APPLICATION OF THE INFINITE MATRIX THEORY TO THE SOLVABILITY OF CERTAIN SEQUENCE SPACES EQUATIONS WITH OPERATORS

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Abstract. In this paper we deal with special sequence spaces equations (SSE) with operators, which are determined by an identity whose each term is a sum or a sum of products of sets of the form \( \chi_a(T) \) and \( \chi_f(x)(T) \) where \( f \) maps \( U^+ \) to itself, and \( \chi \) is any of the symbols \( s, s^0, \) or \( s^{(c)} \). We solve the equation \( \chi_x(\Delta) = \chi_b \) where \( \chi \) is any of the symbols \( s, s^0, \) or \( s^{(c)} \) and determine the solutions of (SSE) with operators of the form \( \chi_a * \chi_b(\Delta) = \chi_0 \) and \( [\chi_a * (\chi_x)^2 + \chi_b * \chi_x](\Delta) = \chi_0 \) and \( \chi_a + \chi_x(\Delta) = \chi_x \) where \( \chi \) is any of the symbols \( s, s^0, \) or \( s^{(c)} \).

1. Introduction

In the book entitled Summability through Functional Analysis [15], Wilansky introduced sets of the form \( 1/a * E \) where \( E \) is a BK space, where \( a = (a_n)_{n\geq1} \) is a sequence satisfying \( a_n \neq 0 \) for all \( n \). Recall that \( \xi = (\xi_n)_{n\geq1} \) belongs to \( 1/a * E \). In [12, 3] the sets \( s_r, s^0_r \) and \( s^{(c)}_r \) were defined by \( ((1/r^n)_{n\geq1})^{-1} * E \) with \( r > 0 \), where \( E \) is \( \ell_\infty, c_0 \) and \( c \) respectively and the sets \( s_a, s^0_a \) and \( s^{(c)}_a \) by \( (1/a)^{-1} * E \) with \( a_n > 0 \) for all \( n \) and \( E \) is \( \ell_\infty, c_0 \) and \( c \) respectively. The aim was to study an infinite linear system represented by the matrix equation \( M \xi = \beta \) where \( \xi \) was the unknown and \( \xi, \beta \) were column matrices, and \( M = (\mu_{nm})_{n,m\geq1} \) was an infinite matrix mapping from \( (1/a)^{-1} * E \) to itself, \( (\text{cf. [12]}) \). In [4, 13] the sum \( \chi_a + \chi_b^{(c)} \) and the product \( \chi_a * \chi_b^{(c)} \) were defined, where \( \chi, \chi^{(c)} \) are any of the symbols \( s, s^0, \) or \( s^{(c)} \), among other things characterizations of matrix transformations mapping in the sets \( s_a + s^0_a(\Delta^q) \) and \( s_a + s^{(c)}_a(\Delta^q) \) were given, where \( \Delta \) is the operator of the first difference. In [7] characterizations of the sets \( s_a(\Delta^q, F) \) can be found, where \( F \) is any of the sets \( c_0, c \) and \( \ell_\infty \). In [13] characterizations of matrix transformations mapping were given in the set \( \tilde{s}_{a,\beta} = s^0_a(\Delta - \lambda I)^h + s^{(c)}_a(\Delta - \mu I)^l \) in some cases the set \( \tilde{s}_{a,\beta, a} \) that can be reduced to a set of the form \( S_{a,\gamma} \). Also cite Hardy’s results [9] extended by Móricz and Rhoades, \( (\text{cf. [10, 11]}), \) de Malafosse and

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characterizations of or equations of the form $\xi$ in the form that (transformations between sets of the form sequence spaces. to the study of the operator any integer $s$ such that $x$ sequences $\chi$ and on the sum and the product of the previous sets. In Section 3 we recall characterizations of $\chi_a(T)$ and $\chi(f(x))(T)$, where $f$ mapa $U^+$ to itself, and $\chi$ is any of the symbols $s$, or $s^0$, the sequence $x$ is the unknown and $T$ is a given triangle. Then we determine the set of all sequences $x \in U^+$ such that

$$u_n = O(a_n) \text{ and } v_n - v_{n-1} = O(x_n)$$

implies $u_n + v_n = O(x_n) \ (n \to \infty)$ for all $u, v \in s$. Conversely, what are the sequences $x$ for which $y_n = O(x_n) \ (n \to \infty)$ implies there are sequences $u$ and $v$ such that $y = u + v$ and (1) holds. This problem leads to the solvability of the equation $s_n + s_x(\Delta) = s_x$. We also determine the set of all sequences $y \in s$ such that $(y_n - y_{n-1})/a_n \to l$ if and only if $y_n/b_n \to l'$. This statement can be written in the form $s_0(c) = s_b(c)$.

This paper is organized as follows. In Section 2 we recall some results on matrix transformations between sets of the form $\chi_\ell$ where $\chi$ is any of the symbols $s$, $s^0$, or $s^{(c)}$ and on the sum and the product of the previous sets. In Section 3 we recall characterizations of $\chi_a(T)$ and determine the solutions of sequence spaces equations of the form $[\chi_a \ast \chi_x + \chi_b](\Delta) = \chi_y$ and $[\chi_a \ast (\chi_x)^2 + \chi_b \ast \chi_x](\Delta) = \chi_y$ and $\chi_a + \chi_x(\Delta) = \chi_x$ where $\chi$ is any of the symbols $s$, or $s^0$.

