x_{m,n} = L \). Clearly, a \( P \)-convergent double sequence is \( A \)-statistically convergent to the same value but its converse is not always true. Also, note that an \( A \)-statistically convergent double sequence need not to be bounded. For example, consider the double sequence \( x = (x_{m,n}) \) given by

\[
x_{m,n} = \begin{cases} \ m n, & \text{if } m \text{ and } n \text{ are squares,} \\ 1, & \text{otherwise.}
\end{cases}
\]

We should note that if we take \( A = C(1,1) := [c_{j,k,m,n}] \), the double Cesàro matrix, defined by

\[
c_{j,k,m,n} = \begin{cases} \ \frac{1}{jk}, & \text{if } 1 \leq m \leq j \text{ and } 1 \leq n \leq k, \\ 0, & \text{otherwise},
\end{cases}
\]

then \( (1,1) \)-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [15, 16]. Finally, if we replace the matrix \( A \) by the identity matrix for four dimensional matrices, then \( A \)-statistical convergence reduces to the Pringsheim convergence, which was introduced in [17].

By \( C(D) \), we denote the space of all continuous real valued functions on \( D \) where \( D = [0,\infty) \times [0,\infty) \). By \( E_2 \), we denote the space of all real valued functions of exponential type on \( D \). More precisely, \( f \in E_2 \) if and only if there are three positive finite constants \( c, d \) and \( a \) with the property \( |f(x,y)| \leq a e^{cx+dy} \). Let \( L \) be a linear operator from \( C(D) \cap E_2 \) into \( C(D) \cap E_2 \). Then, as usual, we say that \( L \) is a positive linear operator provided that \( f \geq 0 \) implies \( L(f) \geq 0 \). Also, we denote the value of \( L(f) \) at a point \((x,y) \in D \) by \( L(f;x,y) \).

Now fix \( a, b > 0 \). For the proof of the our approximation results we use the lattice homomorphism \( H_{a,b} \), which maps \( C(D) \cap E_2 \) into \( C(E) \cap E_2 \), defined by \( H_{a,b}(f) = f|_E \), where \( E = [0,a] \times [0,b] \) and \( f|_E \) denotes the restriction of the domain \( f \) to the rectangle \( E \). The space \( C(E) \) is equipped with the supremum norm

\[
\|f\| = \sup_{(x,y) \in E} |f(x,y)|, \quad (f \in C(E)).
\]

Hence, from the Korovkin-type approximation theorem for double sequences of positive linear operators of two variables which is introduced by Dirik and Demirci [6] the following results follow.

**Theorem 1.** [6] Let \( A = (a_{j,k,m,n}) \) be a non-negative RH-regular summability matrix. Let \( \{L_{m,n}\} \) be a double sequence of positive linear operators acting from \( C(D) \cap E_2 \) into itself. Assume that the following conditions hold:

\[
st^2_{(A)} - \lim_{m,n} L_{m,n}(f_i;x,y) = f_i(x,y), \quad \text{uniformly on } E, \quad (i = 0,1,2,3),
\]

where \( f_0(x,y) = 1, f_1(x,y) = x, f_2(x,y) = y \) and \( f_3(x,y) = x^2 + y^2 \). Then, for all \( f \in C(D) \cap E_2 \), we have

\[
st^2_{(A)} - \lim_{m,n} L_{m,n}(f;x,y) = f(x,y), \quad \text{uniformly on } E.
\]
2. Construction of the operators

Szász-Mirakjan type operators introduced by Favard [9] is the following:

\[
S_{m,n}(f; x, y) = e^{-mx}e^{-ny} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} f \left( \frac{s}{m}, \frac{t}{n} \right) \frac{(mx)^s}{s!} \frac{(ny)^t}{t!}, \tag{2.1}
\]

where \((x, y) \in D\) and \(f \in C(D) \cap E_2\). It is clear that

\[
S_{m,n}(f_0; x, y) = f_0(x, y),
\]

\[
S_{m,n}(f_1; x, y) = f_1(x, y),
\]

\[
S_{m,n}(f_2; x, y) = f_2(x, y),
\]

\[
S_{m,n}(f_3; x, y) = f_3(x, y) + \frac{x}{m} + \frac{y}{n},
\]

where \(f_0(x, y) = 1, f_1(x, y) = x, f_2(x, y) = y\) and \(f_3(x, y) = x^2 + y^2\). Then, we observe that \(P - \lim_{m,n} S_{m,n}(f_i; x,y) = f_i(x,y)\), uniformly on \(E\), where \(i = 0, 1, 2, 3\). If we replace the matrix \(A\) by double identity matrix in Theorem 1, then we immediately get the classical result. Hence, for the \(S_{m,n}\) operators given by (2.1), we have, for all \(f \in C(D) \cap E_2\),

\[
P - \lim_{m,n} S_{m,n}(f; x, y) = f(x, y),\text{ uniformly on } E.
\]

