HARMONIC STARLIKE FUNCTIONS OF COMPLEX ORDER
INVOLVING HYPERGEOMETRIC FUNCTIONS

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Dedicated to my father Prof. P.M. Gangadharan (1938–2011)

Abstract. A family of harmonic starlike functions of complex order in the unit disc has been
introduced and investigated by S.A. Halim and A. Janteng [Harmonic functions starlike of complex
order, Proc. Int. Symp. on New Development of Geometric function Theory and its Applications,
(2008), 132–140]. In this paper we consider a subclass consisting of harmonic parabolic starlike
functions of complex order involving special functions and obtain coefficient conditions, extreme
points and a growth result.

1. Introduction

Let $H$ denote the family of harmonic functions $f = h + \overline{g}$ that are orientation
preserving and univalent in the open disc $\Delta = \{ z : |z| < 1 \}$ with $h$ and $g$ given by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.$$ (1.1)

We note that the family $H$ of orientation preserving, normalized harmonic univalent
functions reduces to the well known class $S$ of normalized univalent functions if the
co-analytic part of $f$ is identically zero, i.e. $g \equiv 0$. Also, we denote by $\mathcal{H}$ the
subfamily of $H$ consisting of harmonic functions $f = h + \overline{g}$ of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1.$$ (1.2)

The seminal work of Clunie and Sheil-Small [2] on harmonic mappings gave rise
to many studies of subclasses of complex-valued harmonic univalent functions. In
particular, Silverman [18], Jahangiri [7] Rosy et al. [17], Halim and Janteng [6]
and others (see [10,11,12]) have investigated properties of various subclasses of $\mathcal{H}$
related to harmonic starlike functions.

2010 AMS Subject Classification: 30C45, 30C50.

Keywords and phrases: Harmonic functions; harmonic starlike functions; hypergeometric
functions; Dziok-Srivastava operator.

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The Hadamard product (or convolution) of two power series

\( \phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \) \hspace{1cm} (1.3)

and

\( \psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n \) \hspace{1cm} (1.4)

in \( S \) is defined (as usual) by

\[ (\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n. \] \hspace{1cm} (1.5)

For positive real values of \( \alpha_1, \ldots, \alpha_l \) and \( \beta_1, \ldots, \beta_m \) \((\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)\) the generalized hypergeometric function \( _{i}F_{m}(z) \) is defined by

\[ {_{i}F_{m}(z)} \equiv {_{i}F_{m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)} := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} \] \hspace{1cm} (1.6)

\((l \leq m + 1; l, m \in N : = N \cup \{ 0 \}; z \in \Delta), \)

where \( N \) denotes the set of all positive integers and \((a)_n\) is the Pochhammer symbol defined by

\[ (a)_n = \begin{cases} 1, & n = 0 \\ a(a + 1)(a + 2) \ldots (a + n - 1), & n \in N. \end{cases} \] \hspace{1cm} (1.7)

The notation \( _{i}F_{m} \) is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel and Laguerre polynomial. Let

\[ H[\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m] : S \rightarrow S \]

be a linear operator defined by

\[ H[\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m] \phi(z) = H_{m}^{l}[\alpha_1] \phi(z) \]

\[ = z \sum_{n=2}^{\infty} \omega_n(\alpha_1; l; m) \phi_n z^n, \] \hspace{1cm} (1.8)

where

\[ \omega_n(\alpha_1; l; m) = \frac{(\alpha_1)_n-1 \cdots (\alpha_l)_n-1}{(\beta_1)_n-1 \cdots (\beta_m)_n-1} \frac{1}{(n-1)!}. \] \hspace{1cm} (1.9)

It follows from (1.8) that

\[ H_{0}^{1}[1] \phi(z) = \phi(z), \quad H_{0}^{1}[2] \phi(z) = z \phi'(z). \]