1.1. The sets $s_a$, $s_a^0$ and $s_a^{(c)}$ for $a \in U^+$

For a given infinite matrix $M = (\mu_{nm})_{n,m \geq 1}$ we define the operators $A_n$ for any integer $n \geq 1$, by

$$M_n(\xi) = \sum_{m=1}^{\infty} \mu_{nm}\xi_m$$

where $\xi = (\xi_m)_{m \geq 1}$, and the series are assumed convergent for all $n$. So we are led to the study of the operator $M$ defined by $M\xi = (M_n(\xi))_{n \geq 1}$ mapping between sequence spaces.

A Banach space $E$ of complex sequences with the norm $||\xi||_E$ is a BK space if each projection $P_n : \xi \to P_n\xi = \xi_n$ is continuous. A BK space $E$ is said to have AK if every sequence $\xi$ of the form $\xi = (\xi_n)_{n \geq 1} \in E$ has a unique representation $\xi = \sum_{n=1}^{\infty} \xi_n e_n$ where $e_n$ is the sequence with 1 in the $n$-th position and 0 otherwise.

We will denote by $s$ the sets of all sequences. By $c_0$, $c$, $\ell_\infty$ we denote the subsets of $s$ that converge to zero, that are convergent and that are bounded respectively. We shall use the set $U^+ = \left\{ (u_n)_{n \geq 1} \in s : u_n > 0 \text{ for all } n \right\}$. Using Wilansky’s...
notations [15], we define for any sequence \( a = (a_n)_{n \geq 1} \in U^+ \) and for any set of sequences \( E \), the set 
\[
(1/a)^{-1} \ast E = \{ (\xi_n)_{n \geq 1} \in s : (\xi_n/a_n) \in E \}.
\]
To simplify, we use the diagonal infinite matrix \( D_a \) defined by \( [D_a]_{nn} = a_n \) for all \( n \) and write \( D_a \ast E = (1/a)^{-1} \ast E \) and define \( s_a = D_a \ast \ell_\infty \), \( s^0_a = D_a \ast c_0 \) and \( s^{(c)}_a = D_a \ast c \), see [1, 3, 4–6, 10, 13, 14]. Each of the previous spaces \( D_a \ast E \) is a BK space normed by \( \| \xi \|_{s_a} = \sup_{n \geq 1} (|\xi_n|/a_n) \) and \( s^0_a \) has AK, see [6].

Now let \( a = (a_n)_{n \geq 1}, b = (b_n)_{n \geq 1} \in U^+ \). By \( S_{a,b} \) we denote the set of infinite matrices \( M = (\mu_{nm})_{a,m \geq 1} \) such that 
\[
\| M \|_{S_{a,b}} = \sup_{n \geq 1} \left( \frac{1}{b_n} \sum_{m=1}^\infty |\mu_{nm}|a_m \right) < \infty.
\]
The set \( S_{a,b} \) is a Banach space with the norm \( \| M \|_{S_{a,b}} \). Let \( E \) and \( F \) be any subsets of \( s \). When \( M \) maps \( E \) into \( F \) we write \( M \in (E,F) \), see [2]. So for every \( \xi \in E \), we have \( M\xi \in F \), \( (M\xi) \in F \) will mean that for each \( n \geq 1 \) the series defined by \( M_n(\xi) = \sum_{m=1}^\infty \mu_{nm}\xi_m \) is convergent and \( (M_n(\xi))_{n \geq 1} \in F \). It can easily be seen that \( (s_a, s_b) = S_{a,b} \).

When \( s_a = s_b \) we obtain the Banach algebra with identity \( S_{a,b} = S_a \), (see for instance [1, 5, 6]) normed by \( \| M \|_{S_a} = \| M \|_{S_{a,a}} \). We also have \( M \in (s_a, s_a) \) if and only if \( M \in S_a \).

If \( a = (r^n)_{n \geq 1} \), we denote by \( s_r, s^0_r \) and \( s^{(c)}_r \) the sets \( s_a \), \( s^0_a \) and \( s^{(c)}_a \) respectively. When \( r = 1 \), we obtain \( s_1 = \ell_\infty \), \( s^0_1 = c_0 \) and \( s^{(c)}_1 = c \), and putting \( e = (1,1,\ldots) \) we have \( S_1 = S_e \). Recall that \( (\ell_\infty, \ell_\infty) = (c_0, c_\infty) = (c, \ell_\infty) = S_1 \). We have \( M \in (c_0, c_0) \) if and only if \( M \in S_1 \) and \( \lim_{n \to \infty} \mu_{nm} = 0 \) for all \( m \geq 1 \); and \( M \in (c,c) \) if and only if \( M \in S_1 \) and \( \lim_{n \to \infty} M_n(e) = l \) and \( \lim_{n \to \infty} \mu_{nm} = l_m \) for all \( m \geq 1 \) and for some scalars \( l \) and \( l_m \). Finally for any given subset \( F \) of \( s \), we define the domain of \( M \) by 
\[
F_M = F(M) = \{ \xi \in s : M \xi \in F \}.
\]

1.2. Sum of sets of the form \( s_\xi \), or \( s^0_\xi \)

In this subsection among other things we recall some properties of the sum \( E + F \) of sets of the form \( s_\xi \), or \( s^0_\xi \).

Let \( E, F \subset s \) be two linear vector spaces, we write \( E + F \) for the set of all sequences \( w = u + v \) where \( u \in E \) and \( v \in F \). From [4, Proposition 1, p. 244] and [5, Theorem 4, p. 293] we deduce the next results.

