GENERALIZATIONS OF PRIMAL IDEALS
IN COMMUTATIVE RINGS

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Abstract. Let $R$ be a commutative ring with identity. Let $\phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ be a function where $\mathcal{J}(R)$ denotes the set of all ideals of $R$. Let $I$ be an ideal of $R$. An element $a \in R$ is called $\phi$-prime to $I$ if $ra \in I - \phi(I)$ (with $r \in R$) implies that $r \in I$. We denote by $S_\phi(I)$ the set of all elements of $R$ that are not $\phi$-prime to $I$. $I$ is called a $\phi$-primal ideal of $R$ if the set $P := S_\phi(I) \cup \phi(I)$ forms an ideal of $R$. So if we take $\phi_0(Q) = \emptyset$ (resp., $\phi_0(Q) = 0$), a $\phi$-primal ideal is primal (resp., weakly primal). In this paper we study the properties of several generalizations of primal ideals of $R$.

1. Introduction

Throughout, $R$ will be a commutative ring with identity. (However, in most places the existence of an identity plays no role.) By a proper ideal $I$ of $R$ we mean an ideal $I$ with $I \neq R$. Fuchs [5] introduced a new class of ideals of $R$: primal ideals. Later Ebrahimi Atani and the author gave a generalization of primal ideals: weakly primal ideals. Let $I$ be an ideal of $R$. An element $a \in R$ is called prime (resp. weakly prime) to $I$ if $ra \in I$ (resp. $0 \neq ra \in I$) (where $r \in R$) implies that $r \in I$. Denote by $S(I)$ (resp. $w(I)$) the set of elements of $R$ that are not prime (resp. are not weakly prime) to $I$. A proper ideal $I$ of $R$ is said to be primal if $S(I)$ forms an ideal of $R$ (so $0$ is not necessarily primal); this ideal is always a prime ideal, called the adjoint prime ideal $P$ of $I$. In this case we also say that $I$ is a $P$-primal ideal of $R$ [5]. Not that if $r \in R$ and $a \in S(I)$, then clearly $ra \in S(I)$. So what we require for $I$ being primal is that if $a$ and $b$ are not prime to $I$, then their difference is also not prime to $I$. Also, a proper ideal $I$ of $R$ is called weakly primal if the set $P = w(I) \cup \{0\}$ forms an ideal; this ideal is always a weakly prime ideal [4, Proposition 4], where a proper ideal $P$ of $R$ is called weakly prime if whenever $a, b \in R$ with $0 \neq ab \in P$, then either $a \in P$ or $b \in P$ [2]. In this case we also say that $I$ is a $P$-weakly primal ideal. If $R$ is not an integral domain, then $0$ is a 0-weakly primal ideal of $R$ (by definition), so a weakly primal ideal need not be primal.

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Bhatwadekar and Sharma [3] recently defined a proper ideal $I$ of an integral domain $R$ to be almost prime if for $a, b \in R$ with $ab \in I - I^2$, then either $a \in I$ or $b \in I$. This definition can obviously be made for any commutative ring $R$. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is almost prime. The concept of almost primal ideals in a commutative ring was introduced and studied in [6]. Let $I$ be an ideal of $R$, and let $n \geq 2$ be an integer. An element $a \in R$ is called almost prime (resp. $n$-almost prime) to $I$ if $ra \in I - I^2$ (resp. $ra \in I - I^n$) (with $r \in R$) implies that $r \in I$. Denote by $S_2(I)$ (resp. $S_n(I)$), the set of all elements of $R$ that are not almost prime (resp. $n$-almost prime) to $I$. Then $I$ is called almost primal (resp. $n$-almost primal) if the set $P = S_2(I) \cup I^2$ (resp. $P = S_n(I) \cup I^n$) forms an ideal of $R$. This ideal is an almost prime (resp. $n$-almost prime) ideal of $R$ [6, Lemma 4], called the almost (resp. $n$-almost) prime adjoint ideal of $I$. In this case we also say that $I$ is a $P$-almost (resp. $P$-$n$-almost) primal ideal.

In this paper we give some more generalizations of primal ideals and study the basic properties of these classes of ideals.

