THE QUASI-HADAMARD PRODUCTS OF UNIFORMLY CONVEX FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

M. K. Aouf, A. Shamandy, A. O. Mostafa and E. A. Adwan

Abstract. The purpose of this paper is to obtain many interesting results about the quasi-Hadamard products of uniformly convex functions defined by Dziok-Srivastava operator belonging to the class $T_{q,s}(\alpha_1; \alpha, \beta)$.

1. Introduction

Let $T$ denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0),$$

(1.1)

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Let $C(\gamma)$ and $T^*(\gamma)$ denote the subclasses of $T$ which are, respectively, convex and starlike functions of order $\gamma$, $0 \leq \gamma < 1$. For convenience, we write $C(0) = C$ and $T^*(0) = T^*$ (see [9]).

A function $f \in T$ is said to be in $UST(\beta, \gamma)$, the class of $\beta$-uniformly starlike functions of order $\gamma$, $-1 \leq \gamma < 1$, if it satisfies the condition

$$\Re \left\{ \frac{zf'(z)}{f(z)} - \gamma \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad (\beta \geq 0).$$

(1.2)

Replacing $f(z)$ in (1.2) by $zf'(z)$ we have the condition

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} - \gamma \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right|, \quad (\beta \geq 0),$$

required for the function $f$ to be in the subclass $UCT(\beta, \gamma)$ of $\beta$-uniformly convex functions of order $\gamma$ (see [2]).

Let $f_j(z) \in T$ ($j = 1, \ldots, t$) be given by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0; j = 1, 2, \ldots, t).$$

(1.3)

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Then the quasi-Hadamard product (or convolution) of these functions is defined by

\[(f_1 * f_2 * \cdots * f_t)(z) = z - \sum_{n=2}^{\infty} \left( \prod_{j=1}^{t} a_{n,j} \right) z^n. \tag{1.4}\]

For positive real parameters \(\alpha_1, \ldots, \alpha_q \) and \(\beta_1, \ldots, \beta_s \), \(\beta_i \in C \setminus Z^{-}; Z^{-} = \{0, -1, -2, \ldots\} \); \(i = 1, 2, \ldots, s\), the Dziok-Srivastava operator (see [3] and [4]) \(H_{q,s}(\alpha_1): T \to T\) is given by

\[H_{q,s}(\alpha_1)f(z) = z \cdot qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) * f(z) = z - \sum_{n=2}^{\infty} \Psi_n a_n z^n, \tag{1.5}\]

where

\[\Psi_n = \frac{(\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1} (n-1)!}, \tag{1.6}\]

\[(\theta)_n = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} = \begin{cases} 1, & n = 0 \\ \theta(\theta + 1) \cdots (\theta + n - 1), & n \in N. \end{cases}\]

and \(qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \) \((q \leq s + 1; s, q \in N_0 = N \cup \{0\}, N = \{1, 2, \ldots\}; z \in U)\) is the generalized hypergeometric function.

\[qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)n \cdots (\alpha_q)n}{(\beta_1)n \cdots (\beta_s)n!} z^n. \]

For \(-1 \leq \gamma < 1, \beta \geq 0\), and for all \(z \in U\), Aouf and Murugusundaramoorthy [1] defined the subclass \(T_{q,s}([\alpha_1]; \gamma, \beta)\) of functions of \(T\) which satisfy:

\[\text{Re} \left( \frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} - \gamma \right) > \beta \left| \frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} - 1 \right|, z \in U. \tag{1.7}\]

They also proved [1] that the necessary and sufficient condition for functions \(f(z)\) of the form \((1.1)\) to be in the class \(T_{q,s}([\alpha_1]; \gamma, \beta)\) is that:

\[\sum_{n=2}^{\infty} \left( n(1 + \beta) - (\gamma + \beta) \right) \Psi_n a_n \leq 1 - \gamma. \tag{1.8}\]

We note that for suitable choices of \(q, s, \gamma\) and \(\beta\), we obtain the following subclasses studied by various authors.

1. For \(q = 2\) and \(s = \alpha_1 = \alpha_2 = \beta_1 = 1\) in (1.7), the class \(T_{2,1}(1; \gamma, \beta)\) reduces to the class \(S_p T(\gamma, \beta)\) \((-1 \leq \gamma < 1, \beta \geq 0)\) and the class \(S_p T(\gamma, 1)\) which for \(\beta = 1\) reduces to the class \(S_p T(\gamma)\) (see [2]).

