ON CONVERGENCE OF $q$-CHLODOVSKY-TYPE MKZD OPERATORS

Harun Karsli and Vijay Gupta

Abstract. In the present paper, we define a new kind of MKZD operators for functions defined on $[0, b_n]$, named $q$-Chlodovsky-type MKZD operators, and give some approximation properties.

1. Introduction

For a function defined on the interval $[0, 1]$, the Meyer-König and Zeller operators $M_n(f, x)$ [10] are defined as

$$M_n(f; x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n+k}\right)$$

(1.1)

where $m_{n,k} = \binom{n+k}{k} x^k (1-x)^n$. In 1989 Guo [2] introduced the integrated Meyer-König and Zeller operators $\widetilde{M}_n$ by the means of the operators (1.1), to approximate Lebesgue integrable functions on the interval $[0, 1]$. Such operators have been defined as

$$\widetilde{M}_n(f; x) = \sum_{k=0}^{\infty} \tilde{m}_{n,k}(x) \int_{I_k} f(t) \, dt$$

(1.2)

where $I_k = \left[\frac{k}{n+k}, \frac{k+1}{n+k+1}\right]$ and $\tilde{m}_{n,k}(x) = (n+1)\binom{n+k+1}{k} x^k (1-x)^n$. Similar results may be also found in the papers [3, 4].

Recently, Karsli [8] defined the following MKZD operators for functions defined on $[0, b_n]$, named Chlodovsky-type MKZD operators as

$$L_n(f; x) = \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k}\left(\frac{x}{b_n}\right) \int_{0}^{b_n} f(t) b_{n,k}\left(\frac{t}{b_n}\right) \, dt, \quad 0 \leq x, \ t \leq b_n,$$

(1.3)

2010 AMS Subject Classification: 41A25, 41A36

Keywords and phrases: $q$-Chlodovsky-type MKZD operators; modulus of continuity; Peetre-K functional; Lipschitz space.
where \((b_n)\) is a positive increasing sequence with the properties
\[
\lim_{n \to \infty} b_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{n} = 0
\]
and \(b_{n,k}(t) = n\binom{n+k}{k} t^k (1-t)^{n-1}\). We now deal with the \(q\)-analogue of Chlodovsky-type MKZD operators \(L_{n,q}\), defined as
\[
L_{n,q}(f;x) = \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q} \left( \frac{x}{b_n} \right) \int_0^{b_n} q^{-k} f(t) b_{n,k,q} \left( \frac{qt}{b_n} \right) d_q t, \quad 0 \leq x \leq b_n,
\]
where
\[
m_{k,n,q}(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1} (1-q^s x)
\]
and
\[
b_{n,k,q}(t) = [n]_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-2} (1-q^s t) \quad (0 \leq t, x \leq 1),
\]
provided the \(q\)-integral and the infinite series on the r.h.s. of \((1.4)\) are well-defined.

It can be easily verified that in the case \(q = 1\) the operators defined by \((1.4)\) reduce to the Chlodovsky-type MKZD operators defined by \((1.3)\).

Actually the \(q\)-analogue of the linear positive operators was started in the last decade when Phillips [11] first introduced \(q\)-Bernstein polynomials, and later their Durrmeyer variants were studied and discussed in [5, 6]. Very recently Govil and Gupta [1] studied the approximation properties of \(q\)-MKZD operators. Here our aim is to study the \(q\)-analogue of summation-integral-type CMKZD operators. We shall prove that the operators \(L_{n,q}f\) being defined in \((1.4)\) converge to the limit \(f\).

