ON SANDWICH THEOREMS FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING EXTENDED MULTIPLIER TRANSFORMATIONS

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Abstract. In this paper we derive some subordination and superordination results for certain normalized analytic functions in the open unit disc, which are acted upon by a class of extended multiplier transformations. Relevant connections of the results, which are presented in this paper, with various known results are also considered.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and let $H[a,n]$ denote the subclass of the functions $f \in H(U)$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a \in \mathbb{C}).$$

(1.1)

Also, let $A(n)$ be the subclass of the functions $f \in H(U)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,$$

(1.2)

and set $A \equiv A(1)$.

For $f, g \in H(U)$, we say that the function $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$, such that $f(z) = g(w(z))$ for all $z \in U$. In particular, if the function $g(z)$ is univalent in $U$, then we have the following equivalence (cf., e.g., [10]; see also [11, p.4]):

$$f(z) \prec g(z) \iff f(0) \prec g(0) \quad \text{and} \quad f(U) \subset g(U).$$

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Supposing that \( p \) and \( h \) are two analytic functions in \( U \), let
\[
\varphi(r,s,t; z) : \mathbb{C}^3 \times U \rightarrow \mathbb{C}.
\]
If \( p \) and \( \varphi(p(z), zp'(z), z^2 p''(z); z) \) are univalent functions in \( U \) and if \( p \) satisfies the second-order superordination
\[
h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z),
\quad (1.3)
\]
then \( p \) is said to be a solution of the differential superordination (1.3). (If \( f \) is subordinate to \( F \), then \( F \) is superordinate to \( f \).) An analytic function \( q \) is called a subordinant of (1.3) if \( q(z) \prec p(z) \) for all the functions \( p \) satisfying (1.3). A univalent subordinant \( \tilde{q} \) that satisfies \( q \prec \tilde{q} \) for all of the subordinants \( q \) of (1.3), is called the best subordinant (cf., e.g., [10], see also [11]).

Recently, Miller and Mocanu [12] obtained sufficient conditions on the functions \( h, q \) and \( \varphi \) for which the following implication holds:
\[
h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z). \quad (1.4)
\]
Using these results, Bulboaca [5] considered certain classes of first-order differential superordinations as well as superordination preserving integral operators [4]. Ali et al. [17] obtained sufficient conditions for normalized analytic functions \( f(z) \) to satisfy
\[
q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \quad (1.5)
\]
where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = 1 \). Shanmugam et al. [17] obtained sufficient conditions for normalized analytic functions \( f(z) \) to satisfy
\[
q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z), \quad \text{and} \quad q_1(z) \prec \frac{z^2 f'(z)}{(f(z))^2} \prec q_2(z),
\]
where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = 1 \) and \( q_2(0) = 1 \). Liu [9] introduced and studied the class of functions \( B(\beta, \alpha, \rho) \) defined by \( f \in B(\beta, \alpha, \rho) \) if and only if
\[
\text{Re} \left\{ (1 - \beta) \left( \frac{f(z)}{z} \right)^{\alpha} + \beta \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^{\alpha} \right\} > \rho,
\]
where \( f(z) \in A, \beta \geq 0, \alpha > 0 \) and \( \rho \geq 0 \).

Many essentially equivalent definitions of multiplier transformation have been given in literature (see [7], [8], and [20]). In [6] Catas defined the operator \( I^m(\lambda, \ell) \) as follows:

**Definition 1** [6]. Let the function \( f(z) \in A(n) \). For \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0, \ell \geq 0 \), the extended multiplier transformation \( I^m(\lambda, \ell) \) on \( A(n) \) is defined by the following infinite series:
\[
I^m(\lambda, \ell) f(z) = z + \sum_{k=n+1}^{\infty} \left[ \ell + 1 + \lambda(k - 1) \right]^{m} a_k z^k, \quad m \in \mathbb{N}_0, \ z \in U. \quad (1.6)
\]
We can write (1.6) as follows:

\[ I^m(\lambda, \ell) f(z) = (\Phi^m_{\lambda, \ell} + f)(z), \]

where

\[ \Phi^m_{\lambda, \ell}(z) = z + \sum_{k=n+1}^{\infty} \left[ \frac{\ell + 1 + \lambda(k - 1)}{\ell + 1} \right] z^k. \]

It is easily verified from (1.6), that

\[ \lambda z (I^m(\lambda, \ell) f(z))' = (1 + \ell) I^{m+1}(\lambda, \ell) f(z) - [1 - \lambda + \ell] I^m(\lambda, \ell) f(z) (\lambda > 0). \quad (1.7) \]