The linear operator \( H_{m}^{l}[\alpha_1] \) is the Dziok-Srivastava operator (see [4]) which was subsequently extended by Dzioek and Raina [3] by using the Wright generalized hypergeometric function. Recently Srivastava et al. [19] defined the linear operator \( L_{\nu}^{s}[\alpha_1] \) as follows:

\[ L_{\nu}^{0, \alpha_1} \phi(z) = \phi(z), \]

\[ L_{\nu}^{1, \alpha_1, l, m}[\phi(z)] = (1 - \lambda) H_{m}^{l}[\alpha_1] \phi(z) + \lambda z (H_{m}^{l}[\alpha_1] \phi(z))' = L_{\nu}^{1, \alpha_1, l, m}[\phi(z)], \quad (\lambda \geq 0), \] \hspace{1cm} (1.10)

\[ L_{\nu}^{2, \alpha_1, l, m}[\phi(z)] = L_{\nu}^{2, \alpha_1, l, m}[L_{\nu}^{1, \alpha_1, l, m}[\phi(z)] \]

\[ \hspace{1cm} (1.11) \]
and in general,
\[ \mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} \phi(z) = \mathcal{L}_{\lambda,l,m}^{\alpha_1}(\mathcal{L}_{\lambda,l,m}^{\tau-1,\alpha_1} \phi(z)) \] (l ≤ m + 1; l, m ∈ N₀ = N ∪ {0}; z ∈ Δ). \hspace{1cm} (1.12)
If the function \( \phi(z) \) is given by (1.3), then we see from (1.8), (1.9), (1.10) and (1.12) that
\[ \mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} \phi(z) := z + \sum_{n=2}^{\infty} \omega_\tau^n(\alpha_1; \lambda; l; m) \phi_n z^n, \hspace{1cm} (1.13) \]
where
\[ \omega_\tau^n(\alpha_1; \lambda; l; m) = \left( \frac{\lambda(n-1)}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \right)^\tau, \quad (n \in N \setminus \{1\}, \tau \in N_0) \] (1.14)
unless otherwise stated. We note that when \( \tau = 1 \) and \( \lambda = 0 \) the linear operator \( \mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} \) would reduce to the familiar Dziok-Srivastava linear operator [4], includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [1], Owa [14] and Ruscheweyh [16].

In view of the relationship (1.14) and the linear operator (1.13) for the harmonic function \( f = h + \overline{g} \) given by (1.1), Murugusundaramoorthy et al. [11,12] have defined the operator
\[ \mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} f(z) = \mathcal{L}_{\lambda,l,m}^{\alpha_1} h(z) + \overline{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z)}, \hspace{1cm} (1.15) \]
and studied the subclass of \( \mathcal{H} \) in terms of this operator.

Goodman [5] introduced two interesting subclasses of \( S \), namely uniformly convex functions (\( \mathcal{UCV} \)) and uniformly starlike functions (\( \mathcal{UST} \)), and Ronning [15] introduced a subclass of starlike functions \( S_p \) corresponding to the class \( \mathcal{UCV} \). In order to consider extension of the class \( S_p \), we study in this note the class of harmonic starlike functions of complex order based on the earlier works of Nasr and Aouf [13] and Halim and Janeteng [6].

For \( 0 ≤ \alpha < 1 \), \( b \), a non-zero complex number with \( |b| < 1 \), we let \( \mathcal{H}_{\lambda,l,m}^{\tau,\alpha_1}(b, \gamma, \alpha) \) be the subclass of \( \mathcal{H} \) consisting of harmonic functions \( f = h + \overline{g} \) where \( h \) and \( g \) are of the form (1.1), satisfying
\[ \Re(w(z)) = \Re \left( 1 + \frac{1}{b} \left( 1 + e^{i\gamma} \right) \frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z))' - \overline{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z))'}}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z) + \overline{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z)}} - e^{i\gamma} - 1 \right) > \alpha, \hspace{1cm} (1.16) \]
z ∈ Δ, and for all real \( \gamma \). We also let \( \overline{\mathcal{H}}_{\lambda,l,m}^{\tau,\alpha_1}(b, \gamma, \alpha) = \mathcal{H}_{\lambda,l,m}^{\tau,\alpha_1}(b, \gamma, \alpha) \cap \overline{H} \).

