THE EULER THEOREM AND DUPIN INDICATRIX FOR SURFACES AT A CONSTANT DISTANCE FROM EDGE OF REGRESSION ON A SURFACE IN $E^3_1$

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Abstract. In this paper we give the Euler theorem and Dupin indicatrix for surfaces at a constant distance from edge of regression on a surface in $E^3_1$.

1. Introduction

Let $k_1, k_2$ denote principal curvature functions and $e_1, e_2$ be principal directions of a surface $M$, respectively. Then the normal curvature $k_n(v_p)$ of $M$ in the direction $v_p = (\cos \theta)e_1 + (\sin \theta)e_2$ is

$$k_n(v_p) = k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$  

This equation is called Euler’s formulae (Leonhard Euler, 1707–1783). The generalized Euler theorem for hypersurfaces in Euclidean space $E^{n+1}$ can be found in [8]. In 1984, A. Kılıç and H.H. Hacisalıhoğlu gave the Euler theorem and Dupin indicatrix for parallel hypersurfaces in $E^n$ [12]. Also the Euler theorem and Dupin indicatrix are obtained for the parallel hypersurfaces in pseudo-Euclidean spaces $E^{n+1}_1$ and $E^{n+1}_\nu$ in the papers [4, 6, 7].

In 2005 H.H. Hacisalıhoğlu and Ö. Tarakçı introduced surfaces at a constant distance from edge of regression on a surface. These surfaces are a generalization of parallel surfaces in $E^3$. Because the authors took any vector instead of normal vector [15]. Euler theorem and Dupin indicatrix for these surfaces are given [2]. In 2010 we obtained the surfaces at a constant distance from edge of regression on a surface in $E^3_1$ [14].

In this paper we give the Euler theorem and Dupin indicatrix for surfaces at a constant distance from edge of regression on a surface in $E^3_1$.

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DEFINITION 1.1. [3, 9, 10, 11, 13] (i) Hyperbolic angle: Let \( x \) and \( y \) be timelike vectors in the same timecone of Minkowski space. Then there is a unique real number \( \theta \geq 0 \), called the hyperbolic angle between \( x \) and \( y \), such that
\[
\langle x, y \rangle = -\|x\| \|y\| \cosh \theta.
\]
(ii) Central angle: Let \( x \) and \( y \) be spacelike vectors in Minkowski space that span a timelike vector subspace. Then there is a unique real number \( \theta \geq 0 \), called the central angle between \( x \) and \( y \), such that
\[
\langle x, y \rangle = \|x\| \|y\| \cosh \theta.
\]
(iii) Spacelike angle: Let \( x \) and \( y \) be spacelike vectors in Minkowski space that span a spacelike vector subspace. Then there is a unique real number \( \theta \) between 0 and \( \pi \) called the spacelike angle between \( x \) and \( y \), such that
\[
\langle x, y \rangle = \|x\| \|y\| \sin \theta.
\]
(iv) Lorentzian timelike angle: Let \( x \) be a spacelike vector and \( y \) be a timelike vector in Minkowski space. Then there is a unique real number \( \theta \geq 0 \), called the Lorentzian timelike angle between \( x \) and \( y \), such that
\[
\langle x, y \rangle = \|x\| \|y\| \cos \theta.
\]

DEFINITION 1.2. Let \( M \) and \( M' \) be two surfaces in \( E_1^3 \) and \( N_p \) be a unit normal vector of \( M \) at the point \( P \in M \). Let \( T_p(M) \) be the tangent space at \( P \in M \) and \( \{X_p, Y_p\} \) be an orthonormal bases of \( T_p(M) \). Let \( Z_p = d_1X_p + d_2Y_p + d_3N_p \) be a unit vector, where \( d_1, d_2, d_3 \in R \) are constant numbers and \( \varepsilon_1d_1^2 + \varepsilon_2d_2^2 - \varepsilon_3d_3^3 = \pm 1 \). If a function \( f \) exists and satisfies the condition \( f : M \to M' \), \( f(P) = P + rZ_p \), \( r \) constant, \( M' \) is called as the surface at a constant distance from the edge of regression on \( M \) and \( M' \) denoted by the pair \((M, M')\).