We note that:

\[ I^0(\lambda, \ell) f(z) = f(z) \quad \text{and} \quad I^1(1, 0) f(z) = z f'(z). \]

Also by specializing the parameters \( \lambda, \ell \) and \( m \) we obtain the following operators studied by various authors:

(i) \( I^m(1, \ell) = I^m(\ell) f(z) \) (see Cho and Srivastava [8] and Cho and Kim [7]);
(ii) \( I^m(\lambda, 0) f(z) = D^m f(z) \) (see Al-Oboudi [2]);
(iii) \( I^m(1, 0) = D^m f(z) \) (see Salagean [16]);
(iv) \( I^m(1, 1) = I^m f(z) \) (see Uralegaddi and Somanatha [20]);

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and lemmas.

**Definition 2.** [12] Denote by \( Q \) the set of all functions \( f(z) \) that are analytic and injective on \( \overline{U} \setminus E(f) \) where

\[ E(f) = \{ \zeta : \zeta \in \partial U \quad \text{and} \quad \lim_{z \to \zeta} f(z) = \infty \}, \quad (2.1) \]

and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(f) \).

**Lemma 1.** [11] Let the function \( q(z) \) be univalent in the unit disc \( U \), and let \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \), with \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Set \( Q(z) = z \varphi'(z) \varphi(q(z)), h(z) = \theta(q(z)) + Q(z) \) and suppose that

(i) \( Q \) is a starlike function in \( U \),
(ii) \( \Re \left( \frac{zh'(z)}{Q(z)} \right) > 0 \) for \( z \in U \).

If \( p \) is analytic in \( U \) with \( p(0) = q(0) \), \( p(U) \subseteq D \) and

\[ \theta(p(z)) + z p'(z) \varphi(p(z)) < \theta(q(z)) + z q'(z) \varphi(q(z)), \quad (2.2) \]

then \( p(z) < q(z) \), and \( q \) is the best dominant.
Lemma 2. [17] Let \( q \) be a convex function in \( U \) and let \( \psi \in \mathbb{C} \) with \( \delta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) with
\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0; -\frac{\psi}{\delta} \right\}, \quad z \in U.
\]
If \( p(z) \) is analytic in \( U \), and
\[
\psi p(z) + \delta zp'(z) \prec \psi q(z) + \delta zq'(z),
\]
then \( p(z) \prec q(z) \), and \( q \) is the best dominant.

Lemma 3. [4] Let \( q(z) \) be a convex univalent function in the unit disc \( U \) and let \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that
(i) \( \text{Re} \left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} > 0 \) for \( z \in U \);
(ii) \( zq'(z)\varphi(q(z)) \) is starlike in \( U \).
If \( p \in H[q(0), 1] \cap Q \) with \( p(U) \subseteq D \), and \( \theta(p(z)) + zp'(z)\varphi(p(z)) \) is univalent in \( U \), and
\[
\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)),
\]
then \( q(z) \prec p(z) \), and \( q \) is the best subordinant.

Lemma 4. [12] Let \( q \) be convex univalent in \( U \) and let \( \delta \in \mathbb{C} \), with \( \text{Re}(\delta) > 0 \).
If \( p \in H[q(0), 1] \cap Q \) and \( p(z) + \delta zp'(z) \) is univalent in \( U \), then
\[
q(z) + \delta zq'(z) \prec p(z) + \delta zp'(z),
\]
implies \( q(z) \prec p(z) \) (\( z \in U \)), and \( q \) is the best subordinant.

This last lemma gives us a necessary and sufficient condition for the univalence of a special function which will be used in some particular cases:

Lemma 5. [15] The function \( q(z) = (1 - z)^{-2ab} \) is univalent in \( U \) if and only if \( |2ab - 1| \leq 1 \) or \( |2ab + 1| \leq 1 \).

3. Subordination results for analytic functions

Unless otherwise mentioned we shall assume throughout the paper that \( \beta \in \mathbb{C}^* \), \( \alpha > 0 \), \( \lambda > 0 \), \( \ell \geq 0 \), \( n \in \mathbb{N} \), \( m \in \mathbb{N}_0 \) and the powers understood as principle values.