Remark. With the above conditions, if we choose \( \gamma = 0 \), we can define the generalized class of harmonic starlike functions of complex order satisfying the condition
\[ \Re(w(z)) = \Re \left( 1 + \frac{2}{b} \left( z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z))' - \overline{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z))'} \right) \left( \mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z) + \overline{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z)} \right)^{-1} \right) > \alpha. \]
In this note we obtain sufficient coefficient conditions for harmonic functions \( f = h + \overline{g} \) of the form (1.1) to be in \( \mathcal{H}_{\lambda,l,m}^{\tau,\alpha_1}(b, \gamma, \alpha) \). We also show that these
conditions are necessary when \( f \in \mathcal{RL}^{\alpha_1}_{\lambda,l,m}(b, \gamma, \alpha) \). We also obtain extreme points and growth results.

2. Main results

Theorem 1. Let \( f = h + g \) be given by (1.1). If

\[
\sum_{n=2}^{\infty} \frac{2n-2+(1-\alpha)|b|}{1-\alpha}|a_n|\omega^\alpha_n(\alpha_1; \lambda; l; m) \\
+ \sum_{n=1}^{\infty} \frac{2n+2-(1-\alpha)|b|}{1-\alpha}|b_n|\omega^\alpha_n(\alpha_1; \lambda; l; m) \leq 1 \tag{2.1}
\]

where \( a_1 = 1, 0 \leq \alpha < 1 \) and \( b (|b| \leq 1) \) is a non-zero complex number, then \( f \) is harmonic univalent and orientation-preserving in \( \Delta \) and \( f \in \mathcal{HL}^{\alpha_1}_{\lambda,l,m}(b, \gamma, \alpha) \).

Proof. First we establish that \( f \) is orientation preserving in \( \Delta \). This is seen as follows, on using (2.1):

\[
|L^{\alpha_1}_{\lambda,l,m}h(z)'| \geq 1 - \sum_{n=2}^{\infty} |n\omega^\alpha_n(\alpha_1; \lambda; l; m)|a_n|r^{n-1} \\
> 1 - \sum_{n=2}^{\infty} |n\omega^\alpha_n(\alpha_1; \lambda; l; m)|a_n| \\
\geq 1 - \sum_{n=2}^{\infty} \left[ \frac{2n-2+(1-\alpha)|b|}{1-\alpha}|a_n| \right] \omega^\alpha_n(\alpha_1; \lambda; l; m) \\
\geq \sum_{n=1}^{\infty} \left[ \frac{2n+2-(1-\alpha)|b|}{1-\alpha}|b_n| \right] \omega^\alpha_n(\alpha_1; \lambda; l; m) \\
\geq \sum_{n=1}^{\infty} n\omega^\alpha_n(\alpha_1; \lambda; l; m)|b_n| \\
\geq \sum_{n=1}^{\infty} n\omega^\alpha_n(\alpha_1; \lambda; l; m)|b_n|r^{n-1} \geq |L^{\alpha_1}_{\lambda,l,m}g(z)'|.
\]

To show that \( f \) is univalent in \( \Delta \), we show that \( f(z_1) \neq f(z_2) \) when \( z_1 \neq z_2 \). Suppose \( z_1, z_2 \in \Delta \) so that \( z_1 \neq z_2 \). Since the unit disc \( \Delta \) is simply connected and convex, we then have \( z(t) = (1-t)z_1 + tz_2 \) in \( D \) where \( 0 \leq t \leq 1 \). Then we write

\[
L^{\alpha_1}_{\lambda,l,m}f(z_2) - L^{\alpha_1}_{\lambda,l,m}f(z_1) \\
= \int_0^1 [(z_2 - z_1)(L^{\alpha_1}_{\lambda,l,m}h(z(t))') + (z_2 - z_1)(L^{\alpha_1}_{\lambda,l,m}g(z(t)))'] dt.
\]

