ON THE FINE SPECTRUM OF GENERALIZED UPPER DOUBLE-BAND MATRICES $\Delta^{uv}$ OVER THE SEQUENCE SPACE $\ell_1$

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Abstract. The main purpose of this paper is to determine the fine spectrum of generalized upper triangle double-band matrices $\Delta^{uv}$ over the sequence space $\ell_1$.

1. Introduction

In functional analysis, the spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum and the residual spectrum. The calculation of three parts of the spectrum of an operator is called calculating the fine spectrum of the operator.

Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We introduce knowledge in the existing literature concerning the spectrum and the fine spectrum. The fine spectrum of the Cesaro operator on the sequence space $\ell_p$ for $(1 < p < \infty)$ has been studied by Gonzalez [12]. Also, Wenger [20] examined the fine spectrum of the integer power of the Cesaro operator over $c$, and Rhoades [17] generalized this result to the weighted mean methods. Reade [16] worked the spectrum of the Cesaro operator over the sequence space $c_0$. Okutoyi [15] computed the spectrum of the Cesaro operator over the sequence space $bv$. The fine spectrum of the Rhally operators on the sequence spaces $c_0$ and $c$ is studied by Yildirim [22]. The fine spectra of the Cesaro operator over the sequence spaces $c_0$ and $bv_p$ have determined by Akhmedov and Basar [1, 4]. Akhmedov and Basar [2, 3] have studied the fine spectrum of the difference operator $\Delta$ over the sequence spaces $\ell_p$, and $bv_p$, where $(1 \leq p < \infty)$. The fine spectrum of the Zweier matrix as an operator over the sequence spaces $\ell_1$ and $bv_1$ have been examined by Altay and Karakus [6]. Altay and Basar [5, 9] have determined the fine spectrum of the difference operator $\Delta$ over the sequence spaces $c_0$, $c$ and $\ell_p$, where $(0 < p < 1)$. The fine spectrum of the difference operator $\Delta$ over the sequence spaces $\ell_1$ and $bv$ is

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investigated by Kayaduman and Furkan [13]. Altun and Karakaya [7, 8] has been studied the fine spectra of Lacunary matrices and fine spectra of upper triangular double-band matrices. Recently, Srivastava and Kumar [18] has been examined the fine spectrum of the generalised difference operator $\Delta_v$ over the sequence space $c_0$.

In this work, our purpose is to determine the fine spectra of the generalised upper double-band matrices $\Delta^u$ as operators over the sequence space $\ell_1$.

By $w$, we denote the space of all real or complex valued sequences. Any vector subspace of $w$ is called a sequence space. Let $\mu$ and $\nu$ be two sequence spaces and $A = (a_{n,k})$ be an infinite matrix operator of real or complex numbers $a_{n,k}$, where $n, k \in \mathbb{N} = \{0, 1, 2, \ldots \}$. We say that $A$ defines a matrix mapping from $\mu$ into $\nu$ and denote it by $A : \mu \longrightarrow \nu$, if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = ((Ax)_n)$, the $A$-transform of $x$, is in $\nu$, where $(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k$.

Let $X$ and $Y$ be Banach spaces and let $T : X \longrightarrow Y$ be a bounded linear operator. By $R(T)$, we denote the range of $T$, i.e.,

$$R(T) = \{ y \in Y : y = Tx, x \in X \}.$$

By $B(X)$, we denote the set of all bounded linear operators of $X$ into itself. If $X$ is any Banach space and $T \in B(X)$ then the adjoint $T^*$ of $T$ is a bounded linear operator on the dual $X^*$ of $X$ defined by $(T^* \psi)(x) = \psi(Tx)$ for all $\psi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \emptyset$ be a complex normed space and let $T : \mathcal{D}(T) \longrightarrow X$ be a bounded linear operator with domain $\mathcal{D} \subseteq X$. With $T$, we associate the operator $T_\lambda = T - \lambda I$, where $\lambda$ is a complex number and $I$ is the identity operator on $\mathcal{D}(T)$, if $T_\lambda$ has an inverse, which is linear, we denote it by $T^{-1}_\lambda$, that is

$$T^{-1}_\lambda = (T - \lambda I)^{-1}$$

and call it the resolvent operator of $T$.