If \( d_1 = d_2 = 0 \), then we have \( Z_p = N_p \) and \( f(P) = P + rN_p \). In this case \( M \) and \( M' \) are parallel surfaces [14].

THEOREM 1.3. [14] Let the pair \((M, M')\) be given in \( E_1^3 \). For any \( W \in \chi(M) \), we have \( f_\ast(W) = \overline{W} + rD_\overline{W}Z \), where \( \overline{W} = \sum_{i=1}^{3} w_i \frac{\partial}{\partial x_i} \), \( \overline{W} = \sum_{i=1}^{3} \overline{w_i} \frac{\partial}{\partial x_i} \) and \( \forall P \in M, w_i(P) = \overline{w_i}(f(p)), 1 \leq i \leq 3 \).

Let \((\phi, U)\) be a parametrization of \( M \), so we can write that
\[
\phi : U_{(u,v)} \subset E_1^3 \to M_{P=\phi(u,v)}.
\]
In this case \( \{\phi_u|_p, \phi_v|_p\} \) is a basis of \( T_M(P) \). Let \( N_p \) is a unit normal vector at \( P \in M \) and \( d_1, d_2, d_3 \in R \) be a constant numbers then we may write that \( Z_p = d_1\phi_u|_p + d_2\phi_v|_p + d_3N_p \). Since \( M' = \{f(P) \mid f(P) = P + rZ_p\} \), a parametric representation of \( M' \) is \( \psi(u, v) = \phi(u, v) + rZ(u, v) \). Thus we may write
\[
M' = \{\psi(u, v) \mid \psi(u, v) = \phi(u, v) + r(d_1\phi_u(u, v) + d_2\phi_v(u, v) + d_3N(u, v)),
\]
\[
d_1, d_2, d_3, r \text{ are constant, } \varepsilon_1d_1^2 + \varepsilon_2d_2^2 - \varepsilon_3d_3^3 = \pm 1, \}
\]
If we take \( rd_1 = \lambda_1, \ rd_2 = \lambda_2, \ rd_3 = \lambda_3 \) then we have
\[
M^f = \{ \psi(u,v) | \psi(u,v) = \phi(u,v) + \lambda_1 \phi_u(u,v) + \lambda_2 \phi_v(u,v) + \lambda_3 N(u,v), \ \\
\lambda_1, \lambda_2, \lambda_3 \text{ are constant} \}.
\]

Let \( \{ \phi_u, \phi_v \} \) is basis of \( \chi(M^f) \). If we take \( \langle \phi_u, \phi_u \rangle = \varepsilon_1, \langle \phi_v, \phi_u \rangle = \varepsilon_2 \) and \( \langle N, N \rangle = -\varepsilon_1 \varepsilon_2 \), then
\[
\psi_u = (1 + \lambda_3 k_1) \phi_u + \varepsilon_2 \lambda_1 k_1 N, \\
\psi_v = (1 + \lambda_3 k_2) \phi_v + \varepsilon_1 \lambda_2 k_2 N
\]
is a basis of \( \chi(M^f) \), where \( N \) is unit normal vector field on \( M \) and \( k_1, k_2 \) are principal of \( M \) \[14\].