2. Results

Let $R$ be a commutative ring, $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function and $I$ an ideal of $R$. Since $I - \phi(I) = I - (I \cap \phi(I))$, there is no loss of generality in assuming that $I \subseteq \phi(I)$. We henceforth make this assumption throughout this paper.

**Definition 2.1.** Let $R$ be a commutative ring and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function. Let $I$ be an ideal of $R$. An element $a \in R$ is called $\phi$-prime to $I$ if $ra \in I - \phi(I)$ (with $r \in R$) implies that $r \in I$.

**Remarks 2.2.** Let $R$ be a commutative ring, and $I$ a proper ideal of $R$. Denote by $S_\phi(I)$ the set of all elements of $R$ that are not $\phi$-prime to $I$. Then

1. Every element of $\phi(I)$ is $\phi$-prime to $I$.
2. If an element of $R$ is prime to $I$, then it is $\phi$-prime to $I$. So $S_\phi(I) \subseteq S(I)$.
3. The converse of (2) is not necessarily true. For example assume that $\phi = \phi_0$, where $\phi_0(Q) = 0$ for every ideal $Q$ of $R$. Let $R = \mathbb{Z}/24\mathbb{Z}$, and $I = 8\mathbb{Z}/24\mathbb{Z}$. Then, $\overline{6}$ is $\phi$-prime to $I$, but it is not prime to $I$ since $\overline{12}, \overline{6} \in I$ with $\overline{12} \notin I$. Consequently $\overline{6}$ is $\phi$-prime to $I$ while it is not prime to $I$.

**Definition 2.3.** Let $R$ be a commutative ring and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function. A proper ideal $I$ of $R$ is said to be a $\phi$-primal ideal of $R$ if $S_\phi(I) \cup \phi(I)$ forms an ideal of $R$.

Let $R$ be a commutative ring and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function. We recall from [1] that a proper ideal $P$ of $R$ is called $\phi$-prime if for every $x, y \in R$, $xy \in P - \phi(P)$ implies $x \in P$ or $y \in P$.

**Proposition 2.4.** If $I$ is a $\phi$-primal ideal of $R$, then $P = S_\phi(I) \cup \phi(I)$ is a $\phi$-prime ideal of $R$. 
Let } x, y ∈ R \text{ be such that } xy ∈ P − φ(P) \text{ and } x ∉ P. Then } xy ∈ S_φ(I) \text{ so } xy \text{ is not } φ\text{-prime to } I. \text{ Hence } rxy ∈ I − φ(I) \text{ for some } r ∈ R − I. \text{ There exists } r ∈ R − I \text{ with } rxy ∈ I − φ(I). \text{ If } x ∉ P, \text{ then } ry ∈ I − φ(I) \text{ implies that } y \text{ is not } φ\text{-prime to } I. \text{ So } y ∈ S_φ(I) \text{ equals } P. \text{ Since } x \text{ is } φ\text{-prime to } I, \text{ from } x(ry) = rxy ∈ I − φ(I) \text{ we get } ry ∈ I − φ(I). \text{ This implies that } y \text{ is not } φ\text{-prime to } I, \text{ that is } y ∈ S_φ(I) \subseteq P. \text{ So } P \text{ is } φ\text{-prime.}

Notation 2.5. Let } I \text{ be a } φ\text{-prime ideal of } R. \text{ By Lemma 2.4, } P = S_φ(I) ∪ φ(I) \text{ is a } φ\text{-prime ideal of } R. \text{ In this case } P \text{ is called the } φ\text{-prime adjoint ideal (simply adjoint ideal) of } I, \text{ and } I \text{ is called a } P\text{-φ-primal ideal of } R.

Theorem 2.6. Let } R \text{ be a commutative ring with identity. Then every } φ\text{-prime ideal of } R \text{ is } φ\text{-primal.}

Proof. Assume that } P \text{ is a } φ\text{-prime ideal of } R. \text{ It suffices to show that } P − φ(P) \text{ consists exactly of elements of } R \text{ that are not } φ\text{-prime to } P. \text{ By Lemma 2.7, } P ⊆ S_φ(P) ∪ φ(I). \text{ So } P − φ(P) ⊆ S_φ(P). \text{ Now assume that } a ∈ S_φ(P). \text{ Then } ab ∈ P − φ(P) \text{ for some } b ∈ R − P. \text{ Since } P \text{ is } φ\text{-prime we have } a ∈ P − φ(P). \text{ Consequently } P = S_φ(P) ∪ φ(P). \text{ This implies that } P \text{ is a } P\text{-φ-primal ideal of } R. \blacksquare

Lemma 2.7. Let } R \text{ be a commutative ring and } I \text{ an ideal of } R.