Before getting onto the main subject, we first give definitions of \(q\)-integer, \(q\)-binomial coefficient and \(q\)-integral, which are required in this paper. For any fixed real number \(q > 0\) and non-negative integer \(r\) the \(q\)-integer of the number \(r\) is defined by
\[
[r]_q = \begin{cases} 
(1-q^r)/(1-q), & q \neq 1 \\
r, & q = 1.
\end{cases}
\]
The \(q\)-factorial is defined by
\[
[r]_q! = \begin{cases} 
[r]_q[r-1]_q \cdots [1]_q, & r = 1, 2, 3, \ldots \\
1, & r = 0.
\end{cases}
\]
and \(q\)-binomial coefficient is defined as
\[
\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q ![n-r]_q!},
\]
for integers \(n \geq r \geq 0\). The \(q\)-integral is defined as (see [9])
\[
\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n
\]
provided the sum converges absolutely. Note that the series on the right-hand side is guaranteed to be absolutely convergent as the function \( f \) is such that, for some \( M > 0, \alpha > -1, |f(x)| < Mx^\alpha \) in a right neighbourhood of \( x = 0 \).

**Definition 1.1.** A function \( f \) is \( q \)-integrable on \([0, \infty)\) if the series

\[
\int_0^\infty f(x) \, dq \, x = (1 - q) \sum_{n \in \mathbb{Z}} f(q^n) q^n
\]

converges absolutely. We use the notation

\[
(a - b)_q^n = \prod_{j=0}^{n-1} (a - q^j b).
\]

The \( q \)-analogue of Beta function (see [7]) is defined as

\[
B_q(m, n) = \int_0^1 t^{m-1} (1 - qt)^{n-1} \, dq, \quad m, n > 0.
\]

Also

\[
B_q(m, n) = [m-1]![n-1]! \frac{[m+n-1]!}{[m+n-1]!}.
\]

**2. Auxiliary results**

In this section we give certain results, which are necessary to prove our main theorem.

**Lemma 2.1.** For \( s \in \mathbb{N} \),

\[
(L_{n,q}t^s)(x) = b_n^s \sum_{k=0}^{\infty} m_{n,k,q} \left( \frac{x}{b_n} \right) \frac{[n+k]_q!}{[k]_q!} \frac{[k+s]_q!}{[k+n]_q!}.
\]  

**Proof.** We have

\[
(L_{n,q}t^s)(x) = \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q} \left( \frac{x}{b_n} \right) \int_0^{b_n} q^{-k} t^s b_{n,k,q} \left( \frac{qt}{b_n} \right) \, dq \, t
\]

\[
= \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q} \left( \frac{x}{b_n} \right) \int_0^{b_n} t^s \left[ \frac{n+k-1}{k} \right]_q \left( \frac{t}{b_n} \right)^k \left( 1 - qt \right)^{n-1} \, dq \, t.
\]

Setting \( u = t/b_n \), we get

\[
(L_{n,q}t^s)(x) = \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q} \left( \frac{x}{b_n} \right) b_n^{s+1} \left[ \frac{n+k-1}{k} \right]_q \int_0^1 u^{k+s}(1 - qu)^{n-1} \, du
\]

\[
= \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q} \left( \frac{x}{b_n} \right) b_n^{s+1} \left[ \frac{n+k-1}{k} \right]_q B_q(k + s + 1, n)
\]
\[
= \sum_{k=0}^{\infty} k_{n,k,q} \left( \frac{x}{b_n} \right)^{n+k-1} \frac{\Gamma_q(k+s+1) \Gamma_q(n)}{(n-1)_q! [k]_q!} \sum_{k=0}^{\infty} m_{n,k,q} \left( \frac{x}{b_n} \right)^{n+k-1} \frac{[k+s]_q!}{[k+n+s]_q!}. \]

For \( s = 0,1 \) and 2 in (2.1), we get respectively

\[
(L_{n,q}) (x) = \sum_{k=0}^{\infty} m_{n,k,q} \left( \frac{x}{b_n} \right)^{n+k-1} \frac{\prod_{s=0}^{k-1} (1 - q^s \frac{x}{b_n})}{(n+k-1)_q!} \frac{[k]_q!}{[k+n+k+1]_q!} = 1,
\]

since

\[
\prod_{s=0}^{k-1} \frac{1}{\left( 1 - q^s \frac{x}{b_n} \right)} = \sum_{k=0}^{\infty} \left( \frac{x}{b_n} \right)^{n+k-1} \frac{[k]_q!}{[k+n+k+1]_q!}.
\]

