SCORE LISTS IN BIPARTITE MULTI HYPERTOURNAMENTS

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Abstract. Given non-negative integers $m$, $n$, $h$ and $k$ with $m \geq h \geq 1$ and $n \geq k \geq 1$, an $[h, k]$-bipartite multi hypertournament (or briefly $[h, k]$-BMHT) on $m + n$ vertices is a triple $(U, V, A)$, where $U$ and $V$ are two sets of vertices with $|U| = m$ and $|V| = n$ and $A$ is a set of $(h + k)$-tuples of vertices, called arcs with exactly $h$ vertices from $U$ and exactly $k$ vertices from $V$, such that for any $(h + k)$ subset $U_1 \cup V_1$ of $U \cup V$, $A$ contains at least one and at most $(h + k)!$ $(h + k)$-tuples whose entries belong to $U_1 \cup V_1$. If $A$ is a set of $(r + s)$-tuples of vertices, called arcs for $r$ $(1 \leq r \leq h)$ vertices from $U$ and $s$ $(1 \leq s \leq k)$ vertices from $V$ such that $A$ contains at least one and at most $(r + s)!$ $(r + s)$-tuples, then the bipartite multi hypertournament is called an $(h, k)$-bipartite multi hypertournament (or briefly $(h, k)$-BMHT). We obtain necessary and sufficient conditions for a pair of sequences of non-negative integers in non-decreasing order to be losing score lists and score lists of $[h, k]$-BMHT and $(h, k)$-BMHT.

1. Introduction

A $k$-hypertournament is a complete $k$-hypergraph with each $k$-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge. In a $k$-hypertournament, the score $s(v_i)$ (losing score $r(v_i)$) of a vertex $v_i$ is the number of arcs containing $v_i$ and in which $v_i$ is not the last element ($v_i$ is the last element). The score (losing score) sequence is formed by listing the scores (losing scores) in non-decreasing order. Zhou et al. [10] obtained a characterization of score and losing score sequences in $k$-hypertournaments, a result analogous to Landau’s theorem [5] on tournament scores. More results on scores in $k$-hypertournaments can be found in [1, 2, 3, 4, 9, 11].

A bipartite hypergraph is a generalization of a bipartite graph. If $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ are vertex sets of a bipartite hypergraph $G$, the hyper edge of $G$ is a subset of $U$ and $V$, containing at least one vertex from $U$ and at least one vertex from $V$. If an hyperedge has exactly $h$ vertices from $U$ and exactly $k$ vertices from $V$, it is called an $[h, k]$-edge. If all edges of $G$ are $[h, k]$-edges, $G$ is said to be an $[h, k]$-bipartite hypergraph. In case $G$ has all its hyper edges as $[i, j]$-edges for every $i, j$ with $1 \leq i \leq h$ and $1 \leq j \leq k$, $G$ is called an $(h, k)$-bipartite hypergraph.

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An \([h, k]\)-bipartite hypertournament or breifly \([h, k]\)-BHT ((\(h, k\))-bipartite hypertournament or breifly (\(h, k\))-BHT) is a complete \([h, k]\)-bipartite hypergraph (complete (\(h, k\))-bipartite hypergraph) with each \([h, k]\)-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyper edge. The score and losing score of a vertex in \([h, k]\)-BHT or (\(h, k\))-BHT is defined in the same way as in a \(k\)-hypertournament.

Throughout this paper, \(m, n, h\) and \(k\) are non-negative integers with \(m \geq h \geq 1\) and \(n \geq k \geq 1\); \(A = [a_i]_{i=1}^m\), \(B = [b_j]_{j=1}^n\), \(C = [c_i]_{i=1}^m\), \(D = [d_j]_{j=1}^n\) are lists of non-negative integers in non-decreasing order, unless otherwise stated.

The following two results [8] provide characterizations of losing score lists and score lists in \([h, k]\]-BHT.

**Theorem 1.** \((A, B)\) is a pair of losing score lists of an \([h, k]\)-BHT if and only if for each \(p\) and \(q\),
\[
\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \geq \binom{p}{h} \binom{q}{k},
\]
with equality when \(p = m\) and \(q = n\).