Theorem 1. Let \( q(z) \) be convex univalent in \( U \), with \( q(0) = 1 \). Suppose that
\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0; -\frac{\alpha}{\beta} \right\}.
\]
If \( f(z) \in A(n) \) satisfies the subordination:
\[
\Phi(f, m, \lambda, \ell, \beta, \alpha) \prec q(z) + \frac{\beta}{\alpha} zq'(z),
\]

where
\[
\Phi(f, m, \lambda, \beta, \alpha) = \left[ 1 - \beta \left( \ell + \frac{1}{\lambda} \right) \right] \left( \frac{I^m(\lambda, \ell) f(z)}{z} \right)^\alpha + \beta \left( \frac{\ell + 1}{\lambda} \right) \frac{I^{m+1}(\lambda, \ell) f(z)}{I^m(\lambda, \ell) f(z)} \left( \frac{I^m(\lambda, \ell) f(z)}{z} \right)^\alpha,
\]
(3.3)
then
\[
\left( \frac{I^m(\lambda, \ell) f(z)}{z} \right)^\alpha \prec q(z),
\]
(3.4)
and \(q(z)\) is the best dominant of (3.2).

**Proof.** Define the function \(p(z)\) by
\[
p(z) = \left( \frac{I^m(\lambda, \ell) f(z)}{z} \right)^\alpha (z \in U).
\]
(3.5)
Then \(p(z)\) is analytic in \(U\) and \(p(0) = 1\). Differentiating (3.5) logarithmically with respect to \(z\), and using the identity (1.7) in the resulting equation, we have
\[
\left[ 1 - \beta \left( \ell + \frac{1}{\lambda} \right) \right] \left( \frac{I^m(\lambda, \ell) f(z)}{z} \right)^\alpha + \beta \left( \frac{\ell + 1}{\lambda} \right) \frac{I^{m+1}(\lambda, \ell) f(z)}{I^m(\lambda, \ell) f(z)} \times \left( \frac{I^m(\lambda, \ell) f(z)}{z} \right)^\alpha = p(z) + \frac{\beta}{\alpha} z p'(z).
\]
(3.6)
Thus the subordination (3.2) is equivalent to
\[
p(z) + \frac{\beta}{\alpha} z p'(z) \prec q(z) + \frac{\beta}{\alpha} z q'(z).
\]
(3.7)
Applying Lemma 2 with \(\gamma = \frac{\beta}{\alpha} (\alpha > 0)\), the proof of Theorem 1 is completed. ■

**Remark 1.** Putting \(m = \ell = 0, \lambda = n = 1\) and \(\beta \geq 0\) in Theorem 1, we obtain the result obtained by Shanmugam et al. [18, Theorem 3.1].

Putting \(\lambda = 1\) and \(\ell = 0\) in Theorem 1, we obtain the following corollary.

**Corollary 1.** Let \(q(z)\) be convex univalent in \(U\), with \(q(0) = 1\) and suppose that \(q(z)\) satisfies the condition (3.1). If \(f(z) \in A(n)\) satisfies the subordination:
\[
\Phi(f, m, \beta, \alpha) \prec q(z) + \frac{\beta}{\alpha} z q'(z),
\]
where
\[
\Phi(f, m, \beta, \alpha) = (1 - \beta) \left( \frac{D^m f(z)}{z} \right)^\alpha + \beta \frac{D^{m+1} f(z)}{D^m f(z)} \left( \frac{D^m f(z)}{z} \right)^\alpha,
\]
(3.8)
then \(\left( \frac{D^m f(z)}{z} \right)^\alpha \prec q(z)\) and \(q(z)\) is the best dominant.
Putting $\ell = 0$ in Theorem 1, we obtain the following corollary.

**Corollary 2.** Let $q(z)$ be convex univalent in $U$, with $q(0) = 1$ and suppose that $q(z)$ satisfy the condition (3.1). If $f(z) \in A(n)$ satisfies the subordination

$$\Phi(f, m, \lambda, \beta, \alpha) \prec q(z) + \frac{\beta}{\alpha} z q'(z),$$

where

$$\Phi(f, m, \lambda, \beta, \alpha) = \left(1 - \frac{\beta}{\lambda}\right) \left(\frac{D^m_\lambda f(z)}{z}\right)^\alpha + \frac{\beta}{\alpha} \frac{D^{m+1}_\lambda f(z)}{D^m_\lambda f(z)} \left(\frac{D^m_\lambda f(z)}{z}\right)^\alpha,$$  \hspace{1cm} (3.9)

then $\left(\frac{D^m_\lambda f(z)}{z}\right)^\alpha \prec q(z)$ and $q(z)$ is the best dominant.

Putting $\lambda = 1$ in Theorem 1, we obtain the following corollary.