The name resolvent is appropriate, since $T^{-1}_\lambda$ helps to solve the equation $T_\lambda x = y$. Thus, $x = T^{-1}_\lambda y$ provided $T^{-1}_\lambda$ exists. More important, the investigation of properties of $T^{-1}_\lambda$ will be basic for an understanding of the operator $T$ itself. Naturally, many properties of $T_\lambda$ and $T^{-1}_\lambda$ depend on $\lambda$, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all $\lambda$ in the complex plane such that $T^{-1}_\lambda$ exists. Boundedness of $T^{-1}_\lambda$ is another property that will be essential. We shall also ask for what $\lambda$ the domain of $T^{-1}_\lambda$ is dense in $X$, to name just a few aspects. For our investigation of $T$, $T_\lambda$ and $T^{-1}_\lambda$, we shall need some basic concepts in spectral theory which are given as follows (see [13, pp. 370–371]):

**Definition 1.1.** Let $X \neq \emptyset$ be a complex normed space and $T : \mathcal{D}(T) \longrightarrow X$, be a linear operator with domain $\mathcal{D} \subseteq X$. A regular value of $T$ is a complex number $\lambda$ such that

- $(R1)$ $T^{-1}_\lambda$ exists,
- $(R2)$ $T^{-1}_\lambda$ is bounded,
- $(R3)$ $T^{-1}_\lambda$ is defined on a set which is dense in $X$. 

The resolvent set \( \rho(T, X) \) of \( T \) is the set of all regular values \( \lambda \) of \( T \). Its complement \( \sigma(T, X) = \mathbb{C} - \rho(T, X) \) in the complex plane \( \mathbb{C} \) is called the spectrum of \( T \). Furthermore, the spectrum \( \sigma(T, X) \) is partitioned into three disjoint sets as follows:

The point spectrum \( \sigma_p(T, X) \) is the set of all \( \lambda \in \mathbb{C} \) such that \( T - \lambda I \) does not exist. The element of \( \sigma_p(T, X) \) is called eigenvalue of \( T \).

The continuous spectrum \( \sigma_c(T, X) \) is the set of all \( \lambda \in \mathbb{C} \) such that \( T - \lambda I \) exists and satisfies (R3) but not (R2), that is, \( T^{-1}_\lambda \) is unbounded.

The residual spectrum \( \sigma_r(T, X) \) is the set of all \( \lambda \in \mathbb{C} \) such that \( T - \lambda I \) exists but do not satisfy (R3), that is, the domain of \( T^{-1}_\lambda \) is not dense in \( X \). The condition (R2) may or may not holds good.

Goldberg's classification of operator \( T_\lambda = (T - \lambda I) \) (see [11, pp. 58–71]): Let \( X \) be a Banach space and \( T_\lambda = (T - \lambda I) \in B(X) \), where \( \lambda \in \mathbb{C} \). Again let \( R(T_\lambda) \) and \( T^{-1}_\lambda \) denote the range and inverse of the operator \( T_\lambda \), respectively. Then following possibilities may occur:

(A) \( R(T_\lambda) = X \),
(B) \( R(T_\lambda) \neq \overline{R(T_\lambda)} = X \),
(C) \( \overline{R(T_\lambda)} \neq X \),

and

(1) \( T_\lambda \) is injective and \( T^{-1}_\lambda \) is continuous,
(2) \( T_\lambda \) is injective and \( T^{-1}_\lambda \) is discontinuous,
(3) \( T_\lambda \) is not injective.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: \( A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 \) and \( C_3 \). If \( \lambda \) is a complex number such that \( T_\lambda \in A_1 \) or \( T_\lambda \in B_1 \), then \( \lambda \) is in the resolvent set \( \rho(T, X) \) of \( T \) on \( X \). The other classifications give rise to the fine spectrum of \( T \). We use \( \lambda \in B_2 \sigma(T, X) \) to denote that the operator \( T_\lambda \in B_2 \), i.e. \( R(T_\lambda) \neq \overline{R(T_\lambda)} = X \) and \( T_\lambda \) is injective but \( T^{-1}_\lambda \) is discontinuous. Similarly for the others.