**Theorem 2.** \((C, D)\) is a pair of score lists of an \((h, k)\)-BHT if and only if for each \(p\) and \(q\),
\[
\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j \geq \binom{p(m-1)}{h-1} \binom{q}{k} + \binom{p}{h} \binom{n}{k} + \binom{n}{k} - \binom{p}{h} \binom{n}{k} + \binom{m}{h} \binom{n}{k},
\]
with equality when \(p = m\) and \(q = n\).

The next two results [6] provide characterizations of losing score lists and score lists in \((h, k)\)-BHT.

**Theorem 3.** \((A, B)\) is a pair of losing score lists of an \((h, k)\)-BHT if and only if for each \(p\) and \(q\),
\[
\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \geq \sum_{r=1}^{h} \sum_{s=1}^{k} \binom{p}{r} \binom{q}{s},
\]
with equality when \(p = m\) and \(q = n\).

**Theorem 4.** \((C, D)\) is a pair of score lists of an \((h, k)\)-BHT if and only if for each \(p\) and \(q\),
\[
\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j \geq \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ \binom{m-1}{r-1} \binom{n}{s} + \binom{m}{r} \binom{n-1}{s-1} \right] + \binom{m-p}{r} \binom{n-q}{s} - \binom{m}{r} \binom{n}{s},
\]
with equality when \(p = m\) and \(q = n\).

More results on bipartite hypertournaments can be found in [7].
2. Scores of \([h, k]^-\)bipartite multi hypertournaments

An \([h, k]-\)bipartite multi hypertournament (or briefly \([h, k]-\)BMHT) \(H\) on \(m+n\) vertices is a triple \((U, V, A)\), where \(U\) and \(V\) are two sets of vertices with \(|U| = m\) and \(|V| = n\) and \(A\) is a set of \((h+k)\)-tuples of vertices, called arcs with exactly \(h\) vertices from \(U\) and exactly \(k\) vertices from \(V\), such that any \(h+k\) subset \(U_1 \cup V_1\) of \(U \cup V\), \(A\) contains at least one and at most \((h+k)!\) \((h+k)\)-tuples whose entries belong to \(U_1 \cup V_1\). An arc in \(H\) with exactly \(h\) vertices from \(U\) and exactly \(k\) vertices from \(V\) is called an \([h, k]-\)arc.

In an \([h, k]-\)BMHT \(H\), for a given vertex \(u_i \in U\), the score \(d_H^+(u_i)\) or simply \(d^+(u_i)\) (losing score \(d_H^-(u_i)\) or simply \(d^-(u_i)\)) is the number of \([h, k]-\)arcs containing \(u_i\) and in which \(u_i\) is the last element. Similarly, we define by \(d^+(v_j)\) and \(d^-(v_j)\) as the score and losing score of a vertex \(v_j \in V\). If the losing scores (scores) of vertices \(u_i \in U\) and \(v_j \in V\) are arranged in non-decreasing order as a pair of lists \(A\) and \(B\) \((C\) and \(D)\), then \((A, B) (C, D)\) is called the pair of losing score lists (score lists) of an \([h, k]-\)BMHT.

We note that in \([h, k]-\)BMHT if the arc set \(A\) contains exactly one of \((h+k)!\) \((h+k)\)-tuples for each \(h+k\) subset \(U_1 \cup V_1\) of \(U \cup V\), then \([h, k]-\)BMHT becomes \([h, k]-\)BHT. Further, if there are exactly \((h+k)!\) \((h+k)\)-tuples in the arc set \(A\) of \([h, k]-\)BMHT for each \(h+k\) subset \(U_1 \cup V_1\) of \(U \cup V\), then we say \([h, k]-\)BMHT is complete.

Clearly there are exactly \((h+k)!\) \({m\choose h}\){n\choose k} arcs in a complete \([h, k]-\)BMHT and so

\[
\sum_{i=1}^{m} d^-(u_i) + \sum_{j=1}^{n} d^-(v_j) = (h+k)! \binom{m}{h} \binom{n}{k}.
\]

From the above discussion, we have the following observation.