(1) If } I \text{ is proper in } R, \text{ then } I ⊆ S_φ(I) ∪ φ(I).
(2) If } I \text{ is a } P\text{-φ-primal ideal of } R, \text{ then } I ⊆ P.

Proof. (1) For every } a ∈ I − φ(I) \text{ we have } a.1_R = a ∈ I − φ(I) \text{ with } 1_R ∈ R − I. \text{ This implies that } a \text{ is not } φ\text{-prime to } I, \text{ that is } a ∈ S_φ(I).

(2) It follows from (1). \blacksquare

Example 2.8. Let } R \text{ be a commutative ring. Define the following functions } φ_α : I(R) → I(R) ∪ \{∅\} \text{ and the corresponding } φ_α\text{-primal ideals:

(1) } φ_0 \phi(I) = ∅ \text{ a } φ\text{-primal ideal is primal.
(2) } φ_0 \phi(I) = 0 \text{ a } φ\text{-primal ideal is weakly primal.
(3) } φ_2 \phi(I) = I^2 \text{ a } φ\text{-primal ideal is almost primal.
(4) } φ_n(n ≥ 2) \phi(I) = I^n \text{ a } φ\text{-primal ideal is } n\text{-almost primal.
(5) } φ_ω \phi(I) = \bigcap_{n=1}^{∞} I^n \text{ a } φ\text{-primal ideal is } ω\text{-primal.
(6) } φ_1 \phi(I) = I \text{ a } φ\text{-primal ideal is any ideal.

The next result provides several characterizations of } φ\text{-primal ideals of a commutative ring } R.

Theorem 2.9. Let } R \text{ be a commutative ring and } φ : I(R) → I(R) ∪ \{∅\} \text{ a function. Let } I \text{ and } P \text{ be proper ideals of } R. \text{ The following are equivalent.

(1) } I \text{ is } P\text{-φ-primal.
(2) For every } x ∈ P − φ(I), (I :_R x) = I ∪ (φ(I) :_R x); \text{ and for every } x ∈ P − φ(I), (I :_R x) ∩ φ(I) :_R x).

(3) for every } x ∈ P − φ(I), (I :_R x) = I \text{ or } (I :_R x) = (φ(I) :_R x); \text{ and for every } x ∈ P − φ(I), (I :_R x) ∩ φ(I) :_R x).
Proof. 1 $\Rightarrow$ 2) Assume that $I$ is $P,\phi$-primal. Then $P - \phi(I)$ consists entirely of elements of $R$ that are not $\phi$-prime to $I$. Let $x \notin P - \phi(I)$. Then $x$ is $\phi$-prime to $I$. Clearly $I \cup (\phi(I) :_R x) \subseteq (I :_R x)$. For every $r \in (I :_R x)$, if $rx \in \phi(I)$, then $r \in (\phi(I) :_R x)$, and if $rx \notin \phi(I)$, then $x$ $\phi$-prime to $I$ gives $r \in I$. Hence $r \in I \cup (\phi(I) :_R x)$, that is $(I :_R x) \subseteq I \cup (\phi(I) :_R x)$. Therefore $(I :_R x) = I \cup (\phi(I) :_R x)$.

Now assume that $x \notin P - \phi(I)$. Then $x$ is not $\phi$-prime to $I$. So there exists $r \in R - I$ such that $rx \in I - \phi(I)$. Hence $r \in (I :_R x) - (I \cup (\phi(I) :_R x))$.