**Corollary 3.** Let $q(z)$ be convex univalent in $U$, with $q(0) = 1$ and suppose that $q(z)$ satisfy (3.1). If $f(z) \in A(n)$ satisfies the subordination

$$\Phi(f, m, \ell, \beta, \alpha) \prec q(z) + \frac{\beta}{\alpha} z q'(z),$$

where

$$\Phi(f, m, \ell, \beta, \alpha) = [1 - \beta(\ell + 1)] \left(\frac{I^m(\ell)f(z)}{z}\right)^\alpha + \beta(\ell + 1) \frac{I^{m+1}(\ell)f(z)}{I^m(\ell)f(z)} \left(\frac{I^m(\ell)f(z)}{z}\right)^\alpha,$$  \hspace{1cm} (3.10)

then $\left(\frac{I^m(\ell)f(z)}{z}\right)^\alpha \prec q(z)$ and $q(z)$ is the best dominant.

Taking $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Theorem 1, we obtain the following corollary.

**Corollary 4.** Let $-1 \leq B < A \leq 1$ and suppose that

$$\text{Re} \left\{\frac{1-Bz}{1+Bz}\right\} > \max \left\{0, -\frac{\alpha}{\beta}\right\}.$$  

If $f(z) \in A(n)$ satisfies the subordination

$$\Phi(f, m, \lambda, \ell, \beta, \alpha) \prec \frac{1+Az}{1+Bz} + \frac{\beta}{\alpha} \frac{(A-B)z}{(1+Bz)^2},$$

where $\Phi(f, m, \lambda, \ell, \beta, \alpha)$ is given by (3.3), then

$$\left(\frac{I^m(\lambda, \ell)f(z)}{z}\right)^\alpha \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.
Remark 2. Putting \( m = \ell = 0, \lambda = n = 1 \) and \( \beta \geq 0 \) in Corollary 4, we obtain the result obtained by Shanmugam et al. [18, Corollary 3.2].

Theorem 2. Let \( q(z) \) be univalent in \( U \), and \( \alpha, \gamma \in \mathbb{C} \). Suppose that \( q(z) \) satisfies
\[
\text{Re} \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0. \tag{3.11}
\]
If \( f(z) \in A(n) \) satisfies the subordination
\[
\psi(f, m, \lambda, \ell, \beta, \alpha) \prec 1 + \gamma \frac{zq'(z)}{q(z)}, \tag{3.12}
\]
where
\[
\psi(f, m, \lambda, \ell, \beta, \alpha) = 1 + \gamma \alpha \left( \frac{\ell + 1}{\lambda} \right) \left[ \frac{I^{m+1}(\lambda, \ell)f(z)}{I^m(\lambda, \ell)f(z)} - 1 \right], \tag{3.13}
\]
then \( \left( \frac{I^m(\lambda, \ell)f(z)}{z} \right)^{\alpha} < q(z) \) and \( q(z) \) is the best dominant.

Proof. Let \( p(z) \) be defined by (3.5). Then, simple computations show that
\[
\frac{zp'(z)}{p(z)} = \alpha \left( \frac{\ell + 1}{\lambda} \right) \left[ \frac{I^{m+1}(\lambda, \ell)f(z)}{I^m(\lambda, \ell)f(z)} - 1 \right].
\]
Putting \( \theta(w) = 1 \) and \( \varphi(w) = \frac{\gamma}{w} \), we can observe that \( \theta(w) \) is analytic in \( \mathbb{C} \), \( \varphi(w) \) is analytic in \( \mathbb{C}^* \) and \( \varphi(w) \neq 0 \) \((w \in \mathbb{C}^*)\). If
\[
\psi(z) = zq'(z) = \varphi(q(z)) = \gamma \frac{zq'(z)}{q(z)}
\]
and
\[
h(z) = \theta(q(z)) + \psi(z) = 1 + \gamma \frac{zq'(z)}{q(z)},
\]
then, from (3.11), we find that \( \psi(z) \) is starlike univalent in \( U \) and
\[
\text{Re} \left( \frac{zh'(z)}{\psi(z)} \right) = \text{Re} \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0.
\]
Then applying Lemma 1, the proof is completed. \( \blacksquare \)

Remark 3. Taking \( m = \ell = 0 \) and \( \lambda = n = 1 \) in Theorem 2, we obtain the result obtained by Shanmugam et al. [18, Theorem 3.4].

Putting \( \lambda = 1 \) and \( \ell = 0 \) in Theorem 2, we obtain the following corollary.