(2.2)

\[
(L_{n,q}) (x) = b_n \sum_{k=0}^{\infty} m_{n,k,q} \left( \frac{x}{b_n} \right)^{n+k-1} \frac{\prod_{s=0}^{k-1} (1 - q^s \frac{x}{b_n})}{(n+k-1)_q! [k]_q!} \frac{[k]_q!}{[k+n+k+1]_q!} \frac{[k+n]_q}{[k+n+k]_q} \frac{[n+k]_q}{[n+k+1]_q} = 1.
\]

(2.3)
From (2.2), (2.3) and (2.4), an easy computation gives

\[ (L_{n,q}t^2)(x) = b_n^2 \sum_{k=0}^{\infty} m_{n,k,q} \left( \frac{x}{b_n} \right) \frac{[n+k]_q!}{k!} \frac{[k+2]_q!}{[k+2+n]_q!} \]

\[ = b_n^2 \prod_{s=0}^{n-1} \left( 1-q^s \frac{x}{b_n} \right) \sum_{k=0}^{\infty} \frac{[n+k-1]_q!}{[n-1]_q! [k]_q!} \left( \frac{x}{b_n} \right)^{k} \frac{[k+2]_q [k+1]_q}{[k+2+n]_q [k+1+n]_q} \]

\[ = b_n^2 \prod_{s=0}^{n-1} \left( 1-q^s \frac{x}{b_n} \right) \sum_{k=0}^{\infty} \frac{[n+k-1]_q!}{[n-1]_q! [k]_q!} \left( \frac{x}{b_n} \right)^{k} \frac{1+q+q[k]_q+2q^2[k]_q+q^3[k]^2_q}{[k+2+n]_q [k+1+n]_q} \]

\[ \leq b_n^2 \prod_{s=0}^{n-1} \left( 1-q^s \frac{x}{b_n} \right) \frac{1}{[n-1]_q!} \times \]

\[ \sum_{k=0}^{\infty} \frac{[n+k-3]_q!}{[k]_q!} \left( \frac{x}{b_n} \right)^{k} \left( 1+q+q[k]_q+2q^2[k]_q+q^3[k]^2_q \right) \]

\[ = (1+q) b_n^2 \prod_{s=0}^{n-1} \left( 1-q^s \frac{x}{b_n} \right) \frac{1}{[n-1]_q[n-2]_q} \sum_{k=0}^{\infty} \frac{[n+k-3]_q!}{[n-3]_q! [k]_q!} \left( \frac{x}{b_n} \right)^{k} \]

\[ + (q+2q^2) b_n^2 \prod_{s=0}^{n-1} \left( 1-q^s \frac{x}{b_n} \right) \frac{1}{[n-1]_q[n-2]_q} \sum_{k=0}^{\infty} \frac{[n+k-2]_q!}{[n-2]_q! [k]_q!} \left( \frac{x}{b_n} \right)^{k+1} \]

\[ + q^3b_n^2 \prod_{s=0}^{n-1} \left( 1-q^s \frac{x}{b_n} \right) \frac{1}{[n-1]_q[n-2]_q} \sum_{k=0}^{\infty} \frac{[n+k-3]_q!}{[k]_q!} \left( \frac{x}{b_n} \right)^{k} \]

\[ = (1+q) b_n^2 \frac{1}{[n-1]_q[n-2]_q} + (q+2q^2) b_n^2 \frac{1}{[n-1]_q} \frac{x}{b_n} \]

\[ + q^3b_n^2 \prod_{s=0}^{n-1} \left( 1-q^s \frac{x}{b_n} \right) \frac{1}{[n-1]_q[n-2]_q} \sum_{k=0}^{\infty} \frac{[n+k-2]_q!}{[k]_q!} \left( \frac{x}{b_n} \right)^{k+1} \]

\[ + q^4b_n^2 \prod_{s=0}^{n-1} \left( 1-q^s \frac{x}{b_n} \right) \frac{1}{[n-1]_q[n-2]_q} \sum_{k=0}^{\infty} \frac{[n+k-1]_q!}{[k]_q!} \left( \frac{x}{b_n} \right)^{k+2} \]

\[ = \frac{(1+q) b_n^2}{[n-1]_q[n-2]_q} + (q+2q^2+q^3) \frac{b_n}{[n-1]_q} x + q^4x^2. \]

(2.4)

From (2.2), (2.3) and (2.4), an easy computation gives

\[ (L_{n,q}(t-x)^2)(x) \leq \frac{(1+q) b_n^2}{[n-1]_q[n-2]_q} + \frac{(q+2q^2+q^3) b_n}{[n-1]_q} x \]
functions.