**Lemma 5.** If \((A, B)\) is a pair of losing score lists of a complete \([h, k]-\)BMHT, then

\[
\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = (h+k)! \binom{m}{h} \binom{n}{k}.
\]

Evidently, in a complete \([h, k]-\)BMHT, there are exactly \((h+k-1)!\) \({m-1\choose h-1}\){n\choose k} arcs containing a vertex \(u_i \in U\), \((i = 1, 2, \ldots, m)\) as the last entry and there are exactly \((h+k-1)!\) \({m\choose h}\){n-1\choose k-1} arcs containing a vertex \(v_j \in V\), \((i = 1, 2, \ldots, n)\) as the last entry. Therefore the losing scores \(a_i\) and \(b_j\), and scores \(c_i\) and \(d_j\) in a complete \([h, k]-\)BMHT are given as follows.

\[
a_i = (h+k-1)! \binom{m-1}{h-1} \binom{n}{k},
\]

\[
b_j = (h+k-1)! \binom{m}{h} \binom{n-1}{k-1}.
\]
and
\[ c_i = (h + k - 1)! (h + k - 1) \left( \begin{array}{c} m-1 \\ h-1 \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right), \]
\[ d_j = (h + k - 1)! (h + k - 1) \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right). \]

Lemma 5 can also be proved as follows.
\[
\begin{align*}
\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j &= \sum_{i=1}^{m} \left[ (h + k - 1)! \left( \begin{array}{c} m-1 \\ h-1 \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right) \right] + \sum_{j=1}^{n} \left[ (h + k - 1)! \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) \right] \\
&= (h + k - 1)! \left[ m \left( \begin{array}{c} m-1 \\ h-1 \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right) + n \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) \right] \\
&= (h + k - 1)! \left[ \frac{(m-1)!}{(h-1)!(m-h)!} \left( \begin{array}{c} n \\ k \end{array} \right) + n \frac{(n-1)!}{(k-1)!(n-k)!} \left( \begin{array}{c} m \\ h \end{array} \right) \right] \\
&= (h + k - 1)! \left[ \frac{m!}{h!(m-h)!} \left( \begin{array}{c} n \\ k \end{array} \right) + \frac{n!}{k!(n-k)!} \left( \begin{array}{c} m \\ h \end{array} \right) \right] \\
&= (h + k) (h + k - 1)! \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right) \\
&= (h + k)! \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right). 
\end{align*}
\]

Theorem 6. If \((C, D)\) is a pair of score lists of a complete \([h,k]\)-BMHT, then
\[
\sum_{i=1}^{m} c_i + \sum_{j=1}^{n} d_j = (h + k)! (h + k - 1) \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right). 
\]

Proof. Since there are exactly \((h + k)! \left( \begin{array}{c} m-1 \\ h-1 \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right)\)
and exactly \((h + k)! \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right)\)
arcs respectively containing a vertex \(u_i \in U\) and a vertex \(v_j \in V\), we have
\[
\begin{align*}
\sum_{i=1}^{m} c_i + \sum_{j=1}^{n} d_j &= \sum_{i=1}^{m} \left[ (h + k)! \left( \begin{array}{c} m-1 \\ h-1 \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right) \right] + \sum_{j=1}^{n} \left[ (h + k)! \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) \right] \\
&\quad - (h + k)! \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right) \\
&= (h + k)! \left[ m \left( \begin{array}{c} m-1 \\ h-1 \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right) + n \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) - \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right) \right]. 
\end{align*}
\]
\[ (h + k)! \left[ \binom{m}{h} \binom{n}{k} + k \binom{m}{h} \binom{n}{k} - \binom{m}{h} \binom{n}{k} \right] \]

\[ = (h + k)!((h + k - 1) \binom{m}{h} \binom{n}{k}). \]

The following results are immediate consequences of the above observations.