Corollary 5. Assume that (3.11) holds. If \( f(z) \in A(n) \), and
\[
1 + \gamma \alpha \left[ \frac{D^{m+1}f(z)}{D^m f(z)} - 1 \right] < 1 + \gamma \frac{zq'(z)}{q(z)},
\]
then \( \left( \frac{D^m f(z)}{z} \right)^{\alpha} \prec q(z) \) and \( q(z) \) is the best dominant.
Putting $\ell = 0$ in Theorem 2, we obtain the following corollary.

**Corollary 6.** Assume that (3.11) holds. If $f(z) \in A(n)$, and

$$1 + \gamma \alpha \left( \frac{D_{\lambda}^{m+1} f(z)}{D_{\lambda}^{m} f(z)} - 1 \right) \prec 1 + \gamma \frac{z f'(z)}{q(z)},$$

then $\left( \frac{D_{\lambda}^{m} f(z)}{z} \right)^{\alpha} \prec q(z)$ and $q(z)$ is the best dominant.

Putting $\lambda = 1$ in Theorem 2, we obtain the following corollary.

**Corollary 7.** Assume that (3.11) holds. If $f(z) \in A(n)$, and

$$1 + \gamma \alpha \left( \frac{I_{\lambda}^{m+1} (\ell) f(z)}{I_{\lambda}^{m} (\ell) f(z)} - 1 \right) \prec 1 + \gamma \frac{z f'(z)}{q(z)},$$

then $\left( \frac{I_{\lambda}^{m} (\ell) f(z)}{z} \right)^{\alpha} \prec q(z)$ and $q(z)$ is the best dominant.

Taking $q(z) = \frac{1}{(1-z)^{2\alpha}}$ ($\alpha, b \in \mathbb{C}^*$), $\gamma = \frac{1}{\alpha b}$, $\lambda = n = 1$ and $m = \ell = 0$ in Theorem 2, we obtain the next result due to Obradović et al. [13, Theorem 1].

**Corollary 8.** [13] Let $\alpha, b \in \mathbb{C}^*$ such that $|2\alpha b - 1| \leq 1$ or $|2\alpha b + 1| \leq 1$. Let $f(z) \in A$ and suppose that $\frac{f(z)}{z} \neq 0$ for all $z \in U$. If

$$1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1 + z}{1 - z},$$

then $\left( \frac{f(z)}{z} \right)^{\alpha} \prec (1 - z)^{-2\alpha b}$ and $(1 - z)^{-2\alpha b}$ is the best dominant.

**Remark 4.** For $\alpha = 1$, Corollary 8 reduces to the recent result of Srivastava and Lashin [19, Corollary 1].

Taking $q(z) = (1 + Bz)^{\alpha(A-B)}$, $-1 \leq B < A \leq 1$, $B \neq 0$, $\alpha \in \mathbb{C}^*$, $\gamma = 1$, $m = \ell = 0$ and $\lambda = 1$ in Theorem 2, we obtain the following corollary.

**Corollary 9.** Let $-1 \leq B < A \leq 1$, with $B \neq 0$, and suppose that

$$\left| \frac{\alpha(A-B)}{B} - 1 \right| \leq 1 \quad \text{or} \quad \left| \frac{\alpha(A-B)}{B} + 1 \right| \leq 1.$$

If $f(z) \in A(n)$ such that $\frac{f(z)}{z} \neq 0$ for all $z \in U$, and let $\alpha \in \mathbb{C}^*$. If

$$1 + \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1 + [B + \alpha(A-B)]z}{1 + Bz},$$

then

$$\left( \frac{f(z)}{z} \right)^{\alpha} \prec (1 + Bz)^{\frac{\alpha(A-B)}{B}},$$

and $(1 + Bz)^{\frac{\alpha(A-B)}{B}}$ is the best dominant.
Remark 5. For $\alpha = n = 1$, Corollary 9 reduces to the recent result of Obradović and Owa [14].

Putting $q(z) = (1 - z)^{-2ab\cos\lambda e^{-i\lambda}} (\alpha, b \in \mathbb{C}^*; |\lambda| < \frac{\pi}{2})$, $\gamma = \frac{e^{i\lambda}}{ab\cos\lambda}, n = \lambda = 1$ and $m = \ell = 0$ in Theorem 2, we obtain the next result due to Aouf et al. [3, Theorem 1].

