A NEW CHARACTERIZATION OF SPACES WITH LOCALLY COUNTABLE $sn$-NETWORKS

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Abstract. In this paper we prove that a space $X$ is with a locally countable $sn$-network (resp., weak base) if and only if it is a compact-covering (resp., compact-covering quotient) compact and $ss$-image of a metric space, if and only if it is a sequentially-quotient (resp., quotient) $\pi$- and $ss$-image of a metric space, which gives a new characterization of spaces with locally countable $sn$-networks (or weak bases).

1. Introduction

In 2002, Y. Ikeda, C. Liu and Y. Tanaka introduced the notion of $\sigma$-strong networks as a generalization of “development” in developable spaces, and considered certain quotient images of metric spaces in terms of $\sigma$-strong networks. By means of $\sigma$-strong networks, some characterizations for the quotient and compact images of metric spaces are obtained (see in [4, 18, 19], for example).

In this paper, by means of $\sigma$-strong networks, we give a new characterization of spaces with locally countable $sn$-networks (or weak bases). Throughout this paper, all spaces are assumed to be $T_1$ and regular, all maps are continuous and onto, $N$ denotes the set of all natural numbers. Let $P$ and $Q$ be two families of subsets of $X$, and $f : X \to Y$ be a map, we denote $P \wedge Q = \{P \cap Q : P \in P, Q \in Q\}$, $P = \bigcap\{P : P \in P\}$, $P = \bigcup\{P : P \in P\}$, $st(x, P) = \bigcup\{P \in P : x \in P\}$, and $f(P) = \{f(P) : P \in P\}$. For a sequence $\{x_n\}$ converging to $x$ and $P \subset X$, we say that $\{x_n\}$ is eventually in $P$ if $\{x\} \bigcup\{x_n : n \geq m\} \subset P$ for some $m \in N$, and $\{x_n\}$ is frequently in $P$ if some subsequence of $\{x_n\}$ is eventually in $P$.

Definition 1.1. Let $X$ be a space and $P$ be a subset of $X$.

1. $P$ is a sequential neighborhood of $x$ in $X$, if each sequence $S$ converging to $x$ is eventually in $P$.
2. $P$ is a sequentially open subset of $X$, if $P$ is a sequential neighborhood of $x$ in $X$ for all $x \in P$.

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Definition 1.2. Let \( \mathcal{P} \) be a collection of subsets of a space \( X \) and let \( K \) be a subset of \( X \). Then,

1. For each \( x \in X \), \( \mathcal{P} \) is a network at \( x \) [18], if \( x \in P \) for every \( P \in \mathcal{P} \), and if \( x \in U \) with \( U \) is open in \( X \), there exists \( P \in \mathcal{P} \) such that \( x \in P \subset U \).
2. \( \mathcal{P} \) is a network for \( X \) [18], if \( \{ P \in \mathcal{P} : x \in P \} \) is a network at \( x \) in \( X \) for all \( x \in X \).
3. \( \mathcal{P} \) is a \( cs^* \)-network for \( X \) [19], if for each sequence \( S \) converging to a point \( x \in U \) with \( U \) is open in \( X \), \( S \) is frequently in \( P \subset U \) for some \( P \in \mathcal{P} \).
4. \( \mathcal{P} \) is a \( cs \)-network for \( X \) [19], if each sequence \( S \) converging to a point \( x \in U \) is open in \( X \), \( S \) is eventually in \( P \subset U \) for some \( P \in \mathcal{P} \).
5. \( \mathcal{P} \) is a \( cfp \)-cover of \( K \) in \( X \) [13], if \( \mathcal{P} \) is a cover of \( K \) in \( X \) such that it can be precisely refined by some finite cover of \( K \) consisting of compact subsets of \( K \).
6. \( \mathcal{P} \) is a \( cfp \)-cover for \( X \) [13], if whenever \( K \) is a compact subset of \( X \), there exists a finite subfamily \( G \subset \mathcal{P} \) such that \( G \) is a \( cfp \)-cover of \( K \).
7. \( \mathcal{P} \) is locally countable, if for each \( x \in X \), there exists a neighborhood \( V \) of \( x \) such that \( V \) meets only countably many members of \( \mathcal{P} \).
8. \( \mathcal{P} \) is point-countable (resp., point-finite), if each point \( x \in X \) belongs to only countably (resp., finitely) many members of \( \mathcal{P} \).
9. \( \mathcal{P} \) is star-countable [15], if each \( P \in \mathcal{P} \) meets only countably many members of \( \mathcal{P} \).