Since \( z_2 - z_1 \neq 0 \), dividing throughout by \( z_2 - z_1 \) and taking only the real parts we obtain

\[
\Re \left( L^{\alpha_1}_{\lambda,l,m}f(z_2) - L^{\alpha_1}_{\lambda,l,m}f(z_1) \right) \\
= \int_0^1 \Re \left[ (L^{\alpha_1}_{\lambda,l,m}h(z(t)))' + \frac{z_2 - z_1}{z_2 - z_1} (L^{\alpha_1}_{\lambda,l,m}g(z(t)))' \right] dt \\
> \int_0^1 \Re \left[ (L^{\alpha_1}_{\lambda,l,m}h(z(t)))' - (L^{\alpha_1}_{\lambda,l,m}g(z(t)))' \right] dt. \tag{2.2}
\]
On the other hand,
\[
\Re(\mathcal{L}_{\lambda,l,m}^\tau,\alpha_1 h(z(t)))' - |(\mathcal{L}_{\lambda,l,m}^\tau,\alpha_1 g(z(t)))'|
\geq \Re(\mathcal{L}_{\lambda,l,m}^\tau,\alpha_1 h(z(t)))' - \sum_{n=1}^{\infty} n\omega_n\tau(\alpha_1; \lambda; l; m)|b_n|
\geq 1 - \sum_{n=2}^{\infty} n\omega_n\tau(\alpha_1; \lambda; l; m)|a_n| - \sum_{n=1}^{\infty} n\omega_n\tau(\alpha_1; \lambda; l; m)|b_n|
\geq 1 - \sum_{n=2}^{\infty} \left[ \frac{2n-2(1-\alpha)|b|}{(1-\alpha)|\tau|} \right] \omega_n\tau(\alpha_1; \lambda; l; m)|a_n|
\geq 0 \text{ by (2.1)}.}

Therefore this together with inequality (2.2) implies the univalence of \(f\).

Next we show that \(f \in \mathcal{H}_{\lambda,l,m}^{\tau,\alpha_1}(b, \gamma, \alpha)\). To do so, we need to show that when (2.1) holds, then (1.16) also holds true. Using the fact that \(\Re w(z) \geq \alpha\) if and only if \(|1 - \alpha + w| \geq |1 + \alpha - w|\) for \(0 \leq \alpha < 1\) it suffices to show that
\[
|(2b - ab - e^{\gamma}(1)) (\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z) + \mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z))
+ (1 + e^{\gamma})(z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z))^' - (\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z))^') - |(1 + ab + e^{\gamma})(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z) + \mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z))|
- (1 + e^{\gamma})(z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z))^' - \mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z))^')| \geq 0
\]

On substituting for \((\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z))\) and \((\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z))\) we obtain
\[
|(2b - ab - (1 + e^{\gamma})) \left[ z + \sum_{n=2}^{\infty} \omega_n\tau(\alpha_1; \lambda; l; m)a_n z^n + \sum_{n=1}^{\infty} \omega_n\tau(\alpha_1; \lambda; l; m)b_n z^n \right]
+ (1 + e^{\gamma}) \left[ z + \sum_{n=2}^{\infty} n\omega_n\tau(\alpha_1; \lambda; l; m)a_n z^n - \sum_{n=1}^{\infty} n\omega_n\tau(\alpha_1; \lambda; l; m)b_n z^n \right]
- (1 + e^{\gamma}) \left[ z + \sum_{n=2}^{\infty} n\omega_n\tau(\alpha_1; \lambda; l; m)a_n z^n + \sum_{n=1}^{\infty} n\omega_n\tau(\alpha_1; \lambda; l; m)b_n z^n \right]
\geq (2 - \alpha)|b||z| - \sum_{n=2}^{\infty} |(2 - \alpha)b + (1 + e^{\gamma})(n - 1)|\omega_n\tau(\alpha_1; \lambda; l; m)|a_n| |z|^n
- \sum_{n=1}^{\infty} |(1 + e^{\gamma})(n + 1) - (2 - \alpha)b|\omega_n\tau(\alpha_1; \lambda; l; m)|b_n| |z|^n
- \alpha|b||z| - \sum_{n=2}^{\infty} |(n - 1)(1 + e^{\gamma}) - \alpha|\omega_n\tau(\alpha_1; \lambda; l; m)|a_n| |z|^n
- \sum_{n=1}^{\infty} |(n + 1)(1 + e^{\gamma}) + \alpha|\omega_n\tau(\alpha_1; \lambda; l; m)|b_n| |z|^n
\geq 2(1 - \alpha)|b||z| \left\{ 1 - \sum_{n=2}^{\infty} \left[ \frac{2[n - 2 + (1 - \alpha)|b|]}{2(1 - \alpha)|\tau|} \right] \omega_n\tau(\alpha_1; \lambda; l; m)|a_n| \right\}
\]
\[-2(1 - \alpha)|b||z| \sum_{n=1}^{\infty} \left[ \frac{2n+2-(1-\alpha)|b|}{2(1-\alpha)|b|} \omega_n^r(\alpha_1; \lambda; l; m)|b_n| \right] \geq 0, \text{ by (2.1).} \]