2. For \(q = 2, s = 1, \alpha_1 = a (a > 0)\), \(\alpha_2 = 1\) and \(\beta_1 = c (c > 0)\) in (1.7), the class \(T_{2,1}(a; \gamma, \beta)\) reduces to the class \(S_p T(a, c; \gamma, \beta)\) \((-1 \leq \gamma < 1, \beta \geq 0)\) (see [5]).

3. For \(q = 2, s = 1, \alpha_1 = \lambda + 1 (\lambda > -1)\), \(\alpha_2 = 1\) and \(\beta_1 = 1\) in (1.7), the class \(T_{2,1}(\lambda + 1, 1; \gamma, \beta)\) reduces to the class \(S_p T(\lambda; \gamma, \beta)\) \((-1 \leq \gamma < 1, \beta \geq 0)\) (see [8]).
(4) For $q = 2$, $s = 1$, $\alpha_1 = v + 1 (v > -1)$, $\alpha_2 = 1$ and $\beta_1 = v + 2$ in (1.7), the class $T_{2,1}(v + 1; v + 2; \gamma, \beta)$ reduces to the class $S_\mu T(v; \gamma, \beta) (-1 \leq \gamma < 1, \beta \geq 0)$ (see [1]).

(5) For $q = 2$, $s = 1$, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\beta_1 = 2 - \mu (\mu \neq 2, 3, \ldots)$ in (1.7), the class $T_{2,1}(2, 1; 2 - \mu; \gamma, \beta)$ reduces to the class $S_\mu T(\mu; \gamma, \beta) (-1 \leq \gamma < 1, \beta \geq 0)$ (see [1]).

2. Main results

Unless otherwise mentioned, we shall assume in the remainder of this paper that the parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ are positive real numbers, $-1 \leq \gamma < 1$, $\beta \geq 0$, $z \in U$, $\Psi_n$ is defined by (1.6), $\Psi_n \geq 1$ and $j = 1, 2, \ldots, t$.

**Theorem 1.** Let the functions $f_j(z)$ defined by (1.3) be in the class $T_{q,s}([\alpha_1]; \gamma_j, \beta)$. Then we have $(f_1 \ast \cdots \ast f_t)(z) \in T_{q,s}([\alpha_1]; \delta, \beta)$, where

$$
\delta = 1 - \frac{(1 + \beta) \prod_{j=1}^t (1 - \gamma_j)}{\prod_{j=1}^t (2 + \beta - \gamma_j) \Psi_2^{\mu - 1} - \prod_{j=1}^t (1 - \gamma_j)}.
$$

(2.1)
The result is sharp for the functions $f_j(z)$ given by

$$
f_j(z) = z - \frac{(1 - \gamma_j)}{(2 + \beta - \gamma_j) \Psi_2} z^2.
$$

(2.2)

**Proof.** Employing the technique used earlier by Schild and Silverman [7] and Owa [6], we prove Theorem 1 by using mathematical induction on $t$. For $t = 2$, (1.8) gives

$$
\sum_{n=2}^{\infty} \frac{|n(1 + \beta) - (\gamma_j + \beta)| \Psi_n}{(1 - \gamma_j)} a_{n,j} \leq 1 \ (j = 1, 2).
$$

(2.3)

By the Cauchy-Schwarz inequality, we have

$$
\sum_{n=2}^{\infty} \sqrt{\frac{2}{\prod_{j=1}^t \frac{|n(1 + \beta) - (\gamma_j + \beta)| \Psi_n}{(1 - \gamma_j)}}} \sqrt{a_{n,1} a_{n,2}} \leq 1.
$$

(2.4)

To prove the case when $t = 2$, we need to find the largest $\delta (-1 \leq \delta < 1)$ such that

$$
\sum_{n=2}^{\infty} \frac{|n(1 + \beta) - (\delta + \beta)| \Psi_n}{(1 - \delta)} a_{n,1} a_{n,2} \leq 1,
$$

(2.5)

thus, it suffices to show that

$$
\frac{|n(1 + \beta) - (\delta + \beta)| \Psi_n}{(1 - \delta)} a_{n,1} a_{n,2} \leq \frac{\sqrt{\prod_{j=1}^t |n(1 + \beta) - (\gamma_j + \beta)| \Psi_n}}{\sqrt{\prod_{j=1}^t (1 - \gamma_j)}} \sqrt{a_{n,1} a_{n,2}}
$$