Definition 1.3. Let \( \mathcal{P} = \bigcup \{ \mathcal{P}_x : x \in X \} \) be a family of subsets of a space \( X \) satisfying that, for every \( x \in X \), \( \mathcal{P}_x \) is a network at \( x \) in \( X \), and if \( U, V \in \mathcal{P}_x \), then \( W \subset U \cap V \) for some \( W \in \mathcal{P}_x \).

1. \( \mathcal{P} \) is a weak base for \( X \) [1], if whenever \( G \subset X \) satisfying for every \( x \in G \), there exists \( P \in \mathcal{P}_x \) with \( P \subset G \), then \( G \) is open in \( X \). Here, \( \mathcal{P}_x \) is a weak neighborhood base at \( x \) in \( X \).
2. \( \mathcal{P} \) is an \( sn \)-network for \( X \) [10], if each member of \( \mathcal{P}_x \) is a sequential neighborhood of \( x \) for all \( x \in X \). Here, \( \mathcal{P}_x \) is an \( sn \)-network at \( x \) in \( X \).

Definition 1.4. Let \( f : X \rightarrow Y \) be a map.

1. \( f \) is a sequence-covering map [16], if for every convergent sequence \( S \) in \( Y \), there exists a convergent sequence \( L \) in \( X \) such that \( f(L) = S \). Note that a sequence-covering map is a strong sequence-covering map in the sense of [9].
2. \( f \) is a compact-covering map [14], if for each compact subset \( K \) of \( Y \), there exists a compact subset \( L \) of \( X \) such that \( f(L) = K \).
3. \( f \) is a pseudo-sequence-covering map [8], if for each convergent sequence \( S \) in \( Y \), there exists a compact subset \( K \) of \( X \) such that \( f(K) = S \). Note that a pseudo-sequence-covering map is a sequence-covering map in the sense of [7].
4. \( f \) is a subsequence-covering map [12], if for each convergent sequence \( S \) in \( Y \), there exists a compact subset \( K \) of \( X \) such that \( f(K) \) is a subsequence of \( S \).
(5) $f$ is a sequentially-quotient map [2], if for each convergent sequence $S$ in $Y$, there exists a convergent sequence $L$ in $X$ such that $f(L)$ is a subsequence of $S$.

(6) $f$ is a quotient map [3], if whenever $U \subset Y$, $U$ is open in $Y$ if and only if $f^{-1}(U)$ is open in $X$.

(7) $f$ is an ss-map [18], if for each $y \in Y$, there exists a neighborhood $U$ of $y$ such that $f^{-1}(U)$ is separable in $X$.

(8) $f$ is a compact map [19], if $f^{-1}(y)$ is compact in $X$ for all $y \in Y$.

(9) $f$ is a $\pi$-map [1], if for every $y \in Y$ and for every neighborhood $U$ of $y$ in $Y$, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where $X$ is a metric space with a metric $d$.

**Definition 1.5.** Let $X$ be a space. Then,

(1) $X$ is a $g$-first countable space [1] (resp., an $sn$-first countable space [3], if there is a countable weak neighborhood base (resp., $sn$-network) at $x$ in $X$ for all $x \in X$.

(2) $X$ is an $\aleph_0$-space [14], if it has a countable cs-network.

(3) $X$ is a sequential space [19], if every sequential open subset of $X$ is open in $X$.

(4) $X$ is a Fréchet space, if for each $x \in X$, there exists a sequence in $A$ converging to $x$ in $X$.

**Definition 1.6.** [8] Let $\{P_n : n \in \mathbb{N}\}$ be a sequence of covers of a space $X$ such that $P_{n+1}$ refines $P_n$ for every $n \in \mathbb{N}$. $\bigcup\{P_n : n \in \mathbb{N}\}$ is a $\sigma$-strong network for $X$, if $\{st(x, P_n) : n \in \mathbb{N}\}$ is a network at $x$ for all $x \in X$.