The function
\[ f(z) = z + \sum_{n=2}^{\infty} \left[ \frac{(1-\alpha)|b|}{2n+2-(1-\alpha)|b|} \right] x_n z^n + \sum_{n=1}^{\infty} \left[ \frac{(1-\alpha)|b|}{2n+2-(1-\alpha)|b|} \right] g_n|z^n, \]
where \(\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1\), shows that the coefficient bound given by (2.1) is sharp.

The next theorem shows that condition (2.1) is necessary for \(f \in \mathcal{HL}_{\lambda,l,m}^{r,\alpha_1}(b, \gamma, \alpha)\).

**Theorem 2.** Let \(f = h + g\) be given by (1.2). Then \(f \in \mathcal{HL}_{\lambda,l,m}^{r,\alpha_1}(b, \gamma, \alpha)\) if and only if
\[
\sum_{n=1}^{\infty} \frac{2n+2-(1-\alpha)|b|}{(1-\alpha)|b|} \omega_n^r(\alpha_1; \lambda; l; m)|a_n| + \sum_{n=1}^{\infty} \frac{2n+2-(1-\alpha)|b|}{(1-\alpha)|b|} \omega_n^r(\alpha_1; \lambda; l; m)|b_n| \leq 2 \quad (2.3)
\]
where \(a_1 = 1, 0 \leq \alpha < 1, b\) is a non-zero complex number such that |b| < 1.

**Proof.** Since \(\mathcal{HL}_{\lambda,l,m}^{r,\alpha_1}(b, \gamma, \alpha) \subset \mathcal{HL}_{\lambda,l,m}^{r,\alpha_1}(b, \gamma, \alpha)\), the if part of the Theorem 2 follows from Theorem 1. To prove the only if part, we show that when (2.3) does not hold then \(f\) is not in \(\mathcal{HL}_{\lambda,l,m}^{r,\alpha_1}(b, \gamma, \alpha)\).