For each integer \(k \in \mathbb{N}\), let \(r_k : [0, \infty) \times X \to \mathbb{R}\) be the function defined by

\[
r_k(\gamma, z) := \frac{-(k\gamma + 1) + \sqrt{(k\gamma + 1)^2 + 4k^2(2^2 + \gamma z)}}{2k} \tag{2.2}
\]

where if \(z\) is the first variable of the following operator, then \(X = [0, a]\) and if \(z\) is the second variable of the following operator, then \(X = [0, b]\). Let

\[
H_{m,n}^{\alpha, \beta}(f; x, y) = S_{m,n}(f; r_m(\alpha, x), r_n(\beta, y))
\]

\[
= e^{-mr_m(\alpha, x)}e^{-nr_n(\beta, y)} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} f \left( \frac{s}{m}, \frac{t}{n} \right) \frac{(mr_m(\alpha, x))^s}{s!} \frac{(nr_n(\beta, y))^t}{t!} \tag{2.3}
\]

where \(\alpha, \beta \in [0, \infty)\), for \(f \in C(D) \cap E_2\).

Hence, in the special case \(\lim_{\alpha \to \infty} r_m(\alpha, x) = x\) and \(\lim_{\alpha \to \infty} r_n(\beta, y) = y\), the operator \(H_{m,n}^{\alpha, \beta}\) becomes the classical Szász-Mirakjan type operators which is given by (2.1).

It is clear that \(H_{m,n}^{\alpha, \beta}\) are positive and linear. It is easy to see that

\[
H_{m,n}^{\alpha, \beta}(f_0; x, y) = f_0(x, y),
\]

\[
H_{m,n}^{\alpha, \beta}(f_1; x, y) = r_m(\alpha, x),
\]

\[
H_{m,n}^{\alpha, \beta}(f_2; x, y) = r_n(\beta, y),
\]

\[
H_{m,n}^{\alpha, \beta}(f_3; x, y) = r_m^2(\alpha, x) + \frac{r_m(\alpha, x)}{m},
\]

\[
H_{m,n}^{\alpha, \beta}(f_2^2; x, y) = r_n^2(\beta, y) + \frac{r_n(\beta, y)}{n}. \tag{2.4}
\]

From the definition of \(r_k\) one can check the validity of the following.
Proposition 1. The operators $H_{m,n}^{\alpha,\beta}$ hold fixed the polynomials $f_1^2 + \alpha f_1$ and $f_2^2 + \beta f_2$, i.e.
\[H_{m,n}^{\alpha,\beta}(f_1^2 + \alpha f_1; x, y) = x^2 + \alpha x \text{ and } H_{m,n}^{\alpha,\beta}(f_2^2 + \beta f_2; x, y) = y^2 + \beta y.\]

Now, we give the following result using Theorem 1 for $A = I$, which is the double identity matrix.

Theorem 2. Let $H_{m,n}^{\alpha,\beta}$ denote the sequence of positive linear operators given by (2.3). If
\[P \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_1; x, y) = x, \quad P \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_2; x, y) = y, \text{ uniformly on } E,
\]
then, for all $f \in C(D) \cap E_2$,
\[P \lim_{m,n} H_{m,n}^{\alpha,\beta}(f; x, y) = f(x, y), \text{ uniformly on } E,
\]
where $\alpha, \beta \in [0, \infty)$.

Proof. For $\alpha, \beta \in [0, \infty)$, $H_{m,n}^{\alpha,\beta}(f_1; x, y)$ converges to $x$ as $m, n$ (in any manner) tends to infinity. Also, we get
\[r_{m,n}(\alpha) = \sup_{(x,y) \in E} |x - H_{m,n}^{\alpha,\beta}(f_1; x, y)|
\]
\[= a - \frac{(ma + 1) + \sqrt{(ma + 1)^2 + 4m^2(a^2 + aa)}}{2m}.
\]
Since $r_{m,n}(\alpha)$ and $r_{m,n}(\beta)$ converge to 0 as $m, n \to \infty$, the convergence is uniform on $E$. From (2.4), Proposition 1 and Theorem 1 for $A = I$, which is the double identity matrix, the proof is completed. □

3. Comparison with Szász-Mirakjan type operators

In this section, we estimate the rates of convergence of the operators $H_{m,n}^{\alpha,\beta}(f; x, y)$ to $f(x, y)$ by means of the modulus of continuity. Thus, we show that our estimations are more powerful than those obtained by the operators given by (2.1) on $D$.