or, equivalently, to

$$
\sqrt{a_{n,1} a_{n,2}} \leq \frac{(1 - \delta) \sqrt{\prod_{j=1}^t |n(1 + \beta) - (\gamma_j + \beta)| \Psi_n}}{|n(1 + \beta) - (\delta + \beta)| \Psi_n \sqrt{\prod_{j=1}^t (1 - \gamma_j)}}.
$$
By noting that
\[ \sqrt{a_{n,1} a_{n,2}} \leq \sqrt{\prod_{j=1}^{2} (1 - \gamma_j)} \sqrt{\prod_{j=1}^{2} [n(1 + \beta) - (\gamma_j + \beta)] \Psi_n}, \]
consequently, we need only to prove that
\[ \frac{\prod_{j=1}^{2} (1 - \gamma_j)}{\prod_{j=1}^{2} [n(1 + \beta) - (\gamma_j + \beta)] \Psi_n} \leq \frac{(1 - \delta)}{[n(1 + \beta) - (\delta + \beta)]} \]
which is equivalent to
\[ \delta \leq 1 - \frac{(n - 1)(1 + \beta) \prod_{j=1}^{2} (1 - \gamma_j)}{\prod_{j=1}^{2} [n(1 + \beta) - (\gamma_j + \beta)] \Psi_n - \prod_{j=1}^{2} (1 - \gamma_j)}. \]
Since
\[ B(n) = 1 - \frac{(n - 1)(1 + \beta) \prod_{j=1}^{2} (1 - \gamma_j)}{\prod_{j=1}^{2} [n(1 + \beta) - (\gamma_j + \beta)] \Psi_n - \prod_{j=1}^{2} (1 - \gamma_j)}, \]
is an increasing function of \( n \ (n \geq 2) \), then
\[ \delta \leq B(2) = 1 - \frac{(1 + \beta) \prod_{j=1}^{2} (1 - \gamma_j)}{\prod_{j=1}^{2} (2 + \beta - \gamma_j) \Psi_2 - \prod_{j=1}^{2} (1 - \gamma_j)}. \]
Therefore, the result is true for \( t = 2 \).

Suppose that the result is true for any positive integer \( t = k \). Then we have
\[ (f_1 \ast \cdots \ast f_k \ast f_{k+1})(z) \in T_{q,s}(\alpha_1; \lambda, \beta), \]
where
\[ \lambda = 1 - \frac{(1 + \beta)(1 - \gamma_{k+1})(1 - \delta)}{(2 + \beta - \gamma_{k+1})(2 + \beta - \delta) \Psi_2 - (1 - \gamma_{k+1})(1 - \delta)}, \]
and \( \delta \) is given by (2.2). After simple calculations, we have
\[ \lambda = 1 - \frac{(1 + \beta) \prod_{j=1}^{k+1} (1 - \gamma_j)}{\prod_{j=1}^{k+1} (2 + \beta - \gamma_j) \Psi_2 - \prod_{j=1}^{k+1} (1 - \gamma_j)}. \quad (2.6) \]
This shows that the result is true for \( t = k + 1 \). Therefore, by mathematical induction, the result is true for any positive integer \( t \ (t \geq 2) \).

Taking the functions \( f_t(z) \) given by (2.2), we have
\[ (f_1 \ast \cdots \ast f_t)(z) = z - \prod_{j=1}^{t} \frac{1 - \gamma_j}{(2 + \beta - \gamma_j) \Psi_2} z^2 = z - H_2 z^2, \quad (2.7) \]
which shows that
\[ \sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\delta + \beta)] \Psi_n H_2}{(1 - \delta)} = \frac{(2 + \beta - \delta) \Psi_2}{(1 - \delta)} \cdot \prod_{j=1}^{t} \frac{(1 - \gamma_j)}{(2 + \beta - \gamma_j) \Psi_2} = 1; \]
Consequently, the result is sharp for functions $f_j(z)$ given by (2.2). This completes the proof of Theorem 1.

Letting $\gamma_j = \gamma$ in Theorem 1, we obtain the following corollary.

**Corollary 1.** Let the functions $f_j(z)$ defined by (1.3) be in the class $T_{q,s}([\alpha_1]; \gamma, \beta)$. Then we have $(f_1 \ast \cdots \ast f_t)(z) \in T_{q,s}([\alpha_1]; \delta, \beta)$,

$$
\delta = 1 - \frac{(1 + \beta)(1 - \gamma)^t}{(2 + \beta - \gamma)^t \Psi_2^{t-1} - (1 - \gamma)^t}. \tag{2.8}
$$

The result is sharp for the functions $f_j(z)$ given by

$$
f_j(z) = z - \frac{(1 - \gamma)}{(2 + \beta - \gamma) \Psi_2} z^2. \tag{2.9}
$$

Putting $t = 2$ in Corollary 1, we obtain the following corollary.