Corollary 10. [3] Let $\alpha, b \in \mathbb{C}^*$ and $|\lambda| < \frac{\pi}{2}$, and suppose that $|2ab\cos\lambda e^{-i\lambda} - 1| \leq 1$ or $|2ab\cos\lambda e^{-i\lambda} + 1| \leq 1$. Let $f(z) \in A$ such that $f(z) \neq 0$ for all $z \in U$. If

$$1 + \frac{e^{i\lambda}}{b\cos\lambda} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec 1 + \frac{1 + z}{1 - z},$$

then

$$\left( \frac{f(z)}{z} \right)^{\alpha} \prec (1 - z)^{-2ab\cos\lambda e^{-i\lambda}},$$

and $(1 - z)^{-2ab\cos\lambda e^{-i\lambda}}$ is the best dominant.

4. Superordination and Sandwich results

Theorem 3. Let $q(z)$ be convex in $U$ with $q(0) = 1$, and $\beta \in \mathbb{C}, \text{Re}\beta > 0$. If $f(z) \in A(n)$ such that $\left( \frac{I^m(\lambda,\ell)f(z)}{f(z)} \right)^{\alpha} \in H[q(0), 1] \cap \mathbb{Q}$ and $\Phi(f, m, \lambda, \ell, \beta, \alpha)$ is univalent in $U$ and satisfies the superordination:

$$q(z) + \frac{\beta}{\alpha} z q'(z) \prec \Phi(f, m, \lambda, \ell, \beta, \alpha),$$

(4.1)

where $\Phi(f, m, \lambda, \ell, \beta, \alpha)$ is given by (3.3), then

$$q(z) \prec \left( \frac{I^m(\lambda,\ell)f(z)}{z} \right)^{\alpha}$$

and $q(z)$ is the best subordinant.

Proof. Let $p(z)$ be given by (3.5) and proceeding as in the proof of Theorem 1, the subordination (4.1) becomes

$$q(z) + \frac{\beta}{\alpha} z q'(z) \prec p(z) + \frac{\beta}{\alpha} z p'(z).$$

The proof follows by an application of Lemma 4. 

Theorem 4. Let $q(z)$ be convex univalent in $U$, $\beta \in \mathbb{C}, \text{Re}\beta > 0$, and $\left( \frac{I^m(\lambda,\ell)f(z)}{f(z)} \right)^{\alpha} \in H[q(0), 1] \cap \mathbb{Q}$. If $f(z) \in A(n)$ and

$$1 + \gamma z q'(z) \prec 1 + \gamma \alpha \left( \frac{\ell + 1}{\lambda} \right) \left[ \frac{I^{m+1}(\lambda,\ell)f(z)}{I^m(\lambda,\ell)f(z)} - 1 \right],$$

(4.2)

then

$$q(z) \prec \left( \frac{I^m(\lambda,\ell)f(z)}{z} \right)^{\alpha}$$

and $q(z)$ is the best subordinant.
REMARK 6. Putting $m = \ell = 0, \lambda = n = 1$ in Theorem 4, we obtain the result obtained by Shanmugam et al. [18, Theorem 4.3].

Combining Theorem 1 with Theorem 3 and Theorem 2 with Theorem 4, we state the following “Sandwich results”.

**Theorem 5.** Let $q_1, q_2$ be convex in $U$ with $q_1(0) = q_2(0) = 1, \beta \in \mathbb{C}, \text{Re} \beta > 0$ and satisfies (3.1). If $f(z) \in A(n), \left(\frac{I^m(\lambda, \ell)f(z)}{z}\right)^\alpha \in H[q(0), 1] \cap Q, \Phi(f, m, \lambda, \ell, \beta, \alpha)$ is univalent in the unit disc $U$, where $\Phi(f, m, \lambda, \ell, \beta, \alpha)$ is defined by (3.3) and

$$q_1(z) + \frac{\beta}{\alpha}zq_1'(z) < \Phi(f, m, \lambda, \ell, \beta, \alpha) < q_2(z) + \frac{\beta}{\alpha}zq_2'(z), \quad (4.3)$$

then

$$q_1(z) \prec \left(\frac{I^m(\lambda, \ell)f(z)}{z}\right)^\alpha \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and best dominant.

Putting $\lambda = 1$ and $\ell = 0$ in Theorem 5, we obtain the following corollary.