**Theorem 1.4.** \[14\] Let the pair \( (M, M^f) \) be given. Let \( \{ \phi_u, \phi_v \} \) (orthonormal and principal vector fields on \( M \)) be basis of \( \chi(M) \) and \( k_1, k_2 \) be principal curvatures of \( M \). The matrix of the shape operator of \( M^f \) with respect to the basis
\[
\{ \psi_u = (1 + \lambda_3 k_1) \phi_u + \varepsilon_2 \lambda_1 k_1 N, \ \\
\psi_v = (1 + \lambda_3 k_2) \phi_v + \varepsilon_1 \lambda_2 k_2 N \}
\]
of \( \chi(M^f) \) is
\[
S^f = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix}
\]
where
\[
\mu_1 = \frac{(1 + \lambda_3 k_2)}{A^3} \left( \epsilon_1 \lambda_1 \frac{\partial k_1}{\partial u} (\lambda_2^2 k_2^2 - \epsilon_1 (1 + \lambda_3 k_2)^2) + k_1 A^2 \right) \\
\mu_2 = \frac{\epsilon_1 \lambda_1^2 \lambda_2 k_1 k_2 (1 + \lambda_3 k_2) \partial k_1}{A^3} \\
\mu_3 = \frac{-\epsilon_1 \lambda_1 \lambda_2^2 k_1 k_2 (1 + \lambda_3 k_1) \partial k_2}{A^3} \\
\mu_4 = \frac{(1 + \lambda_3 k_1)}{A^3} \left( -\epsilon_2 \lambda_2 \frac{\partial k_2}{\partial v} (\lambda_1^2 k_1^2 - \epsilon_2 (1 + \lambda_3 k_1)^2) + k_2 A^2 \right)
\]
and \( A = \sqrt{\epsilon_1 (1 + \lambda_3 k_2)^2 + \epsilon_2 \lambda_2^2 k_2 (1 + \lambda_3 k_1)^2 - \epsilon_1 \epsilon_2 (1 + \lambda_3 k_1)^2 (1 + \lambda_3 k_2)^2} \).

**Definition 1.5.** \[6\] Let \( M \) be a pseudo-Euclidean surface in \( E^3_1 \) and \( p \) is nonumbilic point in \( M \). A function \( k_n \) which is defined in the following form
\[
k_n : T_p M \to \mathbb{R}, \ 
k_n(X_p) = -\frac{1}{\|X_p\|^2} \langle S(X_p), X_p \rangle
\]
is called a normal curvature function of \( M \) at \( p \).

**Definition 1.6.** \[7\] Let \( M \) be a pseudo-Euclidean surface in \( E^3_1 \) and \( S \) be shape operator of \( M \). Then the Dupin indicatrix of \( M \) at the point \( p \) is
\[
D_p = \{ X_p \ | \ \langle S(X_p), X_p \rangle = \pm 1, \ X_p \in T_p M \}.
\]
2. The Euler theorem for surfaces at a constant distance from edge of regression on a surface in $E^3_1$

**Theorem 2.1.** Let $M^f$ be a surface at a constant distance from edge of regression on a $M$ in $E^3_1$. Let $k_1$ and $k_2$ denote principal curvature function of $M$ and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that $\phi_u$ and $\phi_v$ are principal directions on $M$. Let $Y_p \in T_p M$ and we denote the normal curvature by $k_n^f(f_*(Y_p))$ of $M^f$ in the direction $f_*(Y_p)$.

$$k_n^f(f_*(Y_p)) = \frac{\mu_1^s y_1^2 + \varepsilon_1 \varepsilon_2 \mu_2^s y_1 y_2 + \mu_3^s y_2^2}{|\lambda_1^s y_1^2 - 2 \varepsilon_1 \varepsilon_2 \lambda_2^s k_1 y_1 y_2 + \lambda_3^s y_2^2|} \tag{2.1}$$

where

$y_1 = \langle Y_p, \phi_u \rangle$, $y_2 = \langle Y_p, \phi_v \rangle$,

$\lambda_i^s = \varepsilon_i(1 + \lambda_3 k_i)^2 - \varepsilon_1 \varepsilon_2 \lambda_2^s k_i^2$, $(i = 1, 2),$

$\mu_i^s = \varepsilon_1 \mu_1^1 (1 + \lambda_3 k_1)^2 - \lambda_1 \varepsilon_1 \varepsilon_2 \mu_1 \lambda_1 k_1 + \mu_2 \lambda_2 k_2),$