**Corollary 2.** Let the functions $f_j(z)$ ($j = 1, 2$) defined by (1.3) be in the class $T_{q,s}([\alpha_1]; \gamma, \beta)$. Then we have $(f_1 \ast f_2)(z) \in T_{q,s}([\alpha_1]; \delta, \beta)$, where

$$
\delta = 1 - \frac{(1 + \beta)(1 - \gamma)^2}{(2 + \beta - \gamma)^2 \Psi_2 - (1 - \gamma)^2}. \tag{2.10}
$$

The result is sharp.

Next, similarly by applying the method of proof of Theorem 1, we easily get the following result.

**Theorem 2.** Let the functions $f_j(z)$ defined by (1.3) be in the class $T_{q,s}([\alpha_1]; \gamma, \zeta_j)$, $\zeta_j \geq 0$. Then we have $(f_1 \ast \cdots \ast f_t)(z) \in T_{q,s}([\alpha_1]; \gamma, \eta)$, where

$$
\eta = \prod_{j=1}^{t} \frac{(2 + \zeta_j - \gamma) \Psi_2^{t-1}}{(1 - \gamma)^{t-1}} + \gamma - 2. \tag{2.10}
$$

The result is sharp for the functions $f_j(z)$ given by

$$
f_j(z) = z - \frac{(1 - \gamma)}{(2 + \zeta_j - \gamma) \Psi_2} z^2. \tag{2.11}
$$

Let $\zeta_j = \beta$ ($j = 1, 2, \ldots, t$) in Theorem 2, we obtain the following corollary.

**Corollary 3.** Let the functions $f_j(z)$ defined by (1.3) be in the class $T_{q,s}([\alpha_1]; \gamma, \beta)$. Then we have $(f_1 \ast \cdots \ast f_t)(z) \in T_{q,s}([\alpha_1]; \gamma, \eta)$, where

$$
\eta = \frac{(2 + \beta - \gamma)^t \Psi_2^{t-1}}{(1 - \gamma)^{t-1}} + \gamma - 2.
$$

The result is sharp for the functions $f_j(z)$ given by (2.9).
Putting \( t = 2 \) in Corollary 3, we obtain the following corollary.

**Corollary 4.** Let the functions \( f_j(z) \) \((j = 1, 2)\) defined by (1.3) be in the class \( T_{q,s}([\alpha_1]; \gamma, \beta) \). Then we have \((f_1 * f_2)(z) \in T_{q,s}([\alpha_1]; \gamma, \eta)\), where

\[
\eta = \frac{(2 + \beta - \gamma)^2 \Psi_2}{(1 - \gamma)} + \gamma - 2.
\]

The result is sharp.

**Theorem 3.** Let the functions \( f_j(z) \) defined by (1.3) be in the class \( T_{q,s}([\alpha_1]; \gamma_j, \beta) \). Then the function

\[
F(z) = z - \sum_{n=2}^{\infty} \left( \sum_{j=1}^{t} a_{n,j}^m \right) z^n \quad (m > 1)
\]

belongs to the class \( T_{q,s}([\alpha_1]; \delta_t, \beta) \), where

\[
\delta_t = 1 - \frac{t(1 + \beta)(1 - \gamma)^m}{(2 + \beta - \gamma)^m \Psi_2^{m-1} - t(1 - \gamma)^m} \quad (\gamma = \min_{1 \leq j \leq t} \{\gamma_j\}),
\]

and \((2 + \beta - \gamma)^m \Psi_2^{m-1} \geq t(2 + \beta)(1 - \gamma)^m\). The result is sharp for the functions \( f_j(z) \) \((j = 1, 2, \ldots, t)\) given by (2.2).

**Proof.** By virtue of (1.8), we have

\[
\sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\gamma_j + \beta)] \Psi_n a_{n,j}}{(1 - \gamma_j)} \leq 1.
\]

By the Cauchy-Schwarz inequality, we have

\[
\sum_{n=2}^{\infty} \left( \frac{[n(1 + \beta) - (\gamma_j + \beta)] \Psi_n}{(1 - \gamma_j)} \right)^m a_{n,j}^m \leq \left( \sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\gamma_j + \beta)] \Psi_n}{(1 - \gamma_j)} a_{n,j} \right)^m \leq 1.
\]

It follows from (2.14) that

\[
\sum_{n=2}^{\infty} \left( \frac{1}{t} \sum_{j=1}^{t} \left( \frac{[n(1 + \beta) - (\gamma_j + \beta)] \Psi_n}{(1 - \gamma_j)} \right)^m a_{n,j}^m \right) \leq 1.
\]