**Lemma 1.2.** [11, p. 59] A linear operator \( T \) has a dense range if and only if the adjoint \( T^* \) is one to one.

**Lemma 1.3.** [11, p. 60] The adjoint operator \( T^* \) is onto if and only if \( T \) has a bounded inverse.

**Lemma 1.4.** The matrix \( A = (a_{nk}) \) gives rise to a bounded linear operator \( T \in B(c_0) \) from \( c_0 \) to itself if and only if

(1) the rows of \( A \) in \( \ell_1 \) and their \( \ell_1 \) norms are bounded.
(2) the columns of \( A \) are in \( c_0 \).

**Note:** The operator norm of \( T \) is the supremum of the \( \ell_1 \) norms of rows.

**Lemma 1.5.** The matrix \( A = (a_{nk}) \) gives rise to a bounded linear operator \( T \in B(\ell_1) \) from \( \ell_1 \) to itself if and only if the supremum of \( \ell_1 \) norms of the columns of \( A \) is bounded.
COROLLARY 1.6. [11, Corollary II.5.3] \( \sigma_r(T, X) \subseteq \sigma_p(T^*, X^*) \subseteq \sigma_r(T, X) \cup \sigma_p(T, X) \).

In this paper, we introduce a class of generalized upper triangular double-band matrices \( \Delta^{uv} \) over space \( \ell_1 \).

Let \( (u_k) \) be a sequence of positive real numbers such that \( u_k \neq 0 \) for each \( k \in \mathbb{N} \) with \( u = \lim_{k \to \infty} u_k \neq 0 \) and \( (v_k) \) is either constant or strictly decreasing sequence of positive real numbers with \( v = \lim_{k \to \infty} v_k \neq 0 \), and \( \sup_k v_k < u + v \).

We define the operator \( \Delta^{uv} \) on sequence space \( \ell_1 \) as follows:

\[
\Delta^{uv}x = \Delta^{uv}(x_n) = (v_n x_n + u_{n+1} x_{n+1})_{n=0}^{\infty}.
\]

It is easy to verify that the operator \( \Delta^{uv} \) can be represented by the matrix,

\[
\Delta^{uv} = \begin{bmatrix}
v_0 & u_1 & 0 & 0 & 0 & \cdots \\
0 & v_1 & u_2 & 0 & 0 & \cdots \\
0 & 0 & v_2 & u_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

2. Main results

In this section, we compute spectrum, the point spectrum, the continuous spectrum and the residual spectrum of the generalized upper double-band matrices \( \Delta^{uv} \) over the sequence space \( \ell_1 \).

THEOREM 2.1. The operator \( \Delta^{uv} : \ell_1 \to \ell_1 \) is a bounded linear operator and

\[
\|\Delta^{uv}\| = \sup_k (|v_k| + |u_{k+1}|).
\]

Proof. It is elementary. \( \blacksquare \)

If \( T : \ell_1 \to \ell_1 \) is a bounded linear operator with matrix \( A \), then it is known that the adjoint operator \( T^* : \ell_1^* \to \ell_1^* \) is defined by the transpose of the matrix \( A \). The dual space of \( \ell_1 \) is isomorphic to \( \ell_\infty \), the space of all bounded sequences, with the norm \( \|x\| = \sup_k |x_k| \).

We now give the theorem about the point spectrum of the dual operator \( (\Delta^{uv})^* \) of \( \Delta^{uv} \) over the space \( \ell_1^* \).

THEOREM 2.2. The point spectrum of the operator \( (\Delta^{uv})^* \) over \( \ell_1^* \) is

\[
\sigma_p((\Delta^{uv})^*, \ell_1^*) = \emptyset
\]

Proof. The proof of this theorem is divided into two cases.