$\mu_2^s = \varepsilon_1 \mu_2(1 + \lambda_3 k_2)^2 - \lambda_2 k_2 \mu_1 \lambda_1 k_1 + \varepsilon_1 \varepsilon_2 \mu_2 \lambda_2 k_2) + \varepsilon_1 \mu_3(1 + \lambda_3 k_1)^2 - \lambda_1 \varepsilon_1 \varepsilon_2 \mu_3 \lambda_1 k_1 + \mu_4 \lambda_2 k_2)$,

$\mu_3^s = \varepsilon_2 \mu_3(1 + \lambda_3 k_2)^2 - \lambda_2 k_2 \mu_3 \lambda_1 k_1 + \varepsilon_1 \varepsilon_2 \mu_4 \lambda_2 k_2)$.

**Proof.** Let $f_*(Y_p) \in T_{f(p)}M^f$. Then

$$k_n^f(f_*(Y_p)) = \frac{1}{\|f_*(Y_p)\|^2} \langle S^f(f_*(Y_p)), f_*(Y_p) \rangle \tag{2.3}$$

Let us calculate $f_*(Y_p)$ and $S^f(f_*(Y_p))$. Since $\phi_u$ and $\phi_v$ are orthonormal we have

$$Y_p = \varepsilon_1 \langle Y_p, \phi_u \rangle \phi_u + \varepsilon_2 \langle Y_p, \phi_v \rangle \phi_v = \varepsilon_1 y_1 \phi_u + \varepsilon_2 y_2 \phi_v$$

Further without lost of generality, we suppose that $Y_p$ is a unit vector. Then

$$f_*(Y_p) = \varepsilon_1 y_1 f_*(\phi_u) + \varepsilon_2 y_2 f_*(\phi_v) = \varepsilon_1 y_1 \psi_u + \varepsilon_2 y_2 \psi_v. \tag{2.4}$$

On the other hand we find that

$$S^f(f_*(Y_p)) = \varepsilon_1 y_1 S^f(\psi_u) + \varepsilon_2 y_2 S^f(\psi_v)$$

$$= \varepsilon_1 y_1 (\mu_1(1 + \lambda_3 k_1)\phi_u + \mu_2(1 + \lambda_3 k_2)\phi_v + (\mu_1 \varepsilon_2 \lambda_1 k_1 + \mu_2 \varepsilon_1 \lambda_2 k_2) N)$$

$$+ \varepsilon_2 y_2 (\mu_3(1 + \lambda_3 k_1)\phi_u + \mu_4(1 + \lambda_3 k_2)\phi_v + (\mu_3 \varepsilon_2 \lambda_1 k_1 + \mu_4 \varepsilon_1 \lambda_2 k_2) N) \tag{2.5}$$

Thus using equations (2.4) and (2.5) in equation (2.3) we obtain (2.1).

**Corollary 2.2.** Let $M^f$ be a surface at a constant distance from edge of regression on $M$ in $E^3_1$. Let $k_1$ and $k_2$ denote principal curvature function of $M$ and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that $\phi_u$ and $\phi_v$ are principal directions on $M$. Let us denote the angle between $Y_p \in T_p M$ and $\phi_u, \phi_v$ by $\theta_1$ and $\theta_2$ respectively. Thus the normal curvature of $M^f$ in the direction $f_*(Y_p)$
(a) Let $N_p$ be a timelike vector then
\[ k^f_n(f_*(Y_p)) = \frac{\mu_1^* \cos^2 \theta_1 + \mu_2^* \cos \theta_1 \cos \theta_2 + \mu_3^* \cos^2 \theta_2}{|\lambda_1^* \cos^2 \theta_1 + \lambda_2^* \cos^2 \theta_2 - 2\lambda_1 \lambda_2 k_1 k_2 \cos \theta_1 \cos \theta_2|} \]

(b) Let $N_p$ be a spacelike vector.