**Theorem 7.** \((A, B)\) is a pair of losing score lists of a complete \([h, k]\)-BMHT if and only if for each \(p (h \leq p \leq m)\) and each \(q (k \leq q \leq n)\)

\[ \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j = (h + k - 1)! \left[ p \binom{m-1}{h-1} \binom{n}{k} + q \binom{m}{h} \binom{n-1}{k-1} \right]. \]

**Theorem 8.** \((C, D)\) is a pair of score lists of a complete \([h, k]\)-BMHT if and only if for each \(p (h \leq p \leq m)\) and each \(q (k \leq q \leq n)\)

\[ \sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j = (h + k - 1)!((h + k - 1) \left[ p \binom{m-1}{h-1} \binom{n}{k} + q \binom{m}{h} \binom{n-1}{k-1} \right]. \]

The next two results give the necessary and sufficient conditions for a pair of lists \((A, B)\) or \((C, D)\) of non-negative integers in non-decreasing order to be the pair of losing score lists (score lists) of some \([h, k]\)-BMHT.

**Theorem 9.** \((A, B)\) is a pair of losing score lists of an \([h, k]\)-BMHT if and only if for each \(p (h \leq p \leq m)\) and each \(q (k \leq q \leq n)\)

\[ p \binom{m-1}{h-1} \binom{n}{k} + q \binom{m}{h} \binom{n-1}{k-1} \leq \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \leq (h + k - 1)! \left[ \binom{m-1}{h-1} \binom{n}{k} \right]. \] (3)

and

\[ 0 \leq a_i \leq (h + k - 1)! \binom{m-1}{h-1} \binom{n}{k}; \]
\[ 0 \leq b_j \leq (h + k - 1)! \binom{m}{h} \binom{n-1}{k-1}. \]

**Theorem 10.** \((C, D)\) is a pair of score lists of an \([h, k]\)-BMHT if and only if for each \(p (h \leq p \leq m)\) and each \(q (k \leq q \leq n)\)

\[ p \binom{m-1}{h-1} \binom{n}{k} + q \binom{m}{h} \binom{n-1}{k-1} + (m-p) \binom{n-q}{k} - (m) \binom{n}{k} \]

\[ \leq \sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j \]
\[ \leq (h + k - 1)!((h + k - 1) \left[ p \binom{m-1}{h-1} \binom{n}{k} + q \binom{m}{h} \binom{n-1}{k-1} \right]. \] (4)
and

\[ 0 \leq c_i \leq (h + k - 1)!(h + k - 1) \binom{m}{h-1} \binom{n}{k}, \]

\[ 0 \leq d_j \leq (h + k - 1)!(h + k - 1) \binom{m}{h} \binom{n}{k-1}. \]

Before coming to the proofs of Theorems 9 and 10, we make the following important observation.

We define an order relation \( \rightarrow \) on all pairs of lists of non-negative integers in non-decreasing order satisfying (3) and therefore including all pairs of lists satisfying (1) as follows.

Let \( A = [a_i]_m^m \) and \( B = [b_j]_n^n \), and let \( t \) be the smallest index for which \( a_t = a_m = \max\{a_i : 1 \leq i \leq m\} \). Let \( A' = [a'_i]_m^m \), where

\[
a'_i = \begin{cases} 
a_i, & \text{for all } i, i \neq t, 1 \leq i \leq m, \\
a_i - 1, & \text{for } i = t.
\end{cases}
\]

Then we say that the pair \( (A, B) \) strictly covers the pair \( (A', B) \). Here the list \( B \) has been kept fixed. Note that the same argument can be used by fixing the list \( A \) and choosing \( B' = [b'_i]_n^n \), as was chosen in \( A' \).

Clearly, if \( t > 1 \), then

\[ a_{t-1} < a_t = a_{t+1} = \cdots = a_m \]

and if \( t = 1 \), then

\[ a_1 = a_2 = \cdots = a_m. \]

Further, if \( (A, B) \) covers \( (A', B) \), then

\[ \sum_{i=1}^{m} a'_i = \left( \sum_{i=1}^{m} a_i \right) - 1. \]

This implies by Theorem 1, that if the pair \( (X, Y) \) of non-negative integers in non-decreasing order satisfies (3), then \( (X, Y) \) is the losing pair of score lists of some \([h, k]-BHT\) if and only if \( (X, Y) \) covers no pair of lists satisfying (3). In case \( (A, B) \) satisfies (3) and \( (A, B) \) is not a pair of losing score lists of any \([h, k]-BHT\), then \( (A, B) \) covers exactly one pair \( (A', B) \) satisfying (3).