Case (i): Suppose \( (v_k) \) is a constant sequence, say \( v_k = v \) for all \( k \). Consider \( (\Delta^{uv})^* f = \lambda f \), for \( f \neq 0 = (0, 0, 0, \ldots) \) in \( \ell_1^* \equiv \ell_\infty \), where

\[
(\Delta^{uv})^* = \begin{bmatrix}
v_0 & 0 & 0 & 0 & 0 & \cdots \\
u_1 & v_1 & 0 & 0 & 0 & \cdots \\
0 & u_2 & v_2 & 0 & 0 & \cdots \\
0 & 0 & u_2 & v_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

and \( f = \begin{bmatrix} f_0 \\
f_1 \\
f_2 \\
\vdots \end{bmatrix} \).
This gives
\begin{align*}
v_0f_0 &= \lambda f_0 \\
u_1f_0 + v_1f_1 &= \lambda f_1 \\
u_2f_1 + v_2f_2 &= \lambda f_2 \\
&\vdots \\
u_kf_{k-1} + v_kf_k &= \lambda f_k \\
&\vdots
\end{align*}

Let \(f_m\) be the first non-zero entry of the sequence \((f_n)\). So we get \(u_mf_{m-1} + vf_m = \lambda f_m\) which implies \(\lambda = v\) and from the equation \(u_{m+1}f_m + vf_{m+1} = \lambda f_{m+1}\) we get \(f_m = 0\), which is a contradiction to our assumption. Therefore,

\[\sigma_p((\Delta^{uv})^*, \ell^*_1) = \emptyset.\]

**Case (ii):** Suppose \((v_k)\) is a strictly decreasing sequence. Consider \((\Delta^{uv})^*f = \lambda f\), for \(f \neq 0 = (0,0,0,\ldots)\) in \(\ell^*_1 \cong \ell_\infty\), which gives above system of equations. Hence, for all \(\lambda \notin \{v_0, v_1, v_2, \ldots\}\), we have \(f_k = 0\) for all \(k\), which is a contradiction. So \(\lambda \notin \sigma_p((\Delta^{uv})^*, \ell^*_1)\). This shows that

\[\sigma_p((\Delta^{uv})^*, \ell^*_1) \subseteq \{v_0, v_1, v_2, \ldots\}.\]

Let \(\lambda = v_m\) for some \(m\). Then \(f_0 = f_1 = \cdots = f_{m-1} = 0\). Now if \(f_m = 0\), then \(f_k = 0\) for all \(k\), which is a contradiction. Also if \(f_m \neq 0\), then

\[f_{k+1} = \frac{u_{k+1}}{v_m - v_{k+1}} f_k, \quad \text{for all } k \geq m,
\]

and hence

\[
\lim_{k \to \infty} \left| \frac{f_{k+1}}{f_k} \right| = \lim_{k \to \infty} \left| \frac{u_{k+1}}{v_m - v_{k+1}} \right| = \left| \frac{u}{v_m - v} \right| > 1 \quad \text{for all } k \geq m,
\]

since \(v_m < v + u\). Then, \(f \notin \ell^*_1\). Thus

\[\sigma_p((\Delta^{uv})^*, \ell^*_1) = \emptyset.\]

**Theorem 2.3.** For any \(\lambda \in C\), \(\Delta^{uv}_\lambda : \ell_1 \to \ell_1\) has a dense range.

**Proof.** By Theorem 2.2, \(\sigma_p((\Delta^{uv})^*, \ell^*_1) = \emptyset\). Hence \((\Delta^{uv})^* - \lambda I\) is one to one for all \(\lambda\). By applying Lemma 1.2, we get the result. \(\blacksquare\)

**Corollary 2.4.** Residual spectrum \(\sigma_r(\Delta^{uv}, \ell_1)\) of operator \(\Delta^{uv}\) over \(\ell_1\) is

\[\sigma_r(\Delta^{uv}, \ell_1) = \emptyset\]