**Corollary 11.** Let $q_1(z), q_2(z)$ be convex in $U$ with $q_1(0) = q_2(0) = 1, \beta \in \mathbb{C}, \text{Re} \beta > 0$ and satisfies (3.1). If $f(z) \in A(n), \left(\frac{D^m(f(z))}{z}\right)^\alpha \in H[q(0), 1] \cap Q, \Phi(f, m, \beta, \alpha)$ is univalent in the unit disc $U$, where $\Phi(f, m, \beta, \alpha)$ is defined by (3.8) and

$$q_1(z) + \frac{\beta}{\alpha}zq_1'(z) < \Phi(f, m, \beta, \alpha) < q_2(z) + \frac{\beta}{\alpha}zq_2'(z),$$

then

$$q_1(z) \prec \left(\frac{D^m(f(z))}{z}\right)^\alpha \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and best dominant.

Putting $\ell = 0$ in Theorem 5, we obtain the following corollary.

**Corollary 12.** Let $q_1(z), q_2(z)$ be convex in $U$ with $q_1(0) = q_2(0) = 1, \beta \in \mathbb{C}, \text{Re} \beta > 0$ and satisfies (3.1). If $f(z) \in A(n), \left(\frac{D^m(f(z))}{z}\right)^\alpha \in H[q(0), 1] \cap Q, \Phi(f, m, \beta, \alpha)$ is univalent in the unit disc $U$, where $\Phi(f, m, \beta, \alpha)$ is defined by (3.9) and

$$q_1(z) + \frac{\beta}{\alpha}zq_1'(z) < \Phi(f, m, \beta, \alpha) < q_2(z) + \frac{\beta}{\alpha}zq_2'(z),$$

then

$$q_1(z) \prec \left(\frac{D^m(f(z))}{z}\right)^\alpha \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and best dominant.

Putting $\lambda = 1$ in Theorem 5, we obtain the following corollary.

**Corollary 13.** Let $q_1(z), q_2(z)$ be convex in $U$ with $q_1(0) = q_2(0) = 1, \beta \in \mathbb{C}, \text{Re} \beta > 0$ and satisfies (3.1). If $f(z) \in A(n), \left(\frac{I^m(\ell)f(z)}{z}\right)^\alpha \in H[q(0), 1] \cap Q,
\( \Phi(f, m, \ell, \beta, \alpha) \) is univalent in the unit disc \( U \), where \( \Phi(f, m, \ell, \beta, \alpha) \) is defined by (3.10) and

\[
q_1(z) + \frac{\beta}{\alpha} z q_1'(z) \prec \Phi(f, m, \ell, \beta, \alpha) \prec q_2(z) + \frac{\beta}{\alpha} z q_2'(z),
\]

then

\[
q_1(z) \prec \left( \frac{I^m(\ell) f(z)}{z} \right)^\alpha \prec q_2(z)
\]

and \( q_1(z) \) and \( q_2(z) \) are, respectively, the best subordinant and best dominant.

**Theorem 6.** Let \( q_1(z), q_2(z) \) be convex in \( U \) with \( q_1(0) = q_2(0) = 1, \beta \in \mathbb{C}, \) \( \text{Re} \beta > 0 \) and satisfies (3.1). If \( f(z) \in A(n), (I^m(\lambda, \ell) f(z) )^\alpha \in H[q(0), 1] \cap Q, \) \( \psi(f, m, \lambda, \ell, \beta, \alpha) \) is univalent in the unit disc \( U \), where \( \psi(f, m, \lambda, \ell, \beta, \alpha) \) is defined by (3.13) and

\[
1 + \gamma \frac{z q_1'(z)}{q_1(z)} \prec \psi(f, m, \lambda, \ell, \beta, \alpha) \prec 1 + \gamma \frac{z q_2'(z)}{q_2(z)},
\]

then

\[
q_1(z) \prec \left( \frac{I^m(\lambda, \ell) f(z)}{z} \right)^\alpha \prec q_2(z)
\]

and \( q_1(z) \) and \( q_2(z) \) are, respectively, the best subordinant and the best dominant.

Putting \( \lambda = 1 \) and \( \ell = 0 \) in Theorem 6, we obtain the following corollary.