2 $\Rightarrow$ 3) Let $x \notin P - \phi(I)$. Since $(I :_R x)$ is an ideal of $R$ and $(I :_R x) = I \cup (\phi(I) :_R x)$, either $(\phi(I) :_R x) \subseteq I$ or $I \subseteq (\phi(I) :_R x)$. So either $(I :_R x) = I$ or $(I :_R x) = (\phi(I) :_R x)$. Moreover, for every $x \in P - \phi(I)$, $(I :_R x) \nsubseteq I \cup (\phi(I) :_R x)$. Hence $(I :_R x) \nsubseteq I$ and $(I :_R x) \nsubseteq (\phi(I) :_R x)$.

3 $\Rightarrow$ 1) By (3), $P - \phi(I)$ consists exactly of all elements of $R$ that are not $\phi$-prime to $I$. Hence $I$ is $P,\phi$-primal.

Example 2.10. In this example we show that the concepts “primal ideal” and “$\phi$-primal ideal” are different. In fact we show that neither implies the other. Let $R$ be a commutative ring and assume that $\phi = \pi_0$. Then

(1) Let us to denote the set of all zero-divisors of $R$ by $Z(R)$. If $R$ is not an integral domain such that $Z(R)$ is not an ideal of $R$ (for example the ring $\mathbb{Z}/2\mathbb{Z}$), then the zero ideal of $R$ is a $\phi$-primal ideal which is not primal. Hence a $\phi$-primal ideal need not be primal.

(2) Let $R = \mathbb{Z}/24\mathbb{Z}$, and consider the ideal $I = 8\mathbb{Z}/24\mathbb{Z}$ of $R$. It is not difficult to show that $I$ is not a $\phi$-primal ideal of $R$. Now set $P = 2\mathbb{Z}/24\mathbb{Z}$. Then every element of $P$ is prime to $I$. Assume that $\bar{a} \notin P$. If $\bar{a}, \bar{n} \in I$ for some $n \in R$, then 8 divides $n$, that is $\bar{n} \in I$. Hence $\bar{a}$ is prime to $I$. We have shown that $S(I) = P$, that is $I$ is $P$-primal. This example shows that a primal ideal need not be $\phi$-primal.

According to Example 2.10, a $\phi$-primal ideal need not necessarily be primal. In Theorems 2.11 and 2.12 we provide some conditions under which a $\phi$-primal ideal is primal.

Theorem 2.11. Let $R$ be a commutative ring and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ a function. Suppose that $I$ is a $P,\phi$-primal ideal of $R$ with $I^2 \nsubseteq \phi(I)$. If $P$ is a prime ideal of $R$, then $I$ is primal.

Proof. Assume that $a \in P$. Then either $a \in \phi(I)$ or $a \in S_\phi(I)$. If the former case holds, then $a \in \phi(I) \subseteq I \subseteq S(I)$, and if the latter case holds, then $a \in S_\phi(I) \subseteq S(I)$ by Remark 2.2. So in any case $a$ is not prime to $I$. Now assume that $b \in R$ is not prime to $I$. So $rb \in I$ for some $r \in R - I$. If $rb \notin \phi(I)$, then $b$ is not $\phi$-prime to $I$, so $b \notin P$. Thus assume that $rb \in \phi(I)$. First suppose that $bI \nsubseteq \phi(I)$. Then, there exists $r_0 \in I$ such that $br_0 \notin \phi(I)$. Then $b(r + r_0) = br + br_0 \in I - \phi(I)$ with $r + r_0 \in R - I$, implies that $b$ is not $\phi$-prime to $I$, that is $b \notin P$. Now we may assume that $bI \subseteq \phi(I)$. If $rI \nsubseteq \phi(I)$, then $rc \notin \phi(I)$ for some $c \in I$. In this case $b(c + r) = br + cr \in I - \phi(I)$ with $r \in R - I$, that is $b \in P$. So we can assume that $rI \subseteq \phi(I)$. Since $I^2 \nsubseteq \phi(I)$, there are $a_0, b_0 \in I$ with $a_0b_0 \notin \phi(I)$.
(b + a_0)(r + b_0) ∈ I - φ(I) with r + b_0 ∈ R - I implies that b + a_0 ∈ P. On the other hand a_0 ∈ I ⊆ P by Lemma 2.7. So that b ∈ P. We have already shown that P is exactly the set of all elements of R that are not prime to I. Hence I is P-primal.