By setting \( \gamma = \min_{1 \leq j \leq t} \{\gamma_j\} \), the last inequality gives

\[
\sum_{n=2}^{\infty} \left( \frac{1}{t} \left( \frac{[n(1 + \beta) - (\gamma + \beta)] \Psi_n}{(1 - \gamma)} \right)^m \sum_{j=1}^{t} a_{n,j}^m \right) \leq 1.
\]

Therefore, to prove our result we need to find the largest \( \delta_t \) such that

\[
\sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\delta_t + \beta)] \Psi_n}{(1 - \delta_t)} \sum_{j=1}^{t} a_{n,j}^m \leq 1,
\]
that is, that
\[
\frac{[n(1 + \beta) - (\delta_t + \beta)]\Psi_n}{(1 - \delta_t)} \leq \frac{1}{t} \left( \frac{[n(1 + \beta) - (\gamma + \beta)]\Psi_n}{(1 - \gamma)} \right)^m
\]
which leads to
\[
\delta_t \leq 1 - \frac{t(n - 1)(1 + \beta)(1 - \gamma)^m}{[n(1 + \beta) - (\gamma + \beta)]^m \Psi_n - t(1 - \gamma)^m}.
\]

Now let
\[
R(n) = 1 - \frac{t(n - 1)(1 + \beta)(1 - \gamma)^m}{[n(1 + \beta) - (\gamma + \beta)]^m \Psi_n - t(1 - \gamma)^m} \quad (n \geq 2).
\]
Since \(R(n)\) is an increasing function of \(n\) \((n \geq 2)\), then we have
\[
\delta_t \leq R(2) = 1 - \frac{t(1 + \beta)(1 - \gamma)^m}{(2 + \beta - \gamma)^m \Psi_2 - t(1 - \gamma)^m},
\]
and by noting that \((2 + \beta - \gamma)^m \Psi_2 - t(2 + \beta)(1 - \gamma)^m\), we can see that \(0 \leq \delta_t < 1\).

The result is sharp for the functions \(f_j(z)\) \((j = 1, 2, \ldots, t)\) given by (2.2). This completes the proof of Theorem 3. \(\blacksquare\)

Putting \(m = 2\) and \(\gamma_j = \gamma\) \((j = 1, \ldots, t)\) in Theorem 3, we obtain the following corollary.

**Corollary 5.** Let the functions \(f_j(z)\) \((j = 1, 2, \ldots, t)\) defined by (1.3) be in the class \(T_{q,s}([\alpha_1]; \gamma, \beta)\). Then the function
\[
G(z) = z - \sum_{n=2}^{\infty} \left( \sum_{j=1}^{t} a_{n,j}^2 \right) z^n,
\]
belongs to the class \(T_{q,s}([\alpha_1]; \delta_t, \beta)\), where
\[
\delta_t = 1 - \frac{t(1 + \beta)(1 - \gamma)^2}{(2 + \beta - \gamma)^2 \Psi_2 - t(1 - \gamma)^2} \quad (2.16)
\]
and \((2 + \beta - \gamma)^2 \Psi_2 \geq t(2 + \beta)(1 - \gamma)^2\). The result is sharp for the functions \(f_j(z)\) \((j = 1, 2, \ldots, t)\) given by (2.9).

Similarly by applying the method of proof of Theorem 3, we easily get the following result.

**Theorem 4.** Let the functions \(f_j(z)\) defined by (1.3) be in the class
\(T_{q,s}([\alpha_1]; \gamma, \zeta_j), \zeta_j \geq 0\). Then the function \(F(z)\) defined by (2.12) belongs to the class \(T_{q,s}([\alpha_1]; \gamma, \eta_t)\), where
\[
\eta_t = \frac{(2 + \beta - \gamma)^m \Psi_2^{m-1}}{t(1 - \gamma)^{m-1}} + \gamma - 2 \quad (\beta = \min_{1 \leq j \leq t} \{\zeta_j\}),
\]
\(\eta_t \geq 0\).
and \((2 + \beta - \gamma)^m \Psi_2^{m-1} \geq t(2 - \gamma)(1 - \gamma)^{m-1}\). The result is sharp for the functions \(f_j(z)\) \((j = 1, 2, \ldots, t)\) given by (2.11).

Putting \(m = 2\) and \(\zeta_j = \beta\) \((j = 1, 2, \ldots, t)\) in Theorem 4, we obtain the following corollary.