**Corollary 14.** Let \( q_1(z), q_2(z) \) be convex in \( U \) with \( q_1(0) = q_2(0) = 1, \beta \in \mathbb{C}, \) \( \text{Re} \beta > 0 \) and satisfies (3.1). If \( f(z) \in A(n), (\frac{D^{m+1} f(z)}{D^m f(z)} )^\alpha \in H[q(0), 1] \cap Q, \)

\[
1 + \gamma \alpha \left[ \frac{D^{m+1} f(z)}{D^m f(z)} - 1 \right]
\]

is univalent in \( U \) and

\[
1 + \gamma \frac{z q_1'(z)}{q_1(z)} < 1 + \gamma \alpha \left[ \frac{D^{m+1} f(z)}{D^m f(z)} - 1 \right] < 1 + \gamma \frac{z q_2'(z)}{q_2(z)},
\]

then

\[
q_1(z) \prec \left( \frac{D^m f(z)}{z} \right)^\alpha \prec q_2(z)
\]

and \( q_1(z) \) and \( q_2(z) \) are, respectively, the best subordinant and the best dominant.

Putting \( \ell = 0 \) in Theorem 6, we obtain the following corollary.

**Corollary 15.** Let \( q_1(z), q_2(z) \) be convex in \( U \) with \( q_1(0) = q_2(0) = 1, \beta \in \mathbb{C}, \) \( \text{Re} \beta > 0 \) and satisfies (3.1). If \( f(z) \in A(n), (\frac{D^{m+1} f(z)}{D^m f(z)} )^\alpha \in H[q(0), 1] \cap Q, \)

\[
1 + \gamma \alpha \left[ \frac{D^{m+1} f(z)}{D^m f(z)} - 1 \right]
\]

is univalent in \( U \) and

\[
1 + \gamma \frac{z q_1'(z)}{q_1(z)} < 1 + \gamma \alpha \left[ \frac{D^{m+1} f(z)}{D^m f(z)} - 1 \right] < 1 + \gamma \frac{z q_2'(z)}{q_2(z)},
\]

then

\[
q_1(z) \prec \left( \frac{D^m f(z)}{z} \right)^\alpha \prec q_2(z)
\]

and \( q_1(z) \) and \( q_2(z) \) are, respectively, the best subordinant and the best dominant.
is univalent in $U$ and
\[
1 + \gamma \frac{zq_1'(z)}{q_1(z)} < 1 + \gamma \frac{\alpha}{\lambda} \left[ \frac{D_{\lambda}^{m+1}f(z)}{D_{\lambda}^{\alpha}f(z)} - 1 \right] < 1 + \gamma \frac{zq_2'(z)}{q_2(z)}
\]
then
\[
q_1(z) < \left( \frac{D_{\lambda}^m(f(z))^\alpha}{z} \right) < q_2(z)
\]
and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Putting $\lambda = 1$ in Theorem 6, we obtain the following corollary.

**Corollary 16.** Let $q_1(z), q_2(z)$ be convex in $U$ with $q_1(0) = q_2(0) = 1$, $\beta \in \mathbb{C}$, $\text{Re}\beta > 0$ and satisfies (3.1). If $f(z) \in A(n)$, $(\frac{I_m^{(\ell)} f(z)}{z})^\alpha \in H[q(0), 1] \cap Q$,
\[
1 + \gamma \alpha (\ell + 1) \left[ \frac{I_{\ell+1}^m f(z)}{I_{\ell}^m f(z)} - 1 \right]
\]
is univalent in $U$ and
\[
1 + \gamma \frac{zq_1'(z)}{q_1(z)} < 1 + \gamma \alpha (\ell + 1) \left[ \frac{I_{\ell+1}^m f(z)}{I_{\ell}^m f(z)} - 1 \right] < 1 + \gamma \frac{zq_2'(z)}{q_2(z)}
\]
then
\[
q_1(z) < \left( \frac{I_{\ell}^m(f(z))^\alpha}{z} \right) < q_2(z)
\]
and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

**Remark 7.** Putting $m = \ell = 0, \lambda = n = 1$ and $\beta \geq 0$ in Theorem 6, we obtain the following result which improves the result of Shanmugam et al. [18, Theorem 5.2].

**Corollary 17.** Let $q_1(z), q_2(z)$ be convex in $U$ with $q_1(0) = q_2(0) = 1$, $\beta \in \mathbb{C}$, $\text{Re}\beta > 0$ and satisfies (3.1). If $f(z) \in A(n)$, $(\frac{f(z)}{z})^\alpha \in H[q(0), 1] \cap Q$, $1 + \gamma \alpha (\frac{zf'(z)}{f(z)} - 1)$ is univalent in $U$ and
\[
1 + \gamma \frac{zq_1'(z)}{q_1(z)} < 1 + \gamma \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) < 1 + \gamma \frac{zq_2'(z)}{q_2(z)}
\]
then
\[
q_1(z) < \left( \frac{f(z)}{z} \right)^\alpha < q_2(z)
\]
and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

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