First, if \(f \in \mathcal{HL}_{\lambda,l,m}^{r,\alpha_1}(b, \gamma, \alpha)\) then
\[
\Re \left( 1 + \frac{1}{b} \left( (1 + e^{i\gamma}) z \frac{(\mathcal{HL}_{\lambda,l,m}^{r,\alpha_1}(b, \gamma, \alpha) h(z)'}{\mathcal{HL}_{\lambda,l,m}^{r,\alpha_1}(b, \gamma, \alpha) g(z')} - z \left( (1 + e^{i\gamma}) - (1 + e^{i\gamma}) \right) \right) \right) = \Re \left( (1 - \alpha)bz - \sum_{n=2}^{\infty} (1 - \alpha)b(n+1) + (1 + e^{i\gamma})|\omega_n^r(\alpha_1; \lambda; l; m)|a_n|z^n \right) - \Re \left( \sum_{n=2}^{\infty} (1 + e^{i\gamma}) - (1 - \alpha)b|\omega_n^r(\alpha_1; \lambda; l; m)|b_n|z^n \right)
\]
\[
= \Re \left( (1 - \alpha)|b|^2 - \sum_{n=2}^{\infty} (1 - \alpha)b(n+1) + (1 + e^{i\gamma})|\omega_n^r(\alpha_1; \lambda; l; m)|a_n|z^{n-1} \right)
\]
\[
- \frac{1}{b} \left( (1 + e^{i\gamma}) - (1 - \alpha)b|\omega_n^r(\alpha_1; \lambda; l; m)|b_n|z^n \right)
\]
\[
= \Re \left( (1 - \alpha)|b|^2 - \sum_{n=2}^{\infty} (1 - \alpha)b(n+1) + (1 + e^{i\gamma})|\omega_n^r(\alpha_1; \lambda; l; m)|a_n|z^{n-1} \right) - \Re \left( \sum_{n=2}^{\infty} (1 + e^{i\gamma}) - (1 - \alpha)b|\omega_n^r(\alpha_1; \lambda; l; m)|b_n|z^n \right)
\]
\[
\geq 0.
\]
The above condition need hold for all values of \( \gamma, |z| = r < 1 \) and any \( b \) such that 
\( 0 < |b| < 1 \). Choose \( \gamma = 0 \), \( b \) real and positive so that \( |b| = b \) and \( z = r < 1 \) on positive real axis. Thus the above condition becomes

\[
(1 - \alpha)|b|^2 - \sum_{n=2}^{\infty} [(2n - 2) + (1 - \alpha)b]|\omega_n^\tau(\alpha_1; \lambda; l; m)|a_n|r^{n-1}
\]

\[
|b|^2 \left( 1 - \sum_{n=1}^{\infty} \omega_n^\tau(\alpha_1; \lambda; l; m)|a_n|r^{n-1} + \sum_{n=1}^{\infty} \omega_n^\tau(\alpha_1; \lambda; l; m)|b_n|r^{n-1} \right)
\]

\[ - \frac{\sum_{n=1}^{\infty} [(2n + 2) - (1 - \alpha)b]|\omega_n^\tau(\alpha_1; \lambda; l; m)|a_n|r^{n-1}}{|b|^2 \left( 1 - \sum_{n=1}^{\infty} \omega_n^\tau(\alpha_1; \lambda; l; m)|a_n|r^{n-1} + \sum_{n=1}^{\infty} \omega_n^\tau(\alpha_1; \lambda; l; m)|b_n|r^{n-1} \right)} \geq 0. \tag{2.4}
\]

We need to show that the numerator is positive since the denominator is positive.

The numerator is

\[
(1 - \alpha)|b|^2 - b \left[ \sum_{n=2}^{\infty} [(2n - 2) + (1 - \alpha)b]|\omega_n^\tau(\alpha_1; \lambda; l; m)|a_n|r^{n-1}
\]

\[ - \sum_{n=1}^{\infty} [(2n + 2) - (1 - \alpha)b]|b_n|r^{n-1} \]

which is negative if condition (2.3) does not hold. Thus, there exist some point \( z_0 = r_0 \) in \((0,1)\) and some real positive \( b \) for which the quotient in the above inequalities are negative, which contradicts the condition that \( f \in \mathcal{HL}^\tau,a_1(\lambda, l, m)(b, \gamma, \alpha) \).

Hence the proof is complete. \( \blacksquare \)

Next, extreme points of the closed convex hull \( \text{clco} \mathcal{HL}^\tau,a_1(\lambda, l, m)(b, \gamma, \alpha) \) of \( \mathcal{HL}^\tau,a_1(\lambda, l, m)(b, \gamma, \alpha) \) are determined.