**Corollary 6.** Let the functions \(f_j(z)\) \((j = 1, 2, \ldots, t)\) defined by (1.3) be in the class \(T_{q,s}([\alpha_1]; \gamma, \beta)\). Then the function \(G(z)\) defined by (2.15) belongs to the class \(T_{q,s}([\alpha_1]; \gamma, \eta_t)\), where
\[
\eta_t = \frac{(2 + \beta - \gamma)^2 \Psi_2}{t(1 - \gamma)} + \gamma - 2
\]
and \((2 + \beta - \gamma)^2 \Psi_2 \geq t(2 - \gamma)(1 - \gamma)\). The result is sharp for the functions \(f_j(z)\) given by (2.9).

**Theorem 5.** Let the functions \(f_j(z)\) \((j = 1, 2, \ldots, t)\) defined by (1.3) be in the class \(T_{q,s}([\alpha_1]; \gamma_j, \beta)\) \((j = 1, 2, \ldots, t)\) and let the functions \(g_m(z)\) defined by
\[
g_m(z) = z - \sum_{n=2}^{\infty} b_{n,m} z^n \quad (b_{n,m} \geq 0; \ m = 1, 2, \ldots, s), \quad (2.17)
\]
be in the class \(T_{q,s}([\alpha_1]; \gamma_m, \beta)\) \((m = 1, 2, \ldots, s)\), then
\[
(f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in T_{q,s}([\alpha_1]; \Omega, \beta),
\]
where
\[
\Omega = 1 - \frac{(1 + \beta) \prod_{j=1}^{t} (1 - \gamma_j) \prod_{m=1}^{s} (1 - \gamma_m)}{\prod_{j=1}^{t} (2 + \beta - \gamma_j) \prod_{m=1}^{s} (2 + \beta - \gamma_m) \Psi_2^{t+s-1} - \prod_{j=1}^{t} (1 - \gamma_j) \prod_{m=1}^{s} (1 - \gamma_m)}.
\]
The result is sharp for the functions \(f_j(z)\) given by (2.2) and the functions \(g_m(z)\) given by
\[
g_m(z) = z - \frac{(1 - \gamma_m)}{(2 + \beta - \gamma_m) \Psi_2^2} z^2 \quad (m = 1, 2, \ldots, s). \quad (2.19)
\]

**Proof.** From Theorem 1 we note that, if \(f(z) \in T_{q,s}([\alpha_1]; \delta, \beta)\) and \(g(z) \in T_{q,s}([\alpha_1]; \mu, \beta)\), then \((f * g)(z) \in T_{q,s}([\alpha_1]; \Omega, \beta)\), where
\[
\Omega = 1 - \frac{(1 + \beta)(1 - \delta)(1 - \mu)}{(2 + \beta - \delta)(2 + \beta - \mu)(2 - (1 - \delta)(1 - \mu))}. \quad (2.20)
\]
Since Theorem 1 leads to \((f_1 * f_2 * \cdots * f_t)(z) \in T_{q,s}([\alpha_1]; \delta, \beta)\), where \(\delta\) is defined by (2.1) and \((g_1 * g_2 * \cdots * g_s)(z) \in T_{q,s}([\alpha_1]; \mu, \beta)\) with
\[
\mu = 1 - \frac{(1 + \beta) \prod_{m=1}^{s} (1 - \gamma_m)}{\prod_{m=1}^{s} (2 + \beta - \gamma_m) \Psi_2^{s-1} - \prod_{m=1}^{s} (1 - \gamma_m)}. \quad (2.21)
\]
Then, we have $(f_1 * f_2 \cdots * f_t * g_1 * g_2 \cdots * g_s)(z) \in T_{q,s}([\alpha_1]; \Omega, \beta)$, where $\Omega$ is given by (2.18), this completes the proof of Theorem 5.

Letting $\gamma_j = \gamma$ ($j = 1, 2, \ldots, t$) and $\gamma_m = \gamma$ ($m = 1, 2, \ldots, s$) in Theorem 5, we obtain the following corollary.

**Corollary 7.** Let the functions $f_j(z)$ ($j = 1, 2, \ldots, t$) defined by (1.3) and let the functions $g_m(z)$ defined by (2.17) be in the class $T_{q,s}([\alpha_1]; \gamma, \beta)$. Then we have $(f_1 * f_2 \cdots * f_t * g_1 * g_2 \cdots * g_s)(z) \in T_{q,s}([\alpha_1]; \Omega, \beta)$, where

$$\Omega = 1 - \frac{(1 + \beta)(1 - \gamma)^{t+s}}{(2 + \beta - \gamma)^{t+s} - (1 - \gamma)^{t+s}}.$$  

The result is sharp for the functions $f_j(z)$ given by (2.9) and the functions $g_m(z)$ given by

$$g_m(z) = z - \frac{(1 - \gamma)}{(2 + \beta - \gamma)} z^2 \quad (m = 1, 2, \ldots, s).$$

Letting $t = s = 2$ in Corollary 7, we obtain the following corollary.