**Notation 1.7.** Let $\bigcup\{P_n : n \in \mathbb{N}\}$ be a $\sigma$-strong network for a space $X$. For each $n \in \mathbb{N}$, put $P_n = \{P_\alpha : \alpha \in \Lambda_n\}$ and endow $\Lambda_n$ with the discrete topology. Then,

$$M = \{\alpha = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{P_\alpha\} \text{ forms a network at some point } x_\alpha \in X\}$$

is a metric space and the point $x_\alpha$ is unique in $X$ for every $\alpha \in M$. Define $f : M \rightarrow X$ by $f(\alpha) = x_\alpha$. Let us call $(f, M, X, P_n)$ a Ponomarev’s system, following [13].

For some undefined or related concepts, we refer the reader to [8, 11, 19].

2. Main results

**Theorem 2.1.** The following are equivalent for a space $X$.

(1) $X$ is an $sn$-first countable space with a locally countable $cs^*$-network;

(2) $X$ has a locally countable $sn$-network;

(3) $X$ has a $\sigma$-strong network $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ satisfying the following:

(a) Each $\mathcal{U}_n$ is a point-finite cfp-cover;
(b) $\mathcal{U}$ is locally countable.

(4) $X$ is a compact-covering compact and ss-image of a metric space;

(5) $X$ is a pseudo-sequence-covering compact and ss-image of a metric space;

(6) $X$ is a subsequence-covering compact and ss-image of a metric space;

(7) $X$ is a sequentially-quotient $\pi$ and ss-image of a metric space.

Proof. (1) $\implies$ (2). Similar to the proof of (2) $\implies$ (1) in Theorem 2.12 [3].

(2) $\implies$ (3). Let $\mathcal{P} = \bigcup \{ P_x : x \in X \} = \{ P_\alpha : \alpha \in \Lambda \}$ be a locally countable $sn$-network for $X$, where each $P_x$ is an $sn$-network at $x$. Since $X$ is a regular space, we can assume that each element of $\mathcal{P}$ is closed. Then, for each $x \in X$, there exists an open neighborhood $V_x$ of $x$ such that $V_x$ meets only countably many members of $\mathcal{P}$. Let

$$Q = \{ P \in \mathcal{P} : P \subset V_x \text{ for some } x \in X \}.$$ 

Then, $Q$ is a locally countable and star-countable network for $X$. By Lemma 2.1 in [15], $Q = \bigcup_{\alpha \in \Lambda} Q_\alpha$, where each $Q_\alpha$ is a countable subfamily of $Q$ and $(\bigcup Q_\alpha) \cap (\bigcup Q_\beta) = \emptyset$ for all $\alpha \neq \beta$. For each $\alpha \in \Lambda$, let $Q_\alpha = \{ P_{\alpha,n} : n \in \mathbb{N} \}$, and for each $i \in \mathbb{N}$, denote $H_i = \{ P_{\alpha,i} : \alpha \in \Lambda \}$. Then, $Q = \bigcup \{ Q_i : i \in \mathbb{N} \}$. Now, for each $i \in \mathbb{N}$, let

$$A_i = \{ x \in X : H_i \cap P_x = \emptyset \}, \quad G_i = H_i \cup \{ A_i \}.$$ 

Then, we have

(a) $\bigcup \{ G_n : n \in \mathbb{N} \}$ is locally countable.

(b) Each $G_i$ is point-finite.