**Proposition 1.** Let \( a, b \in U^+ \) and let \( \chi \) be either of the symbols \( s \), or \( s^0 \). Then we have

(i) \( \chi_a \subset \chi_b \) if and only if there is \( K > 0 \) such that 
\[
a_n \leq Kb_n \quad \text{for all } n.
\]
(ii) \( a \) \( \chi_a = \chi_b \) if and only if there are \( K_1, K_2 > 0 \) such that 
\[
K_1 \leq \frac{a_n}{b_n} \leq K_2 \quad \text{for all } n.
\]
\[ \beta \, s^{(c)}_a = s^{(c)}_b \] if and only if there is \( l \neq 0 \) such that \( \frac{a_n}{b_n} \to l \) \((n \to \infty)\).

(iii) \( \chi_a + \chi_b = \chi_{a+b} \).

(iv) \( \chi_a + \chi_b = \chi_a \) if and only if \( b/a \in \ell_\infty \).

We immediately deduce the next corollary that will be useful in the following.

**Lemma 2.** The next statements are equivalent.

i) \( a \in s_b \),

ii) \( s_a \subset s_b \),

iii) \( s^0_a \subset s^0_b \),

iv) \( a_n \leq Kb_n \) for all \( n \) and for some \( K > 0 \).

In the following our aim is to determine the set of all sequences \( x = (x_n)_{n \geq 1} \in U^+ \) such that \( \frac{y_n}{b_n} = O(1) \) \((n \to \infty)\) if and only if there are \( u, v \in s \) such that \( y = u + v \) and \( u_n = O(a_n) \) and \( v_n = O(x_n) \) \((n \to \infty)\).

We have the next result.

**Theorem 3.** Let \( a = (a_n)_{n \geq 1}, \ b = (b_n)_{n \geq 1} \in U^+ \) and let \( \chi \) be any of the symbols \( s \), or \( s^0 \). Consider the equation

\[ \chi_a + \chi_x = \chi_b, \]  

where \( x = (x_n)_{n \geq 1} \in U^+ \) is the unknown. Then

(i) if \( a/b \in c_0 \) then equation (3) holds if and only if there are \( K_1, K_2 > 0 \) depending on \( x \), such that \( K_1 b_n \leq x_n \leq K_2 b_n \) for all \( n \), that is \( s_x = s_b \);

(ii) if \( a/b, b/a \in \ell_\infty \) then equation (3) holds if and only if there is \( K > 0 \) depending on \( x \) such that \( 0 < x_n \leq Kb_n \) for all \( n \), that is \( x \in s_b \);

(iii) if \( a/b \notin \ell_\infty \) then equation (3) has no solution in \( U^+ \).

**Proof.** The proof in the case when \( \chi = s \) was given in [1]. When \( \chi = s^0 \) the proof follows exactly the same lines as in the previous case since we have the equivalence of (ii) and (iii) in Lemma 2 and by Proposition 1 we have \( s_\xi = s_\eta \) if and only if \( s_\xi^0 = s_\eta^0 \) for \( \xi, \eta \in U^+ \). We deduce the next corollary.
Corollary 4. Let $r, u > 0$ and let $\chi$ be any of the symbols $s$, or $s^0$. Consider the equation

$$\chi_r + \chi_x = \chi_u$$

where $x = (x_n)_{n \geq 1}$ is the unknown. Then we have

i) If $r < u$ equation (4) is equivalent to

$$K_1 u^n \leq x_n \leq K_2 u^n$$

for some $K_1, K_2 > 0$;

ii) if $r = u$ equation (4) is equivalent to

$$x_n \leq Ku^n$$

for some $K > 0$;

iii) if $r > u$ equation (4) has no solution.

1.3. Product of sequence spaces

In this subsection we will deal with some properties of the product $E * F$ of particular subsets $E$ and $F$ of $s$. For any sequences $\xi \in E$ and $\eta \in F$ we put $\xi \eta = (\xi_n \eta_n)_{n \geq 1}$. Most of the next results were shown in [4]. For any sets of sequences $E$ and $F$, we put

$$E * F = \bigcup_{\xi \in E} (1/\xi)^{-1} * F = \{\xi \eta : \xi \in E \text{ and } \eta \in F\}.$$ 

We immediately have the following results, where we put

$$S = \{s_a : a \in U^+\} \text{ and } S^0 = \{s^0_a : a \in U^+\}.$$ 

Proposition 5. The set $S$, (resp. $S^0$) with multiplication $*$ is a commutative group and $\ell_\infty$, (resp. $c_0$) is the unit element for $S$, (resp. $S^0$).

Proof. We only deal with the set $S$ the case of the set $S^0$ can be treated similarly. By [4, Proposition 1, p. 244] we have $\chi_a * \chi_b = \chi_{ab}$. We deduce that the map $\psi : U^+ \rightarrow S$ defined by $\psi(a) = s_a$ is a surjective homomorphism and since $U^+$ with the multiplication of sequences is a group it is the same for $S$. Then the unit element of $S$ is $\psi(e) = s_1 = \ell_\infty$.

Remark 6. Note that the inverse of $\chi_a$ is $\chi_{1/a}$ where $\chi$ be any of the symbols $s$, or $s^0$.

As a direct consequence of Proposition 5 we deduce the next corollary.

Corollary 7. Let $a, b, b' \in U^+$ and let $\chi$ be any of the symbols $s$, or $s^0$. We successively have

i) $\chi_a * \chi_b = \chi_{ab}$.

ii) $\chi_a * \chi_b = \chi_a * \chi_{b'}$ if and only if $s_b = s_{b'}$.

iii) The sequence $x = (x_n)_{n \geq 1} \in U^+$ satisfies the equation

$$\chi_a * \chi_x = s_b$$

(5)
if and only if
\[ K_1 \frac{b_n}{a_n} \leq x_n \leq K_2 \frac{b_n}{a_n} \text{ for all } n \]  
for some \( K_1, K_2 > 0 \) depending only on \( x \).