Now, if we choose \( \omega \) where

\[
\lim_{n \to \infty} b_n = 0.
\]

### 3. Main results

Now we are ready to obtain some convergence results on \( q \)-CMKZD operators.

**Theorem 3.1.** Let \( (q_n) \) be a sequence of real numbers such that \( 0 < q_n < 1 \) and \( \lim_{n \to \infty} q_n = 1 \). If \( f \in C[0, \infty) \), we have

\[
| (L_{n,q_n} f)(x) - f(x) | \leq 2\omega(f, \sqrt{A_{n,q_n}(x)}),
\]

where \( \omega(f, \cdot) \) is the usual modulus of continuity of \( f \) in the space of continuous functions.

**Proof.** Using (1.4) for \( q = q_n \), we have

\[
| (L_{n,q_n} f)(x) - f(x) | = \left| \sum_{k=0}^{\infty} \frac{[n+k]}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) \int_0^{b_n} q_n^{-k} f(t) b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) d_{q_n} t - f(x) \right|
\]

\[
\leq \sum_{k=0}^{\infty} \frac{[n+k]}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) \int_0^{b_n} q_n^{-k} |f(t) - f(x)| b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) d_{q_n} t
\]

\[
\leq \sum_{k=0}^{\infty} \frac{[n+k]}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) \int_0^{b_n} q_n^{-k} \left( \frac{|t-x|}{\delta} + 1 \right) \omega(f, \delta) b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) d_{q_n} t
\]

\[
= \omega(f, \delta) \sum_{k=0}^{\infty} \frac{[n+k]}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) \int_0^{b_n} q_n^{-k} b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) d_{q_n} t
\]

\[
+ \frac{\omega(f, \delta)}{\delta} \sum_{k=0}^{\infty} \frac{[n+k]}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) \int_0^{b_n} q_n^{-k} \left( \frac{|t-x|}{\delta} \right) b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) d_{q_n} t
\]

\[
\leq \omega(f, \delta) + \frac{\omega(f, \delta)}{\delta} \left( (L_{n,q_n} (t-x)^2)(x) \right)^{1/2}
\]

\[
\leq \omega(f, \delta) + \frac{\omega(f, \delta)}{\delta} \left( A_{n,q_n}(x) \right)^{1/2}
\]

Now, if we choose \( \delta^2 = A_{n,q_n}(x) \), we get

\[
| (L_{n,q_n} f)(x) - f(x) | \leq 2\omega(f, \sqrt{A_{n,q_n}(x)}),
\]

and the proof of Theorem 3.1 is thus complete. \( \square \)
It is easy to see that, the right-hand side of formula (3.1) can diverge. Indeed, for \( x = \frac{b}{2} \) we cannot guarantee \( \delta \to 0 \) as \( n \to \infty \).

From Lemma 2.1 and Theorem 3.1, we can immediately give the following Bohman-Korovkin-type theorem.

**Theorem 3.2.** Let \((q_n)\) be a sequence of real numbers such that \(0 < q_n < 1\) and \(\lim_{n \to \infty} q_n = 1\). Then, for \(f \in C[0, \infty)\), the sequence \(L_n,q_n(f, x)\) converges uniformly to \(f(x)\) on any closed finite subinterval \([0, A]\), where \(A > 0\) being a constant.

**Definition 3.3.** For \(f \in C[a, b]\) and \(t > 0\), the Peetre-K Functional are defined by

\[
K(f, \delta) := \inf_{g \in C^2[a, b]} \left\{ \|f - g\|_{C[a, b]} + t \|g\|_{C^2[a, b]} \right\}.
\]

**Theorem 3.4.** If \(g \in C^2[0, A]\), then

\[
| (L_n,q g)(x) - g(x) | \leq A_{n,q}(x) \|g\|_{C^2[0, A]},
\]

where \(A > 0\) is a constant.