(b.1) If $Y_p$ and $\phi_u$ are timelike vectors in the same timecone then
\[ k^f_n(f_*(Y_p)) = \frac{\mu_1^* \cosh^2 \theta_1 + \delta_1 \mu_2^* \cosh \theta_1 \sinh \theta_2 + \mu_3^* \cosh^2 \theta_2}{|\lambda_1^* \cosh^2 \theta_1 + \lambda_2^* \sinh^2 \theta_2 - 2\delta_1 \lambda_1 \lambda_2 k_1 k_2 \cosh \theta_1 \sinh \theta_2|} \]

(b.2) If $Y_p$ and $\phi_v$ are timelike vectors in the same timecone then
\[ k^f_n(f_*(Y_p)) = \frac{\mu_1^* \sinh^2 \theta_1 + \mu_2^* \sinh \theta_1 \cosh \theta_2 + \mu_3^* \cosh^2 \theta_2}{|\lambda_1^* \sinh^2 \theta_1 + \lambda_2^* \cosh^2 \theta_2 - 2\lambda_1 \lambda_2 k_1 k_2 \sinh \theta_1 \cosh \theta_2|} \]

(b.3) If $Y_p \in T_pM$ is a spacelike vector and $\phi_u$ is timelike vector then
\[ k^f_n(f_*(Y_p)) = \frac{\mu_1^* \sinh^2 \theta_1 - \delta_1 \mu_2^* \sinh \theta_1 \cosh \theta_2 + \mu_3^* \cosh^2 \theta_2}{|\lambda_1^* \sinh^2 \theta_1 + \lambda_2^* \cosh^2 \theta_2 - 2\delta_1 \lambda_1 \lambda_2 k_1 k_2 \sinh \theta_1 \cosh \theta_2|} \]

(b.4) If $Y_p \in T_pM$ is a spacelike vector and $\phi_v$ is timelike vector then
\[ k^f_n(f_*(Y_p)) = \frac{\mu_1^* \cosh^2 \theta_1 - \delta_1 \mu_2^* \cosh \theta_1 \sinh \theta_2 + \mu_3^* \sinh^2 \theta_2}{|\lambda_1^* \cosh^2 \theta_1 + \lambda_2^* \sinh^2 \theta_2 - 2\delta_1 \lambda_1 \lambda_2 k_1 k_2 \cosh \theta_1 \sinh \theta_2|} \]

where $\lambda_1^*, \lambda_2^*, \mu_1^*, \mu_2^*$ and $\mu_3^*$ are given in (2.2) and $\delta_i$, $(i = 1, 2)$ is 1 or $-1$ depending on $y_i$ is positive or negative, respectively.

Proof. (a) Let $N_p$ be a timelike vector. In this case $\theta_1$ and $\theta_2$ are spacelike angle then
\[ y_1 = \langle Y_p, \phi_u \rangle = \cos \theta_1 \]
\[ y_2 = \langle Y_p, \phi_v \rangle = \cos \theta_2. \]

Substituting these equations in (2.1), we get $k^f_n(f_*(Y_p))$.

(b) Let $N_p$ be a spacelike vector.

(b.1) If $Y_p$ and $\phi_u$ are timelike vectors in the same timecone then there is a hyperbolic angle $\theta_1$ and a Lorentzian timelike angle $\theta_2$. Since
\[ y_1 = -\cosh \theta_1 \quad \text{and} \quad y_2 = \delta_2 \sinh \theta_2 \]
the proof is obvious.

(b.2) If $Y_p$ and $\phi_v$ are timelike vectors in the same timecone then there is a Lorentzian timelike angle $\theta_1$ and a hyperbolic angle $\theta_2$. Thus
\[ y_1 = \delta_1 \sinh \theta_1 \quad \text{and} \quad y_2 = -\cosh \theta_2. \]
(b.3) If \( Y_p \in T_pM \) is a spacelike vector and \( \phi_u \) is timelike vector then there is a Lorentzian timelike angle \( \theta_1 \) and a central angle \( \theta_2 \). Thus
\[
y_1 = \delta_1 \sinh \theta_1 \quad \text{and} \quad y_2 = \delta_2 \cosh \theta_2.
\]

(b.4) If \( Y_p \in T_pM \) is a spacelike vector and \( \phi_v \) is timelike vector then there is a central angle \( \theta_1 \) and a Lorentzian timelike angle \( \theta_2 \). Thus
\[
y_1 = \delta_1 \cosh \theta_1 \quad \text{and} \quad y_2 = \delta_2 \sinh \theta_2. \]

As a special case if we take \( \lambda_1 = \lambda_2 = 0, \lambda_3 = r = \text{constant} \), then we obtain that \( M \) and \( M^l \) are parallel surfaces. The following corollary is known the Euler theorem for parallel surfaces in \( E^3 \).