(c) Each $G_i$ is a cfp-cover for $X$. Let $K$ be a non-empty compact subset of $X$. We shall show that there exists a finite subset of $G_i$ which forms a cfp-cover of $K$. In fact, since $X$ has a locally countable $sn$-network, $K$ is metrizable. Note that each $\bigcup Q_\alpha$ is sequentially open in $X$ and $(\bigcup Q_\alpha) \cap (\bigcup Q_\beta) = \emptyset$ for all $\alpha \neq \beta$, so the family $\{ \alpha \in \Lambda : K \cap (\bigcup Q_\alpha) \neq \emptyset \}$ is finite. Thus, $K$ meets only finitely many members of $G_i$. Let $\Gamma_i = \{ \alpha : P_{\alpha} \in H_i, P_{\alpha} \cap K \neq \emptyset \}$. For each $\alpha \in \Gamma_i$, put $K_\alpha = P_{\alpha} \cap K$, then $K_i = K - \bigcup_{\alpha \in \Gamma_i} K_\alpha$. It is obvious that all $K_\alpha$ and $K_i$ are closed subset of $K$, and $K = K_i \cup \bigcup_{\alpha \in \Gamma_i} K_\alpha$. Now, we only need to show $K_i \subset A_i$ for all $i \in \mathbb{N}$. Let $x \in K_i$, then there exists a sequence $\{ x_n \}$ of $K - \bigcup_{\alpha \in \Gamma_i} K_\alpha$ converging to $x$. If $P \in P_x \cap H_i$, then $P$ is a sequential neighborhood of $x$ and $P = P_\alpha$ for some $\alpha \in \Gamma_i$. Thus, $x_n \in P$ for some $m \in \mathbb{N}$. Hence, $x_n \in K_\alpha$ for some $\alpha \in \Gamma_i$, a contradiction. So, $P_x \cap H_i = \emptyset$, and $x \in A_i$. This implies that $K_i \subset A_i$ and $\{ A_i \} \cup \{ P_{\alpha} : \alpha \in \Gamma_i \}$ is a cfp-cover of $K$.

(d) Each $\{ \text{st}(x, G_n) : n \in \mathbb{N} \}$ is a network at $x$. Let $x \in U$ with $U$ is open in $X$. Then, $x \in P \subset U \cap V_x$ for some $P \in P_x$, so $P \in Q$. Thus, there exists a unique $\alpha \in \Lambda$ such that $P \in Q_\alpha$. Hence, $P = P_{\alpha,j} \in H_i$ for some $j \in \mathbb{N}$. Since $P \in H_i \cap P_x$, $x \notin A_i$. Note that $P \cap P_{\alpha,j} = \emptyset$ for all $j \neq i$. Then, $\text{st}(x, G_i) = P \subset U$. Therefore, $\{ \text{st}(x, G_n) : n \in \mathbb{N} \}$ is a network at $x$ for all $x \in X$.

Next, for each $n \in \mathbb{N}$, put $\mathcal{U}_n = \bigwedge \{ G_i : i \leq n \}$. Then, $\bigcup \{ \mathcal{U}_n : n \in \mathbb{N} \}$ is a $\sigma$-strong network and each $\mathcal{U}_n$ is a point-finite cfp-cover for $X$. Now, we shall show
that \( \bigcup \{ \mathcal{U}_n : n \in \mathbb{N} \} \) is locally countable. In fact, since \( \mathcal{P} \) is locally countable, \( \mathcal{V} = (\{A_i : i \in \mathbb{N}\}) \cup \mathcal{P} \) is locally countable. Thus, \( \{\bigcap F : F \text{ is a finite subfamily of } \mathcal{V}\} \) is locally countable. Furthermore, since \( \bigcup \{ \mathcal{G}_i : i \in \mathbb{N} \} \subset \mathcal{V} \), we have

\[
\bigcup \{ \mathcal{U}_n : n \in \mathbb{N} \} \subset \left\{ \bigcap F : F \text{ is a finite subfamily of } \mathcal{V} \right\}.
\]

This implies that \( \bigcup \{ \mathcal{U}_n : n \in \mathbb{N} \} \) is locally countable. Therefore, (3) holds.

(3) \( \implies \) (4). Let \( \mathcal{U} = \bigcup \{ \mathcal{U}_n : n \in \mathbb{N} \} \) be a \( \sigma \)-strong network satisfying (3). Consider the Ponomarev’s system \( (f, M, X, \mathcal{U}_n) \). Because each \( \mathcal{U}_n \) is a point-finite and locally countable \( cfp \)-cover, it follows from Lemma 2.2 [19] that \( f \) is a compact-covering and compact map. We only need to show \( f \) is an \( ss \)-map. Let \( x \in X \), since \( \mathcal{U} \) is locally countable, there exists a neighborhood \( V \) of \( x \) such that \( V \) meets only countably many members of \( \mathcal{U} \). For each \( i \in \mathbb{N} \), let \( \Delta_i = \{\alpha \in \Lambda_i : P_\alpha \cap V \neq \emptyset\} \). Then, each \( \Delta_i \) is countable. On the other hand, since \( f^{-1}(V) \subset \prod_{i \in \mathbb{N}} \Delta_i = f^{-1}(V) \) is separable in \( M \). Therefore, (4) holds.