2. On some sequence spaces equations with operators

In this section we consider among other things the equations \( s^{(c)}(\Delta) = s_b^{(c)} \), \( s_{ax+b}(\Delta) = s_q \) and \( s_n + s_x(\Delta) = s_q \) for given sequences \( a, b \in U^+ \). The resolution of the equation \( s_{ax+b}(\Delta) = s_q \) is equivalent to determine the set of all sequences \( x \in U^+ \) such that
\[ y_n - y_{n-1} = O(a_n x_n + b_n) \]
if and only if \( y_n = O(\eta_n) \) (\( n \to \infty \)) for all \( y \in s \). Solving the equation \( s_n + s_x(\Delta) = s_q \) leads to know the set of all sequences \( x \in U^+ \) such that for each sequence \( y \) we have
\[ y_n = O(x_n) \]
if and only if there are sequences \( u, v \) such that \( y = u + v \) and
\[ u_n = O(a_n) \text{ and } v_n - v_{n-1} = O(x_n) \text{ (} n \to \infty \)).

2.1. On the identities \( \chi_a(\Delta) = \chi_b \) where \( \chi \in \{ s^0, s^{(c)}, s \} \)

To solve the next equations we need additional definitions and properties. The infinite matrix \( T = (t_{nm})_{n,m \geq 1} \) is said to be a triangle if \( t_{nm} = 0 \) for \( m > n \) and \( t_{nn} \neq 0 \) for all \( n \). Now let \( U \) be the set of all sequences \((u_n)_{n \geq 1} \in s \) with \( u_n \neq 0 \) for all \( n \). The infinite matrix \( C(a) \) with \( a = (a_n)_{n \geq 1} \in U \) is defined by
\[ [C(a)]_{nm} = \begin{cases} 1/a_n, & \text{if } m \leq n, \\ 0, & \text{otherwise}. \end{cases} \]

It can be shown that the matrix \( \Delta(a) \) defined by
\[ [\Delta(a)]_{nm} = \begin{cases} a_n, & \text{if } m = n, \\ -a_{n-1}, & \text{if } m = n - 1 \text{ and } n \geq 2, \\ 0, & \text{otherwise}, \end{cases} \]
is the inverse of \( C(a) \), that is \( C(a)(\Delta(a)\xi) = \Delta(a)(C(a)\xi) \) for all \( \xi \in s \). If \( a = e \) we get the well known operator of the first difference represented by \( \Delta(e) = \Delta \). We then have \( \Delta \xi_n = \xi_n - \xi_{n-1} \) for all \( n \geq 1 \), with the convention \( \xi_0 = 0 \). It is usually written
\[ \Sigma = C(e) = \begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]

Note that \( \Delta = \Sigma^{-1} \) and \( \Delta, \Sigma \in SR \) for any \( R > 1 \). Consider the sets where
\[ [C(a)]_n = (\sum_{m=1}^n a_m)/a_n, \]
\[ \widehat{C}_1 = \{ a \in U^+: C(a)a \in \ell_\infty \}, \]
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\[ \hat{C} = \{ a \in U^+ : [C(a)]_n \to l \text{ for some } l \in \mathbb{C} \}, \]

\[ \tilde{\Gamma} = \{ a \in U^+ : \lim_{n \to \infty} \left( \frac{a_{n-1}}{a_n} \right) < 1 \}, \]

\[ \Gamma = \{ a \in U^+ : \limsup_{n \to \infty} \left( \frac{a_{n-1}}{a_n} \right) < 1 \}. \]

and

\[ G_1 = \{ x \in U^+ : x_n \geq k \gamma^n \text{ for all } n \text{ and for some } k > 0 \text{ and } \gamma > 1 \}. \]

By [3, Proposition 2.1, p. 1786] and [6] we obtain the next lemma.

**Lemma 8.** We have

(i) \( \hat{\Gamma} = \hat{C} \).

(ii) \( \Gamma \subset \hat{C}_1 \subset G_1 \).

Since \( \hat{\Gamma} \subset \Gamma \) we deduce \( \hat{\Gamma} = \hat{C} \subset \Gamma \subset \hat{C}_1 \subset G_1 \).

Here among other things we study the equivalence

\[ \frac{y_n - y_{n-1}}{a_n} \to l \text{ if and only if } \frac{y_n}{b_n} \to l' \quad (n \to \infty) \text{ for all } y \in s \text{ and for some } l, l' \in \mathbb{C}. \]

This statement can written in the form \( s_a^{(c)}(\Delta) = s_b^{(c)}. \) We will use the next elementary lemma.

**Lemma 9.** Let \( T_1, T_2 \) be triangles and \( E, F \) be sequence spaces. Then for any triangles \( T \) we have \( T \in (E(T_1), F(T_2)) \) if and only if \( T_2 TT_1^{-1} \in (E, F) \).

The proof is based on the fact that \( T_1, T_2 \) and \( T \) being triangles we have \( E(T_1) = T_1^{-1}E \) and for every \( \xi \in E \) we have

\[ T_2[T(T_1^{-1} \xi)] = (T_2 TT_1^{-1})\xi. \]

Let us state the next results.

**Theorem 10.** Let \( a, b \in U^+ \). We have

(i) The following statements are equivalent

a) \( s_a(\Delta) = s_b \),

b) \( s_0^a(\Delta) = s_0^b \),

c) \( s_a = s_b \) and \( b \in \hat{C}_1 \).