A commutative ring is called decomposable if there exist nontrivial commutative rings $R_1$ and $R_2$ such that $R ≅ R_1 \times R_2$. A ring $R$ that is not decomposable is called indecomposable. An ideal $I$ of $R = R_1 \times R_2$ will have the form $I_1 \times I_2$ where $I_1$ and $I_2$ are ideals of $R_1$ and $R_2$, respectively. It is a well-known, and easily proved, result that $I$ is prime if and only if $I = P_1 \times R_2$ or $I = R_1 \times P_2$ where $P_i$ is a prime ideal of $R_i$. It has also proved in [4, Lemma 13] that if $I_1$ is a primal ideal of $R_1$ and $I_1$ is a primal ideal of $R_2$, then $I_1 \times R_2$ and $R_1 \times I_2$ are primal ideals of $R$. In the following theorem we provide some conditions under which a $φ$-primal ideal of a decomposable ring is primal.

**Theorem 2.12.** Let $R_1$ and $R_2$ be commutative rings and $R = R_1 \times R_2$. Let $ψ_1 : \mathcal{J}(R_1) → \mathcal{J}(R_2) ∪ \{0\}$ ($i = 1, 2$) be functions with $ψ_i(R_i) ≠ R_i$, and set $φ = ψ_1 \times ψ_2$. Assume that $P$ is an ideal of $R$ with $φ(P) ≠ P$. If $I$ is a $P$-$φ$-primal ideal of $R$, then either $I = φ(I)$ or $I$ is primal.

**Proof.** Suppose that $I$ is a $P$-$φ$-primal ideal of $R$. We may assume that $I = I_1 \times I_2$ and $I ≠ φ(I)$. By Proposition 2.4, $P$ is a φ-prime ideal of $R$. Therefore, by [1, Theorem 16], we have the following cases:

**Case 1.** $P = P_1 \times P_2$ where $P_i$ is a proper ideal of $R$ with $ψ_i(P_i) = P_i$. In this case we have $φ(P) = ψ_1(P_1) \times ψ_2(P_2) = P_1 \times P_2 = P$ which is a contradiction.

**Case 2.** $P = P_1 \times R_2$ where $P_1$ is a $ψ_1$-prime ideal of $R_1$. Since $ψ_2(R_2) ≠ R_2$, $P_1$ is a prime ideal of $R_1$ and hence $P$ is a prime ideal of $R$. We show that $I_2 = R_2$. Since $I ≠ φ(I)$, there exists $(a, b) ∈ I - φ(I)$. Then we have $(a, 1)(1, b) = (a, b) ∈ I - φ(I)$. If $(a, 1) ∈ I$, then $(1, b)$ is not φ-prime to $I$. Hence $(1, b) ∈ P = P_1 × R_2$ and so $1 ∈ P_1$ a contradiction. Thus $(a, 1) ∈ I$ and so $1 ∈ I_2$, that is $I_2 = R_2$. Now we prove that $I_1$ is a $P_1$-prime ideal of $R_1$. Pick an element $a_1 ∈ P_1$. Then $(a_1, 0) ∈ P_1 × R_2 = P = S_φ(I) ∪ φ(I)$. If $(a_1, 0) ∈ φ(I) = ψ_1(I_1) \times ψ_2(R_2)$, then $a_1 ∈ ψ_1(I_1) ⊆ I_1 ⊆ S(I_1)$. Hence $a_1$ is not prime to $I_1$. So assume that $(a_1, 0) ∈ S_φ(I)$. In this case $(a_1, 0)(r_1, r_2) ∈ I - φ(I)$ for some $(r_1, r_2) ∈ R - I$. So $a_1 r_1 ∈ I_1 - ψ_1(I_1)$ with $r_1 ∈ R_1 - I_1$ implies that $a_1$ is not $ψ_1$-prime to $I_1$. Hence $a_1$ is not prime to $I_1$ by Remark 2.2. Conversely, assume that $b_1$ is not prime to $I_1$. Then $b_1 c_1 ∈ I_1$ for some $c_1 ∈ R_1 - I_1$. In this case since $1 ∈ R_2 - ψ_2(R_2)$, we have $(b_1, 1)(c_1, 1) = (b_1 c_1, 1) ∈ I_1 × R_2 - (I_1 × ψ_2(R_2) ⊆ I - φ(I))$ with $(c_1, 1) ∈ R - I$. Hence $(b_1, 1)$ is not φ-prime to $I$. Therefore $(b_1, 1) ∈ P = P_1 × R_2$ and hence $b_1 ∈ P_1$. We have already shown that $P_1$ consists exactly of those elements of $R_1$ that are not prime to $I_1$. Hence $I_1$ is a $P_1$ primal ideal of $R_1$. Now $I$ is a $P$-$φ$-primal ideal of $R$ by [4, Lemma 13].