**Corollary 2.3.** Let \( M \) and \( M_r \) be parallel surfaces in \( E^3 \). Let \( k_1 \) and \( k_2 \) denote principal curvature function of \( M \) and let \( \{\phi_u, \phi_v\} \) be orthonormal basis such that \( \phi_u \) and \( \phi_v \) are principal directions on \( M \). Let \( Y_p \in T_pM \) and we denote the normal curvature by \( k_n^p(f_*(Y_p)) \) of \( M_r \), in the direction \( f_*(Y_p) \). Thus
\[
k_n^p(f_*(Y_p)) = \frac{\varepsilon_1 k_1(1 + rk_1)y_1^2 + \varepsilon_2 k_2(1 + rk_2)y_2^2}{|\varepsilon_1(1 + rk_1)^2y_1^2 + \varepsilon_2(1 + rk_2)^2y_2^2|}.
\]

**Proof.** Since
\[
\lambda_i^* = \varepsilon_i(1 + rk_i)^2, \quad (i = 1, 2),
\]
\[
\mu_i^* = \varepsilon_1 k_1(1 + nk_1),
\]
\[
\mu_3^* = 0,
\]
\[
\mu_3^* = \varepsilon_2 k_2(1 + rk_2),
\]
from (2.1) we find \( k_n^p(f_*(Y_p)) \).

**Corollary 2.4.** Let \( M \) and \( M_r \) be parallel surfaces in \( E^3 \). Let \( k_1 \) and \( k_2 \) denote principal curvature function of \( M \) and let \( \{\phi_u, \phi_v\} \) be orthonormal basis such that \( \phi_u \) and \( \phi_v \) are principal directions on \( M \). Let us denote the angle between \( Y_p \in T_pM \) and \( \phi_u, \phi_v \) by \( \theta_1 \) and \( \theta_2 \) respectively. Thus the normal curvature of \( M^l \) in the direction \( f_*(Y_p) \)
(a) Let \( N_p \) be a timelike vector then
\[
k_n^p(f_*(Y_p)) = \frac{k_1(1 + rk_1)\cos^2 \theta_1 + k_2(1 + rk_2)\cos^2 \theta_2}{(1 + rk_1)^2\cos^2 \theta_1 + (1 + rk_2)^2\cos^2 \theta_2}.
\]
(b) Let \( N_p \) be a spacelike vector.
(b.1) If \( Y_p \) and \( \phi_u \) are timelike vectors in the same timecone then
\[
k_n^p(f_*(Y_p)) = \frac{-k_1(1 + rk_1)\cosh^2 \theta_1 + k_2(1 + rk_2)\sinh^2 \theta_2}{(1 + rk_1)^2\cosh^2 \theta_1 - (1 + rk_2)^2\sinh^2 \theta_2}.
\]
(b.2) If \( Y_p \) and \( \phi_v \) are timelike vectors in the same timecone then

\[
k_n^r(f_*(Y_p)) = \frac{k_1(1 + rk_1)\sinh^2 \theta_1 - k_2(1 + rk_2)\cosh^2 \theta_2}{-(1 + rk_1)^2 \sinh^2 \theta_1 + (1 + rk_2)^2 \cosh^2 \theta_2}.
\]