(ii) Assume \( (b_{n-1}/b_n)_n \in c. \) Then

\[ s_0^a(\Delta) = s_0^b \quad (8) \]

if and only if

\[ \frac{a_n}{b_n} \to l \neq 0 \text{ for some } l \in \mathbb{C} \text{ and } b \in \hat{\Gamma}. \]

**Proof.** The statement (i) was shown in [5, Proposition 9, p. 300]. It remains to show (ii). The first identity (8) means that \( \Delta \) is bijective from \( s_a^{(c)} \) to \( s_b^{(c)}. \)
Since $\Delta$ is a triangle and its inverse is equal to $\Sigma$, by Lemma 9 equality (8) is equivalent to $\Sigma \in (s_c, s_c)$ and to $\Delta \in (s_c, s_c)$. Then also by Lemma 9 we have $D_1 \Sigma D_\alpha \in (c, c)$ and $D_1 \Delta D_\beta \in (c, c)$. From the characterization of $(c, c)$ we deduce

$$[C(b)a]_n = \sum_{m=1}^n \frac{a_m}{b_n} \to L \text{ for some } L,$$

and

$$\frac{b_n}{a_n} + \frac{b_{n-1}}{a_n} \leq K \text{ for all } n,$$

Conditions (9) and (10) imply there is $K'$ such that

$$\frac{a_n}{b_n} \leq K' \text{ and } \frac{b_n}{a_n} \leq K \text{ for all } n$$

that is $s_a = s_b$. Then we have $a \in \hat{C}$ since (11) implies

$$[C(a)a]_n = [C(b)a]_n \frac{b_n}{a_n} \leq \frac{1}{K'} [C(b)a]_n \text{ for all } n.$$

Then $b_{n-1}/b_n$ cannot tend to 1. Indeed we have

$$\frac{[C(b)a]_n}{[C(b)a]_{n-1}} = \frac{\sum_{m=1}^{n-1} a_m + a_n b_{n-1}}{\sum_{m=1}^{n-1} a_m} b_n = \left(1 + \frac{a_n}{\sum_{m=1}^{n-1} a_m}\right) \frac{b_{n-1}}{b_n}.$$

Then $L \neq 0$ since

$$[C(b)a]_n \geq \frac{a_n}{K'a_n} = \frac{1}{K'} > 0 \text{ for all } n$$

and

$$\lim_{n \to \infty} \frac{[C(b)a]_n}{[C(b)a]_{n-1}} = \frac{L}{L'} = 1. \text{ So if } b_{n-1}/b_n \text{ tend to 1 we should have}$$

$$1 + \frac{a_n}{\sum_{m=1}^{n-1} a_m} \to 1 \text{ (}n \to \infty\text{)}$$

and

$$[C(a)a]_n = \frac{\sum_{m=1}^{n-1} a_m}{a_n} + 1 \to \infty \text{ (}n \to \infty\text{)}$$

which is contradictory. So we have $b_{n-1}/b_n \to L' \neq 1$. Then

$$\frac{a_n}{b_n} = \frac{1}{b_n} \left(\sum_{m=1}^{n} a_m - \sum_{m=1}^{n-1} a_m\right) = [C(b)a]_n - [C(b)a]_{n-1} \frac{b_{n-1}}{b_n}$$

tends to $L - L'L = L(1 - L') \neq 0$ and $a_n/b_n$ has a nonzero limit $l$. We deduce

$$[C(a)a]_n = [C(b)a]_n \frac{b_n}{a_n} \to \frac{L}{l} \neq 0$$

and $a \in \hat{C} = \hat{\Gamma}$. So $\frac{a_{n-1}}{a_n} \to \chi < 1 (n \to \infty)$ and

$$\frac{b_{n-1}}{b_n} = \frac{b_{n-1}}{a_{n-1}} \frac{a_{n-1}}{a_n} \frac{1}{l} \frac{1}{l} \chi < 1.$$
which implies $b \in \widehat{\Gamma}$. This concludes the proof.

Conversely assume $a_n/b_n \to l \neq 0$ for some $l \in \mathbb{C}$ and $\lim_{n \to \infty} (b_{n-1}/b_n) < 1$. Then $s_a^{(c)} = s_b^{(c)}$ and $b \in \widehat{\Gamma}$ implies $s_a^{(c)}(\Delta) = s_b^{(c)}(\Delta).$

We can state the next result which is a direct consequence of Theorem 10 (i) b).

**Corollary 11.** (i) $s_a^{(c)}(\Delta) = s_a^{(c)}$ if and only if $a \in \widehat{\Gamma}$.
(ii) $c(\Delta) \neq s_a^{(c)}$ for any $a \in U^+$.
(iii) Let $r, u > 0$. Then $s_r^{(c)}(\Delta) = s_u^{(c)}$ if and only if $r = u > 1$.

Let us cite the next lemma where $[\Sigma q]_{nm} = (q + n - m - 1)/n - m$ with $m \leq n$.

**Corollary 12.** [5] Let $q \geq 1$ be an integer. Then the following statements are equivalent

(i) $a \in \widehat{C}_1$,
(ii) $s_a(\Delta) = s_a$,
(iii) $s_a^0(\Delta) = s_a^0$,
(iv) $s_a(\Delta^q) = s_a$,
(v) $s_b^0(\Delta^q) = s_b^0$,
(vi) $\frac{1}{a_n} \sum_{m=1}^{n} \left( q + n - m - 1 \right) a_k = O(1) \ (n \to \infty)$.

2.2. On the (SSE) with operators $(\chi_a * \chi_x + \chi_b)(\Delta) = \chi_{\eta}$ and $[\chi_a * (\chi_x)^2 + \chi_b * \chi_x](\Delta) = \chi_{\eta}$ with $\chi \in \{s^0, s\}$

As consequences of the preceding we can state the next results.