We define the operator \(\Delta_{uv}\) on sequence space \(c_0\) as follows:

\[\Delta_{uv}x = \Delta_{uv}(x_n) = (u_{n-2}x_{n-1} + v_nx_n)_{n=0}^{\infty} \quad \text{with } u_{-2} = u_{-1} = u_0 = 0\]
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It is easy to verify that the operator $\Delta^{uv}$ can be represented by the matrix,

$$
\Delta^{uv} = 
\begin{bmatrix}
v_0 & 0 & 0 & 0 & 0 & \cdots \\
u_1 & v_1 & 0 & 0 & 0 & \cdots \\
0 & u_2 & v_2 & 0 & 0 & \cdots \\
0 & 0 & u_2 & v_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

**Theorem 2.5.** $\sigma_p(\Delta^{uv}, c_0) = \emptyset$

**Proof.** The proof may be obtained by proceeding as in proving Theorem 2.2 so, we omit the details.

If $T : c_0 \rightarrow c_0$ is a bounded linear operator with matrix $A$, then it is known that the adjoint operator $T^* : c_0^* \rightarrow c_0^*$ is defined by the transpose of the matrix $A$. The dual space of $c_0$ is isomorphic to $\ell_1$.

**Theorem 2.6.** $\sigma_r(\Delta^{uv}, c_0) = \{ \lambda \in \mathbb{C} : |\lambda - v| < u \}$

**Proof.** We show that the operator $\Delta^{uv} - \lambda I$ has an inverse and $\mathcal{R}(\Delta^{uv} - \lambda I) \neq c_0$ for $\lambda$ satisfying $|\lambda - v| < u$. If $\lambda \in \{ \lambda \in \mathbb{C} : |\lambda - v| < u \}$, then the operator $\Delta^{uv} - \lambda I$ is a triangle except for $\lambda = v$ (when $(v_k)$ is a constant sequence) and $\lambda = v_k$, for some $k \in \mathbb{N}$ and consequently the operator $\Delta^{uv} - \lambda I$ has an inverse. Further by Theorem 2.5, the operator $\Delta^{uv} - \lambda I$ is one to one for $\lambda = v$ (when $(v_k)$ is a constant sequence) and $\lambda = v_k$, for some $k \in \mathbb{N}$ and hence has an inverse. Now, we show that if $\lambda \in \{ \lambda \in \mathbb{C} : |\lambda - v| < u \}$, then the operator $\Delta^{uv} - \lambda I$ is not one to one.

Suppose $\Delta^{uv}_* y = \lambda y$, for $y \neq 0 = (0, 0, 0, \ldots)$ in $\ell_1$, where $\Delta^{uv}_* = \Delta^{uv}$. This gives

$$
v_0y_0 + u_1y_1 = \lambda y_0 \\
v_1y_1 + u_2y_2 = \lambda y_1 \\
v_2y_2 + u_3y_3 = \lambda y_2 \\
\vdots \\
v_ky_k + u_{k+1}y_{k+1} = \lambda y_k \\
\vdots
$$

If $y_0 = 0$, then $y_k = 0$ for all $k$. Hence $y_0 \neq 0$ and solving the equation above, we get

$$
y_{k+1} = \left( \frac{\lambda - v_k}{u_{k+1}} \right) y_k \text{ for all } k \geq 0,
$$

and consequently

$$
\lim_{k \to \infty} \left| \frac{y_{k+1}}{y_k} \right| = \left| \frac{v - \lambda}{u} \right| < 1 \text{ provided } |v - \lambda| < u.
$$
Hence, $|v - \lambda| < u \Rightarrow y = (y_k) \in \ell_1$, which shows that $\Delta_{uv}^* - \lambda I$ is not one to one. Now Lemma 1.2 yields the fact that the range of the operator $\Delta_{uv} - \lambda I$ is not dense in $c_0$ and this step completes the proof. 

**Theorem 2.7.** $\sigma_p(\Delta_{uv}, \ell_1) = \{ \lambda \in \mathbb{C} : |\lambda - v| < u \}$.

**Proof.** This follows by Corollary 1.6 and Theorems 2.5–2.6.

**Theorem 2.8.** The spectrum of $\Delta_{uv}$ on $\ell_1$ is given by $\sigma(\Delta_{uv}, \ell_1) = \{ \lambda \in \mathbb{C} : |\lambda - v| \leq u \}$.