**Proof.** By Taylor formula with integral remainder term, we write

\[
g(t) = g(x) + (t - x)g'(x) + \int_0^{t-x} (t - x - u)^2 g''(x + u) \, du. \tag{3.2}
\]

If we apply the operator (1.4) to (3.2), we get

\[
| (L_n,q g)(x) - g(x) |
\]

\[
= \left| g'(x)(L_{n,q}(t-x))(x) + \left( L_{n,q} \left( \int_0^{t-x} (t - x - u)^2 g''(x + u) \, du \right) \right)(x) \right|
\]

\[
\leq \|g'\|_{C[0, A]} \|L_{n,q}(t-x)(x)\|
\]

\[
+ \|g''\|_{C[0, A]} \left| L_{n,q} \left( \int_0^{t-x} (t - x - u)^2 \, du \right) \right|(x).
\]

Since

\[
\int_0^{t-x} (t - x - u)^2 \, du = \frac{(t-x)^2}{2},
\]

one gets from (2.5)

\[
| (L_n,q g)(x) - g(x) | \leq \|g'\|_{C[0, A]} \{A_{n,q}(x)\}^{1/2} + \|g''\|_{C[0, A]} A_{n,q}(x).
\]

Now noting that

\[
\|g\|_{C^2[a, b]} = \|g\|_{C[a, b]} + \|g'\|_{C[a, b]} + \|g''\|_{C[a, b]},
\]

we get

\[
| (L_n,q g)(x) - g(x) | \leq A_{n,q}(x) \|g\|_{C^2[0, A]}. 
\]
and this completes the proof of Theorem 3.4. ■

Now, we are ready to prove the following theorem.

**Theorem 3.5.** Let \( (q_n) \) be a sequence of real numbers such that \( 0 < q_n < 1 \) and \( \lim_{n \to \infty} q_n = 1 \). If \( f \in C([0, \infty), \mathbb{R}) \), then

\[
\| (L_{n,q_n} f) - f \|_{C[0,A]} \leq 2K(f, B_{n,q_n}),
\]

where \( B_{n,q_n} \) is the maximum value of \( A_{n,q_n}(x) \) on \([0, A], A > 0\) is a constant; namely,

\[
B_{n,q} = \frac{(1 + q) b_n^2}{\frac{n}{q(n-2)}} + \frac{(q + 2q^2 + q^3) b_n}{\left[ n - 1 \right]_q} A + \left[ q^4 - 2 \frac{n-1}{n+1} q + 1 \right] A^2.
\]

**Proof.** By the linearity property of \( (L_{n,q_n}) \), we get

\[
|(L_{n,q_n} f)(x) - f(x)| = |(L_{n,q_n} f)(x) - (L_{n,q_n} g)(x) + (L_{n,q_n} g)(x) - g(x) + |g(x) - f(x)|
\]

\[
\leq \| f - g \|_{C[0,A]} \| (L_{n,q_n} - 1)(x) \| + \| f - g \|_{C[0,A]} + |(L_{n,q_n} g)(x) - g(x)|.
\]

From Theorem 3.4, one has

\[
\| (L_{n,q_n} f)(x) - f(x) \| \leq 2 \| f - g \|_{C[0,A]} A_{n,q_n}(x) \| g \|_{C^2[0,A]},
\]

and hence

\[
\| (L_{n,q_n} f) - f \|_{C[0,A]} \leq 2 \| f - g \|_{C[0,A]} + B_{n,q_n} \| g \|_{C^2[0,A]}.
\]

(3.3)

If we take the infimum on the right-hand side of (3.3) over all \( g \in C^2[0, A] \), we get

\[
\| (L_{n,q_n} f) - f \|_{C[0,A]} \leq 2K(f, B_{n,q_n}).
\]

This completes the proof. ■

**Theorem 3.6.** Let \( (q_n) \) be a sequence of real numbers such that \( 0 < q_n < 1 \) and \( \lim_{n \to \infty} q_n = 1 \). If \( f \in \text{Lip}^q_{\infty}[0, \infty) \), then for any \( A > 0 \) and \( x \in [0, A] \) the inequality

\[
|(L_{n,q_n} f)(x) - f(x)| \leq M \{ B_{n,q_n} \}^{\frac{2}{q}}
\]

holds with the constant \( M \), which is independent of \( n \) and \( B_{n,q_n} \) is as defined in Theorem 3.5.