(4) \( \implies \) (5) \( \implies \) (6). It is obvious.

(6) \( \implies \) (1). Let \( f : M \rightarrow X \) be a sequentially-quotient \( \pi \) and \( ss \)-map. It follows from Corollary 2.6 [4] that \( X \) has a \( \sigma \)-strong network \( \mathcal{G} = \bigcup \{ \mathcal{G}_n : n \in \mathbb{N} \} \), where each \( \mathcal{G}_n \) is a \( cs^* \)-cover. For each \( x \in X \), let \( \mathcal{G}_x = \{st(x, \mathcal{G}_n) : n \in \mathbb{N}\} \). Since each \( \mathcal{P}_n \) is a \( cs^* \)-cover, it implies that \( \bigcup \{ \mathcal{G}_x : x \in X \} \) is an \( sn \)-network for \( X \). Hence, \( X \) is an \( sn \)-first countable space. Now, let \( \mathcal{B} \) be a point-countable base for \( M \), since \( f \) is a sequentially-quotient and \( ss \)-map, \( f(\mathcal{B}) \) is a locally countable \( cs^* \)-network for \( X \). Therefore, (1) holds. \( \blacksquare \)

**Corollary 2.2.** The following are equivalent for a space \( X \).

1. \( X \) has a locally countable weak base;
2. \( X \) is a sequential space with a \( \sigma \)-strong network \( \mathcal{U} = \bigcup \{ \mathcal{U}_n : n \in \mathbb{N} \} \) satisfying the following:
   (a) Each \( \mathcal{U}_n \) is a point-finite \( cfp \)-cover;
   (b) \( \mathcal{U} \) is locally countable.
3. \( X \) is a compact-covering quotient compact and \( ss \)-image of a metric space;
4. \( X \) is a pseudo-sequence-covering quotient compact and \( ss \)-image of a metric space;
5. \( X \) is a subsequence-covering quotient compact and \( ss \)-image of a metric space;
6. \( X \) is a quotient \( \pi \) and \( ss \)-image of a metric space.

**Example 2.3.** Let \( C_n \) be a convergent sequence containing its limit point \( p_n \) for each \( n \in \mathbb{N} \), where \( C_m \cap C_n = \emptyset \) if \( m \neq n \). Let \( \mathbb{Q} = \{q_n : n \in \mathbb{N}\} \) be the set of all rational numbers of the real line \( \mathbb{R} \). Put \( M = (\bigoplus \{C_n : n \in \mathbb{N}\}) \oplus \mathbb{R} \) and let \( X \) be the quotient space obtained from \( M \) by identifying each \( p_n \) in \( C_n \) with \( q_n \) in \( \mathbb{R} \). Then, by the proof of Example 3.1 [6], \( X \) has a countable weak base and \( X \) is not a sequence-covering quotient and \( \pi \)-image of a metric space. Hence,

1. A space with a locally countable \( sn \)-network \( \Rightarrow \) a sequence-covering and \( \pi \)-image of a metric space.
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(2) A space with a locally countable weak base \( \nRightarrow \) a sequence-covering quotient and $\pi$-image of a metric space.

**Example 2.4.** Using Example 3.1 [5], it is easy to see that $X$ is Hausdorff, non-regular and $X$ has a countable base, but it is not a sequentially-quotient and $\pi$-image of a metric space. This shows that regular properties of $X$ can not be omitted in Theorem 2.1 and Corollary 2.2.

**Example 2.5.** $S_\omega$ is a Fréchet and $\aleph_0$-space, but it is not first countable. Thus, $S_\omega$ has a locally countable $cs$-network. Since $S_\omega$ is not first countable, it has not locally countable $sn$-network. Hence, a space with a locally countable $cs$-network \( \nRightarrow \) a sequentially-quotient and $\pi$-image of a metric space.

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**References**


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