**Proof.** Let $f \in \ell_\infty$ and consider $(\Delta_{uv}^*)^* x = f$. Then we have the linear system of equations

\[
(v_0 - \lambda)x_0 = f_0 \\
u_1x_0 + (v_1 - \lambda)x_1 = f_1 \\
u_2x_1 + (v_2 - \lambda)x_2 = f_2 \\
\vdots \\
u_kx_{k-1} + (v_k - \lambda)x_k = f_k \\
\vdots
\]

Solving the equations, for $x = (x_k)$ in terms of $f$, we get

\[
x_0 = \frac{1}{v_0 - \lambda}, \text{ and } x_k = \frac{1}{v_k - \lambda} \sum_{i=0}^{k-1} \frac{u_{j+1}}{(\lambda - v_j)} f_i, \text{ for } k \geq 1.
\]

Then $|x_k| \leq S_k ||f||_\infty$, where

\[
S_k = \frac{1}{|v_k - \lambda|} + \frac{u_k}{|v_{k-1} - \lambda| |v_k - \lambda|} + \frac{u_{k-1} u_k}{u_k u_1 u_2 \cdots u_k} + \cdots + \frac{1}{|v_0 - \lambda| |v_1 - \lambda| |v_{k-1} - \lambda| |v_k - \lambda|}.
\]

Clearly each $S_k$ is finite. Now we prove that $\text{sup}_k S_k$ is finite. If $u < |\lambda - v|$, then $\lim_{n \to \infty} \frac{u_n}{|v_n - \lambda|} = \frac{u}{|v - \lambda|} = p < 1$. Then there exists $k \in \mathbb{N}$ such that $\frac{u_n}{|v_n - \lambda|} < p_0 < 1$, for all $n \geq k + 1$ and so we get

\[
S_{n+k} \leq \frac{1}{|v_{n+k} - \lambda|} \left( \frac{u_1 u_2 \cdots u_k}{|v_0 - \lambda| |v_1 - \lambda| |v_{k-1} - \lambda|} p_0^n \\
+ \frac{u_2 u_3 \cdots u_k}{|v_1 - \lambda| |v_2 - \lambda| |v_{k-1} - \lambda|} p_0^{n-1} + \cdots + p_0 + 1 \right).
\]

If we put $M = \max \left\{ \frac{u_1 u_2 \cdots u_k}{|v_{j-1} - \lambda| |v_j - \lambda| |v_{k-1} - \lambda|} : 1 \leq j \leq k \right\}$, then we have

\[
S_{n+k} \leq \frac{M}{|v_{n+k} - \lambda|} (1 + p_0 + p_0^2 + \cdots + p_0^n) \leq \frac{M}{|v_{n+k} - \lambda|} (1 + p_0 + p_0^2 + \cdots).
\]
But, for large $n$, we have $\frac{1}{v_{n+k-1}} < d < \frac{1}{u}$ and so $S_{n+k} \leq \frac{M}{d-\rho_0}$, for all $n \geq k+1$. Thus, $\sup_k S_k < \infty$. This shows that $\|x\|_\infty \leq \sup_k S_k \|f\|_\infty < \infty$. Therefore $x \in \ell_\infty$. Hence, for $u < |\lambda - v|$, $(\Delta^u_\lambda)^* \sigma$ is onto, and by Lemma 1.3, $\Delta^u_\lambda$ has a bounded inverse. This means that

$$\sigma_\ell(\Delta^u_\lambda, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| < u\}.$$ Combining this with Theorem 2.2 and Corollary 2.5, we get

$$\{\lambda \in C : |\lambda - v| < u\} \subseteq \sigma(\Delta^u_\lambda, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| \leq u\}.$$ Since the spectrum of any bounded operator is closed, we have

$$\sigma(\Delta^u_\lambda, \ell_1) = \{\lambda \in C : |\lambda - v| \leq u\}. \quad \blacksquare$$