For any two pairs of non-negative integer lists \( (X, Y) \) and \( (X', Y) \) in non-decreasing order satisfying (3), define \( (X', Y) \rightarrow (X, Y) \) if either \( (X', Y) = (X, Y) \) or there is a sequence

\[ (X', Y) = (X_1, Y), (X_2, Y), \ldots, (X_l, Y), (X_0, Y) = (X, Y), \]

\( (1 \leq i \leq l) \) of pairs of non-negative integer lists in non-decreasing order, each satisfying conditions (3) such that \( (X_i, Y) \) covers \( (X_{i-1}, Y) \).
In case the two lists are \((X, Y)\) and \((X, Y')\) then \((X, Y') \rightarrow (X, Y)\) if either \((X, Y') = (X, Y)\) or there is a sequence
\[
(X, Y') = (X, Y_w), (X, Y_{w-1}), \ldots, (X, Y_1), (X, Y_0) = (X, Y),
\]
\((1 \leq i \leq l)\) of pairs of non-negative integer lists in non-decreasing order, each satisfying conditions (3) such that \((X, Y_j)\) covers \((X, Y_{j-1})\).

**Lemma 11.** If \((A, B)\) is a pair of lists satisfying (3), then there exists an \([h, k]\)-BHT \(H\) with a pair of losing score lists \((P, Q)\) where \(P = [p_i]_1^n\), \(p_1 \leq p_2 \leq \cdots \leq p_m\) and \(Q = [q_j]_1^n\), \(q_1 \leq q_2 \leq \cdots \leq q_n\) such that \(p_i \leq a_i\) and \(q_j \leq b_j\) for all \(1 \leq i \leq m\) and \(1 \leq j \leq n\).

**Proof.** Let the pair of lists \((A, B)\) satisfy (3). We use induction
\[
l(A, B) = \sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j - \binom{m}{h} \binom{n}{k}.
\]
If \(l(A, B) = 0\), then by Theorem 1, \((A, B)\) is itself a pair of losing score lists of some \([h, k]\)-BHT \(H^*\). If \(l(A, B) = 1\), then by above observations, \((A, B)\) covers exactly one pair of lists \((X, B)\), with \(X = [x_i]_1^n\), satisfying (3) and such that \(x_i \leq a_i\) for \(1 \leq i \leq m\) and
\[
l(X, B) = \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} b_j - \binom{m}{h} \binom{n}{k} - 1
\]
\[
= l(A, B) - 1.
\]
By induction hypothesis applied to \((X, B)\), there is a pair of losing score lists \((X, B)\) of some \([h, k]\)-BHT \(H^{**}\) such that \(x_i \leq a_i\) for \(1 \leq i \leq m\).

If \(l(A, B) \geq 2\), starting from the pair \((X, B)\), it follows by using similar arguments that there is a pair of losing score lists \((X, Y)\), where \(Y = [y_j]_1^n\) with \(y_j \leq b_j\) for all \(1 \leq j \leq n\), of some \([h, k]\)-BHT \(H^{***}\).

Repeating the above process and by applying transitivity of \(\rightarrow\) it follows that there is a pair of losing score lists \((P, Q)\) of some \([h, k]\)-BHT \(H\), where \(p_i \leq x_i \leq a_i\) and \(q_j \leq y_j \leq b_j\) for all \(1 \leq i \leq m\) and \(1 \leq j \leq n\).

**Lemma 12.** If \((C, D)\) is a pair of lists satisfying (4), then there exists an \([h, k]\)-BHT \(H\) with a pair of score lists \((P, Q)\) where \(P = [p_i]_1^n\) and \(Q = [q_j]_1^n\) such that \(p_i \leq c_i\) and \(q_j \leq d_j\) for all \(1 \leq i \leq m\) and \(1 \leq j \leq n\).

**Proof of Theorem 9. Necessity.** Let \((A, B)\) be a pair of losing score lists of some \([h, k]\)-BMHT \(H\) of order \(m + n\), and let \(U\) and \(V\) be the two sets of vertices of \(H\) with \(|U| = m\) and \(|V| = n\). Then any \(p\) vertices from \(U\) and any \(q\) vertices from \(V\) induce an \([h, k]\)-BMHT of order \(p + q\) which, in turn, contains an \([h, k]\)-BHT \(H^*\) of order \(p + q\). Therefore the sum of the losing scores in \(H^*\) of these
Thus, the same argument as in Theorem 9.