**Corollary 8.** Let the functions $f_j(z)$ ($j = 1, 2$) defined by (1.3) and let the functions $g_m(z)$ ($m = 1, 2$) defined by (2.17) be in the class $T_{q,s}([\alpha_1]; \gamma, \beta)$. Then we have $(f_1 * f_2 * g_1 * g_2)(z) \in T_{q,s}([\alpha_1]; \Omega, \beta)$, where

$$\Omega = 1 - \frac{(1 + \beta)(1 - \gamma)^4}{(2 + \beta - \gamma)^4 - (1 - \gamma)^4}.$$  

The result is sharp.

Putting $q = 2, s = 1$ and $\alpha_1 = \alpha_2 = \beta_1 = 1$ in Theorem 5, we obtain the following corollary.

**Corollary 9.** Let $f_j(z)$ ($j = 1, 2, \ldots, t$) defined by (1.3) be in the class $S_pT(\gamma_j, \beta)$ and let the functions $g_m(z)$ ($m = 1, 2, \ldots, s$) defined by (2.17) be in the class $S_pT(\gamma_m, \beta)$ ($m = 1, 2, \ldots, s$), then

$$(f_1 * f_2 \cdots * f_t * g_1 * g_2 \cdots * g_s)(z) \in S_pT(\tau, \beta),$$

where

$$\tau = 1 - \frac{(1 + \beta) \prod_{j=1}^{t}(1 - \gamma_j) \prod_{m=1}^{s}(1 - \gamma_m)}{\prod_{j=1}^{t}(2 + \beta - \gamma_j) \prod_{m=1}^{s}(2 + \beta - \gamma_m) - \prod_{j=1}^{t}(1 - \gamma_j) \prod_{m=1}^{s}(1 - \gamma_m)}.$$  

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z - \frac{(1 - \gamma_j)}{(2 + \beta - \gamma_j)} z^2 \quad (j = 1, 2, \ldots, t)$$

and the functions $g_m(z)$ given by

$$g_m(z) = z - \frac{(1 - \gamma_m)}{(2 + \beta - \gamma_m)} z^2 \quad (m = 1, 2, \ldots, s).$$
Putting \( q = 2, s = 1, \alpha_1 = a(a > 0), \alpha_2 = 1 \) and \( \beta_1 = c \) in Theorem 5, we obtain the following corollary.

**Corollary 10.** Let \( f_j(z) \) \((j = 1, 2, \ldots, t)\) defined by (1.3) be in the class \( S_pT(a, c; \gamma_j, \beta) \) and let the functions \( g_m(z) \) \((m = 1, 2, \ldots, s)\) defined by (2.17) be in the class \( S_pT(a, c; \gamma_m, \beta) \) \((m = 1, 2, \ldots, s)\), then

\[
(f_1 \ast f_2 \cdots \ast f_t \ast g_1 \ast g_2 \cdots \ast g_s)(z) \in S_pT(a, c; \zeta, \beta),
\]

where

\[
\zeta = 1 - \frac{e^{t+s-1}(1 + \beta) \prod_{j=1}^t (1 - \gamma_j) \prod_{m=1}^s (1 - \gamma_m)}{a^{t+s-1} \prod_{j=1}^t (2 + \beta - \gamma_j) \prod_{m=1}^s (2 + \beta - \gamma_m) - e^{t+s-1} \prod_{j=1}^t (1 - \gamma_j) \prod_{m=1}^s (1 - \gamma_m)}.
\]

The result is sharp for the functions \( f_j(z) \) given by

\[
f_j(z) = z - \frac{(1 - \gamma_j)c}{(2 + \beta - \gamma_j)a} z^2 \quad (j = 1, 2, \ldots, t)
\]

and the functions \( g_m(z) \) given by

\[
g_m(z) = z - \frac{(1 - \gamma_m)c}{(2 + \beta - \gamma_m)a} z^2 \quad (m = 1, 2, \ldots, s).
\]

Putting \( q = 2, s = 1, \alpha_1 = \lambda + 1(\lambda > -1), \alpha_2 = 1 \) and \( \beta_1 = 1 \) in Theorem 5, we obtain the following corollary.