By $C_B(D)$ we denote the space of all continuous and bounded functions on $D$. For $f \in C_B(D) \cap E_2$, the modulus of continuity of $f$, denoted by $\omega(f; \delta)$, is defined as
\[\omega(f; \delta) = \sup \{|f(u, v) - f(x, y)| : \sqrt{(u - x)^2 + (v - y)^2} < \delta, (u, v), (x, y) \in D\}.
\]
Then it is clear that for any $\delta > 0$ and each $(x, y) \in D$
\[|f(u, v) - f(x, y)| \leq \omega(f; \delta) \left(\frac{\sqrt{(u - x)^2 + (v - y)^2}}{\delta} + 1\right).
\]
After some simple calculations, for any double sequence $\{L_{m,n}\}$ of positive linear operators on $C_B(D) \cap E_2$, we can write, for $f \in C_B(D) \cap E_2$,
\[|L_{m,n}(f; x, y) - f(x, y)| \leq \omega(f; \delta) \left\{L_{m,n}(f_0; x, y) + \frac{1}{\delta^2} L_{m,n}((u - x)^2 + (v - y)^2; x, y) \right\} + |f(x, y)||L_{m,n}(f_0; x, y) - f_0(x, y)|.
\]
Now we have the following:

**Theorem 3.** If $H_{m,n}^{\alpha,\beta}$ is defined by (2.1), then for every $f \in C_B(D) \cap E_2$, $(x, y) \in D$ and any $\delta > 0$, we have

$$
|H_{m,n}^{\alpha,\beta}(f; x, y) - f(x, y)| \leq \omega(f, \delta) \left\{ 1 + \frac{1}{\delta^2}(2x^2 + \alpha x - H_{m,n}^{\alpha,\beta}(f_1; x, y)(\alpha + 2x)) + \frac{1}{\delta^2}(2y^2 + \beta y - H_{m,n}^{\alpha,\beta}(f_2; x, y)(\beta + 2y)) \right\}.
$$

(3.2)

Furthermore, when (3.2) holds,

$$
2x^2 + \alpha x - H_{m,n}^{\alpha,\beta}(f_1; x, y)(\alpha + 2x) + 2y^2 + \beta y - H_{m,n}^{\alpha,\beta}(f_2; x, y)(\beta + 2y) \geq 0
$$

for $(x, y) \in D$.

**Remark 1.** For the Szász-Mirakjan type operators given by (2.1), we may write from (3.1) that for every $f \in C_B(D) \cap E_2$, $m, n \in \mathbb{N}$,

$$
|S_{m,n}(f; x, y) - f(x, y)| \leq \omega(f, \delta) \left\{ 1 + \frac{1}{\delta^2}(\frac{x}{m} + \frac{y}{n}) \right\}.
$$

(3.3)

The estimate (3.2) is better than the estimate (3.3) if and only if

$$
2x^2 + \alpha x - H_{m,n}^{\alpha,\beta}(f_1; x, y)(\alpha + 2x) + 2y^2 + \beta y - H_{m,n}^{\alpha,\beta}(f_2; x, y)(\beta + 2y) \leq \frac{x}{m} + \frac{y}{n},
$$

(3.4)

$(x, y) \in D$. Thus, the order of approximation towards a given function $f \in C_B(D) \cap E_2$ by the sequence $H_{m,n}^{\alpha,\beta}$ will be at least as good as that of $S_{m,n}$ whenever the following function $\phi_{m,n}^{\alpha,\beta}(x, y)$ is non-negative:

$$
\phi_{m,n}^{\alpha,\beta}(x, y) = \frac{x}{m} + \frac{y}{n} - 2x^2 - \alpha x + H_{m,n}^{\alpha,\beta}(f_1; x, y)(\alpha + 2x) - 2y^2 - \beta y + H_{m,n}^{\alpha,\beta}(f_2; x, y)(\beta + 2y).
$$

The non-negativity of $\phi_{m,n}^{\alpha,\beta}(x, y)$ is obviously fulfilled at those points $(x, y)$ where simultaneously

$$
H_{m,n}^{\alpha,\beta}(f_1; x, y)(\alpha + 2x) - 2x^2 - \alpha x + \frac{x}{m} \geq 0
$$

and

$$
H_{m,n}^{\alpha,\beta}(f_2; x, y)(\beta + 2y) - 2y^2 - \beta y + \frac{y}{n} \geq 0.
$$

Some calculations state the validity of these inequalities when and only when $(x, y)$ lies in the subset of $D$ given by the rectangle

$$
\left[ 0, \frac{2m + \alpha + 2}{2m + 1} \right] \times \left[ 0, \frac{2n + \beta + 2}{2n + 1} \right].
$$

As $m, n \to \infty$, the endpoints of these intervals decrease to 1 and 1, respectively. As a consequence the order of approximation of $H_{m,n}^{\alpha,\beta}f$ towards $f$ is at least as good as the order of approximation to $f$ given by $S_{m,n}$ whenever $(x, y)$ lies in $[0, 1] \times [0, 1]$. 