**Theorem 2.9.** Continuous spectrum $\sigma_c(\Delta^u_\lambda, \ell_1)$ of operator $\Delta^u_\lambda$ over $\ell_1$ is

$$\sigma_c(\Delta^u_\lambda, \ell_1) = \{\lambda \in C : |\lambda - v| < u\}.$$ 

**Proof.** Since $\sigma_f(\Delta^u_\lambda, \ell_1) = \emptyset$, $\sigma_p(\Delta^u_\lambda, \ell_1) = \{\lambda \in C : |\lambda - v| < u\}$ and $\sigma(\Delta^u_\lambda, \ell_1)$ is the disjoint union of the parts $\sigma_p(\Delta^u_\lambda, \ell_1)$, $\sigma_c(\Delta^u_\lambda, \ell_1)$ and $\sigma_c(\Delta^u_\lambda, \ell_1)$, we deduce that

$$\sigma_c(\Delta^u_\lambda, \ell_1) = \{\lambda \in C : |\lambda - v| = u\}. \quad \blacksquare$$

**Theorem 2.10.** If $|\lambda - v| < u$, then $\lambda \in A_3 \sigma(\Delta^u_\lambda, \ell_1)$. 

**Proof.** Let $|\lambda - v| < u$. Then by Theorem 2.2, $\lambda \in (3)$; it remains to prove that $\Delta^u_\lambda$ is surjective when $|\lambda - v| < u$. Let $y = (y_0, y_1, y_2, \ldots) \in \ell_1$ and consider the equation $\Delta^u_\lambda x = y$. Then we have the linear system of equations

$$(v_0 - \lambda)x_0 + u_1 x_1 = y_0$$

$$(v_1 - \lambda)x_1 + u_2 x_2 = y_1$$

$$(v_2 - \lambda)x_2 + u_3 x_3 = y_2$$

$$\vdots$$

$$(v_k - \lambda)x_k + u_{k+1} x_{k+1} = y_k$$

Now, set $x_0 = 0$ and by solving these equations, we get $x_1 = \frac{1}{u_1} y_0$ and

$$x_k = \frac{1}{u_k} \left( \sum_{i=0}^{k-2} \prod_{j=i+1}^{k-1} \frac{\lambda - v_j}{u_j} \right) y_i + y_k$$

for all $k \geq 2$.

Then $\sum_k |x_k| \leq \sum_k |S_k| |y_k|$, where

$$S_k = \frac{1}{u_{k+1}} + \frac{1}{u_{k+2}} \frac{|v_{k+1} - \lambda|}{u_{k+1}} + \frac{1}{u_{k+3}} \frac{|v_{k+1} - \lambda|}{u_{k+1}} \frac{|v_{k+2} - \lambda|}{u_{k+2}} + \cdots$$ for all $k$. 

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Let
\[ S_{n,k} = \frac{1}{u_{k+1}} + \frac{1}{u_{k+2}} \frac{|v_{k+1} - \lambda|}{u_{k+1}} + \frac{1}{u_{k+3}} \frac{|v_{k+2} - \lambda|}{u_{k+2}} + \cdots + \frac{1}{u_{k+n+1}} \frac{|v_{k+n} - \lambda|}{u_{k+n}} \]
for all \( k, n \).

Then
\[ S_n = \lim_{k \to \infty} S_{n,k} = \frac{1}{u} + \frac{|v - \lambda|}{u^2} + \frac{|v - \lambda|^2}{u^3} + \cdots + \frac{|v - \lambda|^n}{u^{n+1}}. \]

Now for \(|\lambda - v| < u\), we can see that
\[ S = \lim_{n \to \infty} S_n = \frac{1}{u} + \frac{|v - \lambda|}{u^2} + \frac{|v - \lambda|^2}{u^3} + \cdots < \infty, \]
hence \((S_k)\) is a convergent sequence of positive real numbers with the limit \( S \). Therefore, \((S_k)\) is bounded and \( \sup_k S_k < \infty \). Thus \( \sum_k |x_k| \leq \sup_k S_k \sum_k |f_k| < \infty \). This shows that \( x \in \ell_1 \).

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**References**


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Fine spectra of $\Delta^uv$ over the space $\ell_1$  


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