**Corollary 11.** Let \( f_j(z) \) \((j = 1, 2, \ldots, t)\) defined by (1.3) be in the class \( S_pT(\lambda; \gamma_j, \beta) \) and let the functions \( g_m(z) \) \((m = 1, 2, \ldots, s)\) defined by (2.17) be in the class \( S_pT(\lambda; \gamma_m, \beta) \) \((m = 1, 2, \ldots, s)\), then

\[
(f_1 \ast f_2 \cdots \ast f_t \ast g_1 \ast g_2 \cdots \ast g_s)(z) \in S_pT(\lambda; \nu, \beta),
\]

where

\[
\nu = 1 - \frac{(1 + \beta) \prod_{j=1}^t (1 - \gamma_j) \prod_{m=1}^s (1 - \gamma_m)}{(\lambda + 1)^{t+s-1} \prod_{j=1}^t (2 + \beta - \gamma_j) \prod_{m=1}^s (2 + \beta - \gamma_m) - \prod_{j=1}^t (1 - \gamma_j) \prod_{m=1}^s (1 - \gamma_m)}.
\]

The result is sharp for the functions \( f_j(z) \) given by

\[
f_j(z) = z - \frac{(1 - \gamma_j)}{(2 + \beta - \gamma_j)(\lambda + 1)} z^2 \quad (j = 1, 2, \ldots, t)
\]
and the functions $g_m(z)$ given by
\[ g_m(z) = z - \frac{(1 - \gamma_m)}{(2 + \beta - \gamma_m)(\lambda + 1)} z^2 \] (m = 1, 2, ..., s).

Putting $q = 2$, $s = 1$, $\alpha_1 = v + 1 (v > -1)$, $\alpha_2 = 1$ and $\beta_1 = v + 2$ in Theorem 5, we obtain the following corollary.

**Corollary 12.** Let $f_j(z)$ (j = 1, 2, ..., t) defined by (1.3) be in the class $S_\mu T(v; \gamma_j, \beta)$ and let the functions $g_m(z)$ (m = 1, 2, ..., s) defined by (2.17) be in the class $S_\mu T(v; \gamma_m, \beta)$ (m = 1, 2, ..., s), then
\[ (f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in S_\mu T(v; \sigma, \beta), \]
where
\[ \sigma = 1 - \frac{(v + 2)^{t+s-1}(1 + \beta) \prod_{j=1}^t (1 - \gamma_j) \prod_{m=1}^s (1 - \gamma_m)}{(v + 1)^{t+s-1} \prod_{j=1}^t (2 + \beta - \gamma_j) \prod_{m=1}^s (2 + \beta - \gamma_m) - (v + 2)^{t+s-1} \prod_{j=1}^t (1 - \gamma_j) \prod_{m=1}^s (1 - \gamma_m)}. \]
The result is sharp for the functions $f_j(z)$ given by
\[ f_j(z) = z - \frac{(1 - \gamma_j)(v + 2)}{(2 + \beta - \gamma_j)(v + 1)} z^2 \] (j = 1, 2, ..., t)
and the functions $g_m(z)$ given by
\[ g_m(z) = z - \frac{(1 - \gamma_m)(v + 2)}{(2 + \beta - \gamma_m)(v + 1)} z^2 \] (m = 1, 2, ..., s).

Putting $q = 2$, $s = 1$, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\beta_1 = 2 - \mu$ ($\mu \neq 2, 3, \ldots$) in Theorem 5, we obtain the following corollary.

**Corollary 13.** Let $f_j(z)$ (j = 1, 2, ..., t) defined by (1.3) be in the class $S_\mu T(\mu; \gamma_j, \beta)$ and let the functions $g_m(z)$ (m = 1, 2, ..., s) defined by (2.17) be in the class $S_\mu T(\mu; \gamma_m, \beta)$ (m = 1, 2, ..., s), then
\[ (f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in S_\mu T(\mu; \kappa, \beta), \]
where
\[ \kappa = 1 - \frac{(2 - \mu)^{t+s-1}(1 + \beta) \prod_{j=1}^t (1 - \gamma_j) \prod_{m=1}^s (1 - \gamma_m)}{2^{t+s-1} \prod_{j=1}^t (2 + \beta - \gamma_j) \prod_{m=1}^s (2 + \beta - \gamma_m) - (2 - \mu)^{t+s-1} \prod_{j=1}^t (1 - \gamma_j) \prod_{m=1}^s (1 - \gamma_m)}. \]
The result is sharp for the functions $f_j(z)$ given by
\[
f_j(z) = z - \frac{(2 - \mu)(1 - \gamma_j)}{2(2 + \beta - \gamma_j)} z^2 \quad (j = 1, 2, \ldots, t)
\]
and the functions $g_m(z)$ given by
\[
g_m(z) = z - \frac{(2 - \mu)(1 - \gamma_m)}{2(2 + \beta - \gamma_m)} z^2 \quad (m = 1, 2, \ldots, s).
\]

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