**Case 3.** $P = R_1 × P_2$ where $P_2$ is a $ψ_2$-prime ideal of $R_2$. A similar argument as in the Case 2 shows that $I$ is $P$-primal.
Theorem 2.13. Let $R$ be a commutative ring and $\phi : R \to R$ a function. Let $I$ and $J$ be ideals of $R$ with $J \subseteq \phi(I)$. Then $I$ is a $\phi$-prime ideal of $R$ if and only if $I/J$ is a $\phi_J$-prime ideal of $R/J$.

Proof. Assume that $I$ is a $P$-$\phi$-prime ideal of $R$. Suppose that $a + J$ is an element of $R/J$ that is not $\phi_J$-prime to $I/J$. There exists $b \in R - I$ with $(a + J)(b + J) \in I/J - \phi_J(I/J)$. In this case $ab \in I - \phi(I)$ with $b \in R - I$ implies that $a$ is not $\phi$-prime to $I$. Hence $a \in S_\phi(I) \subseteq P$, and so $a + J \in P/J$. Now assume that $c + J \in P/J$. Then $c \in P = S_\phi(I) \cup \phi(I)$. If $c \in \phi(I)$, then $c + J \in \phi_J(I/J)$. So assume that $c \in S_\phi(I)$, that is $c$ is not $\phi$-prime to $I$. Then $cd \in I - \phi(I)$ for some $d \in R - I$. Consequently, $(c + J)(d + J) \in I/J - (\phi(I)/J) = I/J - \phi_J(I/J)$ with $d + J \in R/J - I/J$. This implies that $c + J$ is not $\phi_J$-prime to $I/J$; so $c + J \in S_\phi(I)$, and hence $c \in P - \phi(I)$. It follows that $P = S_\phi(I) \cup \phi(I)$ which implies that $I$ is a $P$-$\phi$-prime ideal of $R$. $lacksquare$

Conversely, suppose that $I/J$ is a $\phi_J$-prime in $R/J$ with the adjoint ideal $P/J$. For every $a \in P - \phi(I)$, we have $a + J \in P/J - \phi_J(I/J)$. So $a + J$ is not $\phi_J$-prime to $I/J$. So $(a + J)(b + J) \in I/J - \phi_J(I/J)$ for some $b + J \in R/J - I/J$. In this case $b \in R - I$ and $ab \in I - \phi(I)$ implies that $a$ is not $\phi$-prime to $I$. Conversely, assume that $c \in R$ is not $\phi$-prime to $I$. In this case $cd \in I - \phi(I)$ for some $d \in R - I$. Then $(c + J)(d + J) \in I/J - \phi_J(I/J)$ with $d + J \notin I/J$, that is $c + J$ is not $\phi_J$-prime to $I/J$. Hence $c + J \in P/J - \phi_J(I/J)$, and hence $c \in P - \phi(I)$. It follows that $P = S_\phi(I) \cup \phi(I)$ which implies that $I$ is a $P$-$\phi$-prime ideal of $R$.

Lemma 2.14. Let $R$ be a commutative ring and $\phi : R \to R$ be a function. Let $L = \{\phi(I) \mid \phi(I) \subseteq R\}$ be a function and define $\phi_T : \mathcal{I}(R_T) \to \mathcal{I}(\mathcal{I}(R_T)) \cup \{\emptyset\}$ by $\phi_T(I) = (\phi(I))_T$ for every ideal $I$ of $R_T$. Note that $\phi_T(I) \subseteq J$. In the remainder of this paper we study the relations between the set of $\phi$-primal ideals of $R$ and $\phi_T$-primal ideals of $R_T$.