**Theorem 3.** \( f \in \text{clco} \mathcal{HL}^\tau,a_1(\lambda, l, m)(b, \gamma, \alpha) \) if and only if

\[
f(z) = \sum_{n=1}^{\infty} (X_nh_n + Y_ng_n) \tag{2.5}
\]

where

\[
h_1(z) = z, h_n(z) = z - \frac{(1 - \alpha)|b|}{2n - 2 + (1 - \alpha)b}|\omega_n^\tau(\alpha_1; \lambda; l; m)|z^n, \quad n = 2, 3, \ldots;
\]

\[
g_n(z) = z + \frac{(1 - \alpha)|b|}{2n + 2 - (1 - \alpha)b}|\omega_n^\tau(\alpha_1; \lambda; l; m)|z^n, \quad n = 1, 2, \ldots;
\]

\[
\sum_{n=1}^{\infty} (X_n + Y_n) = 1, \quad X_n \geq 0 \quad \text{and} \quad Y_n \geq 0. \quad \text{In particular, the extreme points of} \quad \mathcal{HL}^\tau,a_1(\lambda, l, m)(b, \gamma, \alpha) \quad \text{are} \quad \{h_n\} \quad \text{and} \quad \{g_n\}.
\]

**Proof.** For functions \( f \) having the form (2.5), we have

\[
f(z) = \sum_{n=1}^{\infty} (X_nh_n + Y_ng_n)
\]

\[ = \sum_{n=1}^{\infty} (X_n + Y_n)z - \sum_{n=2}^{\infty} \frac{(1 - \alpha)|b|}{2n - 2 + (1 - \alpha)b}|\omega_n^\tau(\alpha_1; \lambda; l; m)|X_nz^n
\]

\[ + \sum_{n=1}^{\infty} \frac{(1 - \alpha)|b|}{2n + 2 - (1 - \alpha)b}|\omega_n^\tau(\alpha_1; \lambda; l; m)|Y_nz^n.
\]
Thus
\[
\sum_{n=2}^{\infty} \left[ \frac{2n - 2 + (1 - \alpha)|b|}{(1 - \alpha)|b|} \omega_n^*(\alpha_1; \lambda; l; m) \right] X_n + \sum_{n=1}^{\infty} \left[ \frac{2n + 2 - (1 - \alpha)|b|}{(1 - \alpha)|b|} \omega_n^*(\alpha_1; \lambda; l; m) \right] Y_n = \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1.
\]

Therefore, \( f \in \text{clco } \mathcal{FL}_{\lambda,l,m}^{\tau,\alpha_1}(b, \gamma, \alpha) \).

Conversely, suppose that \( f \in \text{clco } \mathcal{FL}_{\lambda,l,m}^{\tau,\alpha_1}(b, \gamma, \alpha) \). Set
\[
X_n = \frac{2n - 2 + (1 - \alpha)|b|}{(1 - \alpha)|b|} |a_n| \omega_n^*(\alpha_1; \lambda; l; m), \quad n = 2, 3, \ldots,
\]
and
\[
Y_n = \frac{2n + 2 - (1 - \alpha)|b|}{(1 - \alpha)|b|} |b_n| \omega_n^*(\alpha_1; \lambda; l; m), \quad n = 1, 2, \ldots,
\]
where \( \sum_{n=1}^{\infty}(X_n + Y_n) = 1 \). Then
\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n
\]
\[
= z - \sum_{n=2}^{\infty} \left[ \frac{2n - 2 + (1 - \alpha)|b|}{(1 - \alpha)|b|} \omega_n^*(\alpha_1; \lambda; l; m) \right] X_n z^n
\]
\[
+ \sum_{n=1}^{\infty} \left[ \frac{2n + 2 - (1 - \alpha)|b|}{(1 - \alpha)|b|} \omega_n^*(\alpha_1; \lambda; l; m) \right] Y_n z^n
\]
\[
= z - \sum_{n=2}^{\infty} [X_n(h_n(z) - z)] + \sum_{n=1}^{\infty} [Y_n(g_n(z) - z)]
\]
\[
= \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n).
\]

From Theorem 2, we can deduce that \( 0 \leq X_n \leq 1, \quad (n \geq 2) \) and \( 0 \leq Y_n \leq 1, \quad (n \geq 1) \). We define \( X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n \). Again from Theorem 2, \( X_1 \geq 0 \). Therefore \( \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) = f(z) \) as required in the theorem. ■