If \((A, B)\) is a pair of lists satisfying (3), then by Lemma 11, there is an \([h, k]\)-BMHT \(H^{**}\) of order \(m + n\) with a pair of losing score lists \((P, Q)\) where \(p \in A, b \in B\). In \(H^{**}\), denote the vertex with losing score \(p_i\) by \(u_i\), \(1 \leq i \leq m\), and the vertex with losing score \(q_j\) by \(v_j\), \(1 \leq j \leq n\). Since the number of arcs in which \(u_i\) and \(v_j\) respectively are not at the last entry are

\[
(h + k - 1)! \binom{m - 1}{h - 1} \binom{n}{k} - p_i \geq (h + k - 1)! \binom{m - 1}{h - 1} \binom{n}{k} - a_i
\]

and

\[
(h + k - 1)! \binom{m}{h} \binom{n - 1}{k - 1} \geq (h + k - 1)! \binom{m}{h} \binom{n - 1}{k - 1} - b_j
\]

therefore \((h + k - 1)! \binom{m - 1}{h - 1} \binom{n}{k} - a_i\) and \((h + k - 1)! \binom{m}{h} \binom{n - 1}{k - 1} - b_j\) arcs respectively can be added in which \(u_i\) and \(v_j\) are at the last entry. This produces an \([h, k]\)-BMHT \(H^{**}\) with a pair of losing score lists \((A, B)\).

Proof of Theorem 10 can be now established by using Lemma 12 and by using the same argument as in Theorem 9.

3. Scores of \((h, k)\)-bipartite multi hypertournaments

An \((h, k)\)-bipartite multi hypertournament (or briefly \((h, k)\)-BMHT) \(H\) on \(m + n\) vertices is a triple \((U, V, A)\), where \(U\) and \(V\) are two sets of vertices with \(|U| = m\) and \(|V| = n\) and \(A\) is a set of \((r + s)\)-tuples of vertices, called arcs with \(r\) \((1 \leq r \leq h)\) vertices from \(U\) and \(s\) \((1 \leq s \leq k)\) vertices from \(V\), such that any \(r + s\) subset \(U_1 \cup V_1\) of \(U \cup V\) contains at least one and at most \((h + k)! (h + k-)\) tuples whose entries belong to \(U_1UV_1\). The score and losing score of a vertex in \((h, k)\)-BMHT is defined in the same way as in \([h, k]\)-BMHT.

An \((h, k)\)-BMHT is said to be complete if for every set of vertices \(U_i\) (for all \(h = 1, 2, \ldots, |U_i|\)) and every set of vertices \(V_i\) (for all \(k = 1, 2, \ldots, |V_i|\)) there are exactly \((h + k)! (h + k-)\) tuples in the arc set \(A\). Clearly in a complete \((h, k)\)-BMHT \(H\) there are exactly \(\sum_{r=1}^{h} \sum_{s=1}^{k} \binom{r}{h} \binom{s}{r+s}\) arcs and therefore

\[
\sum_{i=1}^{m} d_H(u_i) + \sum_{j=1}^{n} d_H(v_j) = \sum_{r=1}^{h} \sum_{s=1}^{k} \binom{r}{h} \binom{s}{r+s}.
\]

Thus,

\[
\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = \sum_{r=1}^{h} \sum_{s=1}^{k} \binom{r}{h} \binom{s}{r+s}.
\]

Further, we observe that the maximum arcs in \((h, k)\)-BMHT is

\[
\sum_{r=1}^{h} \sum_{s=1}^{k} \binom{r}{h} \binom{s}{r+s}.
\]
From the above facts, we have the following result.