**Proposition 13.** Let $a, b, \eta \in U^+$. Then

i) a) If $b/\eta \in c_0$ the (SSE) with operator

$$(s_a * s_x + s_b)(\Delta) = s_\eta$$

is equivalent to $s_x = s_{\eta/a}$ and $\eta \in \widehat{C}_1$;

b) If $s_b = s_\eta$ then (SSE) (12) is equivalent to $x \in s_{\eta/a}$ and $\eta \in \widehat{C}_1$;

c) If $b/\eta \notin \ell_\infty$ then (SSE) (12) has no solution.

ii) Assume

$$a \in s_\eta^0$$

and

$$b \in s_a.$$ 

Then the (SSE)

$$[s_a * (s_x)^2 + s_b * s_x](\Delta) = s_\eta$$

is equivalent to $\eta \in \widehat{C}_1$ and $s_x = s_{\sqrt{\eta/a}}$. 
Proof. i) We have $s_\eta s_x + s_b = s_{ax} + s_b = s_{ax+b}$. So $(s_\eta s_x + s_b)(\Delta) = s_{ax+b}(\Delta)$. By Theorem 10 (ii) we have that (12) is equivalent to

$$\begin{cases}
  s_{ax+b} = s_\eta \\
  \eta \in \hat{C}_1,
\end{cases}$$

and $s_{ax+b} = s_\eta$ is equivalent to $s_b + s_{ax} = s_\eta$. For the study of the (SSE) it is enough to apply Theorem 3. If $b/\eta \in c_0$ then $s_{ax} = s_\eta$ and $s_x = s_\eta/a$. The remainder of the proof can be shown similarly.

ii) First show the necessity. Since we have $s_\eta s_x^2 + s_b s_x = s_{ax}^2 + bx$, by Theorem 10 (iii) identity (15) is equivalent to

$$\begin{cases}
  s_{ax}^2 + bx = s_\eta \\
  \eta \in \hat{C}_1.
\end{cases}$$

Then $s_{ax}^2 + bx = s_\eta^2$. Let us show $x_n \to \infty (n \to \infty)$. Since $\eta \in \hat{C}_1$ we have $\eta_n \to \infty$ and by (17) there is $K > 0$ such that $a_n x_n^2 + b_n x_n \geq K \eta_n$ and

$$y_n = x_n^2 + b_n/a_n x_n \geq K \eta_n/a_n$$

for all $n$. Then condition (13) implies $\eta_n/a_n \to \infty (n \to \infty)$ and $y_n \to \infty (n \to \infty)$. Now by the identity $y_n = x_n^2 + (b_n/a_n)x_n$ we have

$$x_n = \frac{1}{2} \left( -\frac{b_n}{a_n} + \sqrt{\left(\frac{b_n}{a_n}\right)^2 + 4y_n} \right)$$

for all $n$, and by (14) we deduce $x_n \to \infty (n \to \infty)$. We then have

$$\frac{a_n x_n^2 + b_n x_n}{a_n x_n^2} = 1 + \frac{b_n}{a_n} \frac{1}{x_n} = 1 + O(1) o(1) = 1 + o(1) (n \to \infty),$$

and $a_n x_n^2 + b_n x_n \to 1 (n \to \infty)$, which shows $s_{ax^2+bx} = s_{ax^2}$. By Corollary 7 iii) we conclude $s_x = s_{\eta/a}$.

Sufficiency. Assume $s_x = s_{\eta/a}$ and $\eta \in \hat{C}_1$. Then $s_{ax^2+bx} = s_\eta$. But (14) implies $s_b \subset s_a$ and

$$s_b \sqrt{\frac{1}{2}} \subset s_{\sqrt{\eta/a}}$$

and by (13) we have $\sqrt{a_n \eta_n}/\eta_n = \sqrt{a_n / \eta_n} = o(1) (n \to \infty)$. We conclude $s_{ax^2+bx} = s_\eta$ and since $\eta \in \hat{C}_1$ we have $s_{ax^2+bx}(\Delta) = s_\eta$. This concludes the proof of i). ■

We deduce the next corollaries.

**Corollary 14.** Let $u, p > 0$ and $R > 1$. Consider the (SSE)

$$(s(u^n x_n)_n + s(n^p)_n)(\Delta) = s_R$$

with $x \in U^+$. (18)

Then
(i) if \( R > u \) then the solutions \( x \) of (18) satisfy \( x_n \to \infty (n \to \infty) \) and for any \( \alpha > 0 \) we have \( \lim_{n \to \infty} \frac{x_n}{n^\alpha} = \infty; \)

(ii) if \( R = u \) then the solutions of (18) satisfy \( x_n = O(1) (n \to \infty) \);

(iii) if \( R < u \) then for any given \( \beta > 0 \) the solutions of (18) satisfy

\[
\lim_{n \to \infty} n^\beta x_n = 0.
\]

**Proof.** (i) We have \( a_n = u^n, \eta_n = R^n \) and \( b_n = n^p \). Since \( n^p R^{-n} \to 0 (n \to \infty) \) we have \( b/\eta \in c_0 \) and (18) is equivalent to \( s_x = s_{R/u} \). Then putting \( R/u = r \) there is \( K_1 \) such that \( x_n R^{-\alpha} \geq K_1 R^n n^{-\alpha} \) and since \( r > 1 \) we have \( R^n n^{-\alpha} \to \infty \) and \( x_n n^{-\alpha} \to \infty (n \to \infty) \).

(ii) We have \( R = u \) and as we have seen above we have \( s_x = s_1 \) which implies \( x_n = O(1) (n \to \infty) \).