**Proof.** For convenience we write \( L_{n,q_n}(f; x) \) instead of \( (L_{n,q_n}) f(x) \). Note that

\[
|L_{n,q_n}(f; x) - f(x)| \leq L_{n,q_n}(\| f(t) - f(x) \|; x)
\]

\[
= \sum_{k=0}^{\infty} \frac{[n+k]_{q_n}}{b_n^{n,k,q_n}} m_{n,k,q_n} \left( \frac{x}{b_n} \right) \int_0^{b_n} q_n^k |f(t) - f(x)| b_{n,k,q_n} \left( \frac{q_bt}{b_n} \right) d_{q_n} t
\]
By Hölder inequality, we have

\[ \leq M \int_0^{b_n} q_n^{-k} |t - x|^{\alpha} \sum_{k=0}^{\infty} \left[ \frac{n+k}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) \right] d_{q_n} t. \]

If we choose \( p_1 = \frac{2}{\alpha} \) and \( p_2 = \frac{2}{2-\alpha} \), then \( \frac{1}{p_1} + \frac{1}{p_2} = 1 \). Therefore

\[ |L_{n,q_n}(f;x) - f(x)| \]

\[ \leq M \int_0^{b_n} \left( |t - x|^2 \sum_{k=0}^{\infty} \left[ \frac{n+k}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) \right] \right)^{\frac{1}{p_1}} \times \]

\[ \times \left\{ q_n^{-k} \sum_{k=0}^{\infty} \left[ \frac{n+k}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) \right] \right\}^{\frac{1}{p_2}} \]

By Hölder inequality, we have

\[ |L_{n,q_n}(f;x) - f(x)| \]

\[ \leq M \left\{ \int_0^{b_n} q_n^{-k} |t - x|^2 \sum_{k=0}^{\infty} \left[ \frac{n+k}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) d_{q_n} t \right] \right\}^{\frac{1}{p_1}} \times \]

\[ \times \left\{ \int_0^{b_n} q_n^{-k} \sum_{k=0}^{\infty} \left[ \frac{n+k}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) \right] \right\}^{\frac{1}{p_2}} \]

\[ = M \left\{ \int_0^{b_n} q_n^{-k} |t - x|^2 \sum_{k=0}^{\infty} \left[ \frac{n+k}{b_n} m_{n,k,q_n} \left( \frac{x}{b_n} \right) b_{n,k,q_n} \left( \frac{q_n t}{b_n} \right) \right] \right\}^{\frac{2}{p_1}}. \]

From (2.5) we obtain

\[ |L_{n,q_n}(f;x) - f(x)| \leq M \left\{ A_{n,q_n}(x) \right\}^{\frac{2}{p_1}}. \]

This implies that for \( x \in [0, A] \)

\[ |(L_{n,q_n} f)(x) - f(x)| \leq M \left\{ B_{n,q_n} \right\}^{\frac{2}{p_1}} \]

which in view of (2.5) and (2.6) tends to zero as \( n \to \infty \).  

**ACKNOWLEDGEMENT.** The authors are thankful to the referees for their valuable remarks and suggestions.

**REFERENCES**


(Received 16.06.2011; in revised form 06.02.2012; available online 15.03.2012)

Abant Izzet Baysal University, Faculty of Science and Arts, Department of Mathematics, 14280 Golkoy Bolu
E-mail: karsli_h@ibu.edu.tr

School of Applied Sciences, Netaji Subhas Institute of Technology, Sector 3 Dwarka New Delhi-110075, India
E-mail: vijaygupta2001@hotmail.com