**Lemma 13.** If \((C, D)\) is a pair of score lists of a complete \((h, k)\)-BMHT \(H\), then

\[
\sum_{i=1}^{m} c_i + \sum_{j=1}^{n} d_j = \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right].
\]

**Proof.** Since there are

\[
\sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right]
\]
arcs containing a vertex \(u_i \in U\) and

\[
\sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right]
\]
arcs containing a vertex \(v_j \in V\), so

\[
\sum_{i=1}^{m} c_i + \sum_{j=1}^{n} d_j = \sum_{i=1}^{m} \left( \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right] \right)
\]

\[
+ \sum_{j=1}^{n} \left( \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right] \right)
\]

\[
= m \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right] + n \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right]
\]

\[
= \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right] + (r + s)! \sum_{r=1}^{m} \binom{m}{r} \binom{n}{s}
\]

\[
= \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right] + (r + s)! \binom{m}{r} \binom{n}{s} - (r + s)! \binom{m}{r} \binom{n}{s}
\]

\[
= \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s)! \binom{m}{r} \binom{n}{s} \right].
\]

Now we have the following results which are immediate consequences of the above facts.

**Theorem 14.** \((A, B)\) is a pair of losing score lists of a complete \((h, k)\)-BMHT if and only if for each \(p(h \leq p \leq m)\) and each \(q(k \leq q \leq n)\)

\[
\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j = \sum_{r=1}^{h} \sum_{s=1}^{k} \left[ (r + s - 1)! \binom{m}{r} \binom{n}{s} + (r + s)! \binom{m}{r} \binom{n}{s} \right].
\]
Theorem 15. \((C, D)\) is a pair of score lists of a complete \((h, k)\)-BMHT if and only if for each \(p(h \leq p \leq m)\) and each \(q(k \leq q \leq n)\)

\[
\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j = \sum_{r=1}^{h} \sum_{s=1}^{k} [(r+s)! (r+s-1)! \left( p \binom{m-1}{r-1} \binom{n}{s} + q \binom{m}{r} \binom{n-1}{s-1} \right)].
\]

From the above discussions and observations, we conjecture the following two necessary and sufficient conditions for a pair of lists \((A, B)\) and \((C, D)\) respectively to be the pair of losing score lists and score lists of some \((h, k)\)-BMHT.

Conjecture 16. \((A, B)\) is a pair of losing score lists of a complete \((h, k)\)-BMHT if and only if for each \(p(h \leq p \leq m)\) and each \(q(k \leq q \leq n)\)

\[
\sum_{r=1}^{h} \sum_{s=1}^{k} \binom{p}{r} \binom{q}{s} \leq \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \\
\leq \sum_{r=1}^{h} \sum_{s=1}^{k} [(r+s-1)! \left( p \binom{m-1}{r-1} \binom{n}{s} + q \binom{m}{r} \binom{n-1}{s-1} \right)]
\]

and

\[
0 \leq a_i \leq \sum_{r=1}^{h} \sum_{s=1}^{k} \binom{r+s-1}{r-1} \left( p \binom{m-1}{r-1} \binom{n}{s} \right), \\
0 \leq b_j \leq \sum_{r=1}^{h} \sum_{s=1}^{k} \binom{r+s-1}{s-1} \left( q \binom{m}{r} \binom{n-1}{s-1} \right).
\]

Conjecture 17. \((C, D)\) is a pair of score lists of a complete \((h, k)\)-BMHT if and only if for each \(p(h \leq p \leq m)\) and each \(q(k \leq q \leq n)\)

\[
\sum_{r=1}^{h} \sum_{s=1}^{k} \left( p \binom{m-1}{r-1} \binom{n}{s} + q \binom{m}{r} \binom{n-1}{s-1} \right) - \left( m \binom{r-1}{r-1} \binom{n-1}{s-1} \right)
\]

\[
\leq \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \\
\leq \sum_{r=1}^{h} \sum_{s=1}^{k} [(r+s-1)! \left( p \binom{m-1}{r-1} \binom{n}{s} + q \binom{m}{r} \binom{n-1}{s-1} \right)]
\]

and

\[
0 \leq c_i \leq \sum_{r=1}^{h} \sum_{s=1}^{k} \binom{r+s-1}{r-1} \left( m \binom{r-1}{r-1} \binom{n}{s} \right), \\
0 \leq d_j \leq \sum_{r=1}^{h} \sum_{s=1}^{k} \binom{r+s-1}{s-1} \left( m \binom{r}{r} \binom{n-1}{s-1} \right).
\]

References


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