(iii) Here we have \( s_x = s_{R/u} = s_r \) with \( r < 1 \) so there is \( K_2 \) such that \( x_n r n^{-\alpha} \leq K_2 R^n n^{-\alpha} \) and since \( R^n n^{-\alpha} \) tends to naught we conclude it is the same for \( n^\beta x_n \).

**Corollary 15.** Let \( x \in U^+ \) satisfy the (SSE) with operator

\[
(s_{(n^p x^2)} + s_{(x n \ln n)}) (\Delta) = s_R
\]

with \( p > 0 \) and \( R > 1 \). Then for every \( \alpha > 0 \) we have \( \lim_{n \to \infty} \frac{x_n}{n^\alpha} = \infty. \)

**Proof.** Here we have \( a_n = n^p, b_n = \ln n, \eta_n = R^n \) and conditions (13) and (14) hold since trivially we have \( n^p R^{-n} = o(1) \) and \( \ln n n^p = O(1) (n \to \infty) \), since \( R > 1 \) we also have \( \eta \in \bar{C}_1 \). Then the solutions of (19) satisfy \( x_n \geq K_1 R \sqrt{n} n^{-\alpha} \) and \( x_n R^{-\alpha} \geq K_2 R^{\sqrt{n}} n^{-\alpha} \) then \( R^{\sqrt{n}} n^{-\alpha} \to \infty \) and \( x_n n^{-\alpha} \to \infty (n \to \infty) \). This concludes the proof.

Using similar arguments we immediately obtain the following result.

**Proposition 16.** Let \( a, b, \eta \in U^+ \). Then

(i) \( \alpha \) If \( b/\eta \in c_0 \) then the (SSE)

\[
s_{ax+b} (\Delta) = s_{\eta}
\]

is equivalent to \( s_{ax+b} = s_{\eta/\alpha} \) and \( \eta \in \bar{C}_1. \)

(ii) \( \beta \) If \( s_{a} = s_{\eta} \) then (SSE) (20) is equivalent to \( x \in s_{\eta} \) and \( \eta \in \bar{C}_1; \)

\( \gamma \) If \( b/\eta \notin \ell_\infty \) then (SSE) (20) has no solution.

(ii) \( \alpha \) Assume \( a \in s_{\eta}^0 \) and \( b \in s_a \). Then the (SSE)

\[
s_{ax^2 + bx} (\Delta) = s_{\eta}^0
\]

is equivalent to \( \eta \in \bar{C}_1 \) and \( s_x = s_{\sqrt{n/a}}. \)

We immediately deduce the following.
Corollary 17. The (SSE) with operator
\[ \chi_{ax^2+x}(\Delta) = s_\eta \text{ with } \chi = s^0, \text{ or } s \]
is equivalent to \( \eta \in \widehat{C}_1 \) and \( s_x = s_\sqrt{\eta} \).

Proof. We only consider the (SSE) (22) where \( \chi = s \), the other case can be shown similarly. We have \( s_{x^2+x}(\Delta) = s_\eta \) equivalent to \( s_{x^2+x} = s_\eta \) and \( \eta \in \widehat{C}_1 \).

So \( a = e \in s^0_\eta \) since \( 1/\eta \in c_0 \) and \( b = e \in s_\eta = \ell_\infty \), then by Proposition 12 we conclude \( s_x = s_\sqrt{\eta} \). Conversely. Assume \( s_x = s_\sqrt{\eta} \) and \( \eta \in \widehat{C}_1 \). Then \( \eta_n \to \infty \), so we have \( (\eta_n + \sqrt{\eta_n})/\eta_n \to 1 \) \( (n \to \infty) \) and \( s_{x^2+x} = s_\eta + \sqrt{\eta} = s_\eta \). We conclude \( s_{x^2+x}(\Delta) = s_\eta(\Delta) = s_\eta \).  

2.3. On the (SSE) \( \chi_{ax^2+x}(\Delta) = \chi_x \) and \( \chi_a + \chi_x(\Delta) = \chi_x \) with \( \chi \in \{s^0, s\} \)

Now we are interested in the study of sequence spaces equations with a second member depending on \( x \) such as the (SSE) \( \chi_{ax^2+x}(\Delta) = s_x \) and \( \chi_a + \chi_x(\Delta) = \chi_x \).

We will see that the last equation is equivalent to the equation \( s^0_a + s^0_x(\Delta) = s^0_x \).

Proposition 18. The (SSE)
\[ \chi_{ax^2+x}(\Delta) = \chi_x \]
where \( \chi \) is any of the symbols \( s^0 \), or \( s \) is equivalent to \( x \in \widehat{C}_1 \) and to
\[ x_n \leq \frac{K}{a_n} \text{ for all } n \text{ and for some } K > 0 \]

Proof. We only show the proposition for \( \chi = s \). The proof being similar for the other case. We have that (23) is equivalent to
\[ \begin{cases} s_{ax^2+x} = s_x, \\ x \in \widehat{C}_1. \end{cases} \]

Since we have \( s_{ax^2+x} = s_{ax^2} + s_x \) the identity \( s_{ax^2+x} = s_x \) is equivalent to \( s_{ax^2} \subset s_x \) and to \( s_x \subset s_{1/a} \) by Proposition 1 i). This concludes the proof of the proposition.  

Using similar arguments we deduce the following result.

Remark 19. We immediately deduce that \( s_{x^2+x}(\Delta) = s_x \) has no solution since we have \( x \in \widehat{C}_1 \) implies \( x_n \to \infty \) \( (n \to \infty) \) and we cannot have \( s_x \subset s_{1/a} = \ell_\infty \). It is the same for the equation \( s^0_{x^2+x}(\Delta) = s^0_x \).