4. \textbf{A-statistical convergence}

Gadjiev and Orhan [10] have investigated the Korovkin-type approximation theory via statistical convergence. In this section, using the concept of A-statistical convergence for double sequence, we give the Korovkin-type approximation theorem for the $H_{m,n}^{\alpha,\beta}$ operators given by (2.3). The Korovkin-type approximation theorem is given by Theorem 1 and Proposition 1 as follows:

**Theorem 4.** Let $A = (a_{j,k,m,n})$ be a non-negative RH-regular summability matrix. Let $H_{m,n}^{\alpha,\beta}$ be the double sequence of positive linear operators given by (2.3). If

$$st^2_m - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_1; x, y) = x, \quad st^2_n - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_2; x, y) = y,$$

then, for all $f \in C(D) \cap E_2$,

$$st^2_m - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f; x, y) = f(x, y), \text{ uniformly on } E.$$

Now, we choose a subset $K$ of $\mathbb{N}^2$ such that $\delta_A^{(2)}(K) = 1$. Define function sequences $\{r^*_m(\alpha, x)\}$ and $\{r^*_n(\beta, y)\}$ by

$$r^*_m(\alpha, x) = \begin{cases} 0, & (m, n) \notin K \\ \frac{-(m\alpha + 1) + \sqrt{(m\alpha + 1)^2 + 4m^2(x^2 + \alpha x)}}{2m}, & (m, n) \in K \end{cases}$$

$$r^*_n(\beta, y) = \begin{cases} 0, & (m, n) \notin K \\ \frac{-(n\beta + 1) + \sqrt{(n\beta + 1)^2 + 4n^2(y^2 + \beta y)}}{2n}, & (m, n) \in K \end{cases}$$

(4.1)

It is clear that $r^*_m(\alpha, x)$ and $r^*_n(\beta, y)$ are continuous and exponential-type on $[0, \infty)$. We now turn our attention to $H_{m,n}^{\alpha,\beta}$ given by (2.3) with $r^*_m(\alpha, x)$ and $r^*_n(\beta, y)$ replaced by $\{r^*_m(\alpha, x)\}$ and $\{r^*_n(\beta, y)\}$ where $r^*_m(\alpha, x)$ and $r^*_n(\beta, y)$ are defined by (4.1). Observe that $H_{m,n}^{\alpha,\beta}$ is a positive linear operator and

$$H_{m,n}^{\alpha,\beta}(f_1; x, y) = r^*_m(\alpha, x), \quad H_{m,n}^{\alpha,\beta}(f_2; x, y) = r^*_n(\beta, y),$$

(4.2)

and

$$H_{m,n}^{\alpha,\beta}(f_1^2; x, y) = \begin{cases} r^2_m(\alpha, x) + \frac{r^*_m(\alpha, x)}{m}, & (m, n) \in K \\ 0, & \text{otherwise} \end{cases}$$

$$H_{m,n}^{\alpha,\beta}(f_2^2; x, y) = \begin{cases} r^2_n(\beta, y) + \frac{r^*_n(\beta, y)}{n}, & (m, n) \in K \\ 0, & \text{otherwise} \end{cases}$$

(4.3)

Since $\delta_A^{(2)}(K) = 1$, we obtain

$$st^2_m - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_1; x, y) = x, \quad st^2_n - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_2; x, y) = y, \text{ uniformly on } E$$

(4.4)

and

$$st^2_m - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_1^2 + f_2^2; x, y) = x^2 + y^2, \text{ uniformly on } E.$$  

(4.5)

The relations (4.2)–(4.5) and Theorem 1 yield the following:
Theorem 5. Let $A = (a_{j,k,m,n})$ be a non-negative RH-regular summability matrix and let $\{H^\alpha,\beta_{m,n}\}$ denote the double sequence of positive linear operators given by (2.3) with $\{r_m(\alpha, x)\}$ and $\{r_n(\beta, y)\}$ replaced by $\{r^*_m(\alpha, x)\}$ and $\{r^*_n(\beta, y)\}$ where $r^*_m(\alpha, x)$ and $r^*_n(\beta, y)$ are defined by (4.1). Then, for all $f \in C(D) \cap E_2$, we have

$$\text{st}_2^{(A)} \lim_{m,n} H^\alpha,\beta_{m,n}(f; x, y) = f(x, y),$$

uniformly on $E$.

We note that $r^*_m(\alpha, x)$ and $r^*_n(\beta, y)$ in Theorem 5 do not satisfy the conditions of Theorem 2.

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