Proof. Assume that $\lambda = a/s$ of $\lambda$ as a formal fraction (with $a \in R$ and $s \in T$) must have its numerator in $I$. Moreover if $(\phi(I))_T \not\subseteq I_T$, then $I = I_T \cap R$.

Proof. Assume that $\lambda = a/s \in I_T - (\phi(I))_T$. Then $a/s = b/t$ for some $b \in I$ and $t \in T$. In this case $uta = usb \in I$ for some $u \in T$. If $uta \in \phi(I)$, then $a/s = (uta)/(uts) \in (\phi(I))_T$ a contradiction. So we have $uta \in I - \phi(I)$. If $a \notin I$, then $ut$ is not $\phi$-prime to $I$; so $ut \in P \cap T$ which contradicts the hypothesis. Therefore $a \in I$.

For the last part, it is clear that $I \subseteq I_T \cap R$. Now pick an element $a \in I_T \cap R$. Then $sa \in I$ for some $s \in T$. If $sa \notin \phi(I)$ and $a \notin I$, then $s$ is not $\phi$-prime to $I$, so $s \in P \cap T$ a contradiction. So $a$ must be in $I$. If $sa \in \phi(I)$, then $a/1 = (sa)/s \in (\phi(I))_T$, and so $a \in (\phi(I))_T \cap R$. Therefore $I_T \cap R = I \cup ((\phi(I))_T \cap R)$. Hence
either $I_T \cap R = I$ or $I_T \cap R = (\phi(I))_T \cap R$. But the latter case does not hold, for otherwise $I_T = (\phi(I))_T$ which is a contradiction.

Let $R$ be a commutative ring and $M$ an $R$-module. An element $a \in R$ is called a zero-divisor on $M$ if $am = 0$ for some $rm = 0$. We denote by $Z_R(M)$ the set of all zero-divisors of $R$ on $M$.

**Theorem 2.15.** Let $R$ be a commutative ring and $\phi : \mathfrak{I}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ a function. Suppose that $T$ is a multiplicatively closed subset of $R$ and $I$ a $P-\phi$-primal ideal of $R$ with $P \cap T = \emptyset$, $T \cap Z_R(R/\phi(I)) = \emptyset$ and $(\phi(I))_T \subseteq \phi_T(I_T)$. Then $I_T$ is a $\phi_T$-primal ideal of $R_T$ with the adjoint ideal $P_T$.

**Proof.** Suppose that $a/s \in P_T - \phi_T(I_T)$. Since $(\phi(I))_T \subseteq \phi_T(I_T)$ we have $a \notin \phi(I)$. Hence, by Theorem 2.6 and 2.14, $a \in P - \phi(I)$. Thus $a$ is not $\phi$-prime to $I$; so $ab \in I - \phi(I)$ for some $b \in R - I$. If $(ab)/s \in \phi_T(I_T)$, then $(ab)/s = c/t$ for some $c \in \phi(I) \cap R$ and $t \in T$. One can shows that $c \in \phi(I)$ and so $utab = usc \in \phi(I)$ shows that $ut \in T \cap Z_R(R/\phi(I))$ a contradiction. So $(ab)/s \notin \phi_T(I_T)$. In this case, by Lemma 2.14, $b/1 \notin I_T$ and $(a/s)(b/1) = (ab)/s \in I_T - \phi_T(I_T)$ implies that $a/s$ is not $\phi_T$-prime to $I_T$. Conversely assume that $a/s \in R_T$ is not $\phi_T$-prime to $I_T$. Then $a/s \notin \phi_T(I_T)$ and $(a/s)(b/t) \in I_T - \phi_T(I_T)$ for some $b/t \in R_T - I_T$. Since $(\phi(I))_T \subseteq \phi_T(I_T)$ we have $(ab)/(st) \in I_T - (\phi(I))_T$. Then, by Lemma 2.14, $ab \in I - \phi(I)$ and $b \in R - I$ implies that $a$ is not $\phi$-prime to $I$. So $a \in P$ and hence $a/s \in P_T - \phi_T(I_T)$. Consequently $P_T = S_{\phi_T}(I_T) \cup \phi_T(I_T)$ shows that $I_T$ is a $P_T-\phi_T$-primal ideal of $R_T$.

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