**Theorem 4.** If \( f \in \mathcal{FL}_{\lambda,l,m}^{\tau,\alpha_1}(b, \gamma, \alpha) \), then for \( |z| = r < 1 \),
\[
|f(z)| \leq (1 + b_1)r + \left( \frac{(1 - \alpha)|b|}{2 + (1 - \alpha)|b|} \omega_2^*(\alpha_1; \lambda; l; m) \right)^2 - \frac{4 - (1 - \alpha)|b|}{2 + (1 - \alpha)|b|} |b_1| r^2
\]
and
\[
|f(z)| \geq (1 - b_1)r - \left( \frac{(1 - \alpha)|b|}{2 + (1 - \alpha)|b|} \omega_2^*(\alpha_1; \lambda; l; m) \right)^2 - \frac{4 - (1 - \alpha)|b|}{2 + (1 - \alpha)|b|} |b_1| r^2
\]
Proof. Let \( f(z) \in \mathcal{H} \mathcal{L}^{r,\alpha_1}_{\lambda, l, m}(b, \gamma, \alpha) \). On taking the absolute value of \( f \), we have

\[
|f(z)| \leq (1 + |b_1|)r + \sum_{n=2}^{\infty} |a_n| + |b_n| \omega_n^r(\alpha_1; \lambda; l; m) r^n
\]

\[
\leq (1 + |b_1|)r + r^2 \sum_{n=2}^{\infty} (|a_n| + |b_n|) \omega_n^r(\alpha_1; \lambda; l; m)
\]

\[
= (1 + |b_1|)r + \frac{(2 + (1 - \alpha)|b| \omega_2^r(\alpha_1; \lambda; l; m)}{(1 - \lambda)|b|} (|a_n| + |b_n|) \omega_2^r(\alpha_1; \lambda; l; m)
\]

\[
\leq (1 + |b_1|)r + \frac{(2 + (1 - \alpha)|b| \omega_2^r(\alpha_1; \lambda; l; m)}{(1 - \lambda)|b|} (|a_n| + |b_n|) \omega_2^r(\alpha_1; \lambda; l; m)
\]

\[
\leq (1 + |b_1|)r + \frac{(1 - \alpha)|b| \omega_2^r(\alpha_1; \lambda; l; m)}{(1 - \lambda)|b|} (1 - \frac{4(1 - \alpha)|b|}{4(1 - \alpha)|b|} |b_1|) r^2
\]

Similarly we can prove the other inequality. The result is sharp for the function

\[
f(z) = z + |b_1|z + \left( \frac{(1 - \alpha)|b|}{2 + (1 - \alpha)|b| \omega_2^r(\alpha_1; \lambda; l; m)} |b_1| \right) z^2, \quad |b_1| \leq \frac{(1 - \alpha)|b|}{4(1 - \alpha)|b|}.
\]

Concluding remarks. By choosing \( \tau = 1; \lambda = 0 \) and specializing the parameters \( \alpha_1, l, m \), the various results presented in this paper would provide interesting analogous results for the class of harmonic functions those considered earlier in [7–10,12,17,18]. In fact, by appropriately selecting these arbitrary sequences, the results presented in this paper would find further applications for the class of harmonic functions which would incorporate a generalized form of the Dziok-Srivastava linear operator [4] involving the Hadamard product (or convolution) of the function in (1.1) with the Fox-Wright generalization \( \psi_m \) (see [3]) of the hypergeometric function \( {_rF_m} \). Theorems 1 to 4 would thus eventually lead us further to new results for the class of functions (defined analogously to the class \( f \in \mathcal{H} \mathcal{L}^{r,\alpha_1}_{\lambda, l, m}(b, \gamma, \alpha) \)), by associating instead the Fox-Wright generalized hypergeometric function \( \psi_m \). Further, it is of interest to note that the results obtained in this paper yield various results studied in the literature by taking \( \gamma = 0 \) with \( \tau = 1; \lambda = 0 \). We choose to skip further details in this regard.

Acknowledgement. The author would like to thank the referees for their suggestions.
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(received 16.03.2011; in revised form 06.12.2011; available online 15.03.2012)

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