In the following we will use the set \( s^*_a = \{x \in U^+ : a/x \in \ell_\infty\} \). We can state the next result.

Proposition 20. Assume
\[ \lim_{n \to \infty} \left( \frac{r^n}{a_n} \right) > 0 \text{ for all } r > 1. \]

Then
\[ \{x \in U^+ : \chi_a + \chi_x(\Delta) = \chi_x\} = \widehat{C}_1 \]
where \( \chi \) is either \( s \), or \( s^0 \).
Proof. First show identity (25) with \( \chi = s \). Let \( A_a \) be the set
\[
A_a = \{ x \in U^+ : s_a + s_x(\Delta) = s_x \}.
\]
Show that \( A_a = \widehat{C}_1 \cap s_a^* \). First let \( x \in A_a \). Then \( s_x(\Delta) \subset s_x \) and \( I \in (s_x(\Delta), s_x) \), by Lemma 9 we have \( \Sigma \in (s_x, s_x) \) that is
\[
\frac{1}{x_n}(x_1 + \cdots + x_n) = O(1) \ (n \to \infty).
\]
We conclude \( A_a \subseteq \widehat{C}_1 \). Then show \( A_a \subseteq s_a^* \). We have \( x \in A_a \) also implies
\[
s_a \subseteq s_a + s_x(\Delta) = s_x
\]
we deduce \( a \in s_a \subset s_x \) and \( x \in s_a^* \). We conclude \( A_a \subseteq \widehat{C}_1 \cap s_a^* \). Now show the inclusion \( \widehat{C}_1 \cap s_a^* \subset A_a \). Let \( x \in \widehat{C}_1 \cap s_a^* \). First \( x \in \widehat{C}_1 \) implies \( s_x(\Delta) = s_x \), then \( x \in s_a^* \) implies \( s_a \subset s_x \) and \( s_a + s_x = s_x \). We conclude \( s_a + s_x(\Delta) = s_x \) and \( x \in A_a \). This shows \( \widehat{C}_1 \cap s_a^* \subset A_a \). Now show \( \widehat{C}_1 \subset s_a^* \). Since by Lemma 8 (ii) we have \( \widehat{C}_1 \subset G_1 \), the condition \( x \in \widehat{C}_1 \) implies there are \( k > 0 \) and \( \gamma > 1 \) such that \( x_n \geq k\gamma^n \). Since we have \( \lim_{n \to \infty}(r^n/a_n) > 0 \) then \( \inf_n(r^n/a_n) > 0 \) for all \( r > 1 \) and there is \( r_0 \in ]1, \gamma[ \) such that
\[
\frac{x_n}{a_n} \geq k \left( \frac{\gamma^n}{a_n} \right) \geq k \inf_n \left( \frac{r_0^n}{a_n} \right) > 0 \text{ for all } n
\]
and \( x \in s_a^* \). So we have shown \( \widehat{C}_1 \subset s_a^* \) and \( A_a = \widehat{C}_1 \). This completes the first part of the proof.

Now show identity (25) holds with \( \chi = s^0 \). Let \( A_a^0 \) be the set
\[
A_a^0 = \{ x \in U^+ : s_a^0 + s_x^0(\Delta) = s_x^0 \}.
\]
Show that \( A_a^0 = \widehat{C}_1 \cap s_a^* \). First let \( x \in A_a^0 \). Again by Lemma 9 we have \( s_x^0(\Delta) \subset s_x^0 \) and \( \Sigma \in (s_x^0, s_x^0) \). So we have
\[
\frac{1}{x_n}(x_1 + \cdots + x_n) = O(1) \text{ and } \frac{1}{x_n} = o(1) \ (n \to \infty).
\]
But since we have \( x \notin \widehat{C}_1 \) implies \( 1/x_n \to 0 \), conditions given by (27) are equivalent to \( x \in \widehat{C}_1 \). So we have \( A_a^0 \subseteq \widehat{C}_1 \). Then show \( A_a^0 \subseteq s_a^* \). We have \( x \in A_a^0 \) implies \( s_x^0 \subset s_a^0 + s_x^0(\Delta) = s_x^0 \) and \( s_a^0 \subset s_x^0 \). By Lemma 2 we deduce \( s_a \subset s_x \) and \( a \in s_a \subset s_x \), this means that \( x \in s_a^* \). We conclude \( A_a^0 \subset \widehat{C}_1 \cap s_a^* \). The proof of the inclusion \( \widehat{C}_1 \cap s_a^* \subset A_a^0 \) follows exactly the same lines that in the proof of \( \widehat{C}_1 \cap s_a^* \subset A_a^0 \). So \( A_a^0 = \widehat{C}_1 \cap s_a^* \). Finally reasoning as above condition (24) permits us to conclude (25) holds with \( \chi = s^0 \). \( \blacksquare \)

The next corollary can be easily deduced.

Corollary 21. We have
(i) \( s_a + s_x(\Delta) \subset s_x \) if and only if \( x \in \widehat{C}_1 \cap s_a^* \);
(ii) if \( x \in \widehat{C}_1 \) then \( s_x \subset s_a + s_x(\Delta) \).
Example 22. Let $\alpha > 0$. Then the set of all sequences $x \in U^+$ such that

$$u_n = O(n^\alpha) \text{ and } v_n - v_{n-1} = O(x_n)$$

implies

$$u_n + v_n = O(x_n) \quad (n \to \infty) \quad \text{for all } u, v \in s,$$

is equal to $\widehat{C}_1$. Indeed for any $r > 1$ we have $\lim_{n \to \infty} (r^n / n^\alpha) > 0$ and $s_a + s_x(\Delta) \subset s_x$.

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