(b.3) If \( Y_p \in T_p M \) is a spacelike vector and \( \phi_u \) is timelike vector then

\[
k_n^r(f_*(Y_p)) = \frac{-k_1(1 + rk_1)\sinh^2 \theta_1 + k_2(1 + rk_2)\cosh^2 \theta_2}{-(1 + rk_1)^2 \sinh^2 \theta_1 + (1 + rk_2)^2 \cosh^2 \theta_2}.
\]

(b.4) If \( Y_p \in T_p M \) is a spacelike vector and \( \phi_v \) is timelike vector then

\[
k_n^r(f_*(Y_p)) = \frac{k_1(1 + rk_1)\cosh^2 \theta_1 - k_2(1 + rk_2)\sinh^2 \theta_2}{(1 + rk_1)^2 \cosh^2 \theta_1 - (1 + rk_2)^2 \sinh^2 \theta_2}.
\]

3. The Dupin indicatrix for surfaces at a constant distance from edge of regression on surfaces in \( E_3^1 \)

**Theorem 3.1.** Let \( M^f \) be a surface at a constant distance from edge of regression on \( M \) in \( E_3^1 \). Let \( k_1 \) and \( k_2 \) denote principal curvature functions of \( M \) and \( \{\phi_u, \phi_v\} \) be orthonormal bases such that \( \phi_u \) and \( \phi_v \) are principal directions on \( M \). Thus

\[
D^f_{f(p)} = \{ f_*(Y_p) \in T_{f(p)}M^f | c_1y_1^2 + \varepsilon_1\varepsilon_2c_2y_1y_2 + c_3y_2^2 = \pm 1 \},
\]

where

\[
f_*(Y_p) = \varepsilon_1y_1(1 + \lambda_3k_1)\phi_u + \varepsilon_2y_2(1 + \lambda_3k_2)\phi_v + \varepsilon_1\varepsilon_2(y_1\lambda_1k_1 + y_2\lambda_2k_2)N
\]

\[
c_1 = \varepsilon_1\mu_1(1 + \lambda_3k_1)^2 - \lambda_1k_1(\varepsilon_1\varepsilon_2\mu_1\lambda_1k_1 + \mu_2\lambda_2k_2),
\]

\[
c_2 = \varepsilon_2\mu_2(1 + \lambda_3k_2)^2 - \lambda_2k_2(\mu_1\lambda_1k_1 + \varepsilon_1\varepsilon_2\mu_2\lambda_2k_2)
\]

\[
+ \varepsilon_1\mu_3(1 + \lambda_3k_1)^2 - \lambda_1k_1(\varepsilon_1\varepsilon_2\mu_3\lambda_1k_1 + \mu_4\lambda_2k_2),
\]

\[
c_3 = \varepsilon_2\mu_4(1 + \lambda_3k_2)^2 - \lambda_2k_2(\mu_3\lambda_1k_1 + \varepsilon_1\varepsilon_2\mu_4\lambda_2k_2).
\]

**Proof.** Let \( f_*(Y_p) \in T_{f(p)}M^f \). Since

\[
D^f_{f(p)} = \{ f_*(Y_p) | \langle S^f(f_*(Y_p)), f_*(Y_p) \rangle = \pm 1 \}
\]

the proof is clear. □

According to this theorem the Dupin indicatrix of \( M^f \) at the point \( f(p) \) in general will be a conic section of the following type:

**Corollary 3.2.** Let \( M^f \) be a surface at a constant distance from edge of regression on \( M \) in \( E_3^1 \). The Dupin indicatrix of \( M^f \) at the point \( f(p) \) is:

(a) an ellipse, if \( c_2^2 - 4c_1c_3 < 0 \),

(b) two conjugate hyperbolas, if \( c_2^2 - 4c_1c_3 > 0 \),

(c) parallel two lines, if \( c_2^2 - 4c_1c_3 = 0 \).
The Euler theorem and Dupin indicatrix

Corollary 3.3. Let $M$ and $M_r$ be parallel surfaces in $E^3_1$. Let $k_1$ and $k_2$ denote principal curvature functions of $M$ and $\{\phi_u, \phi_v\}$ be orthonormal bases such that $\phi_u$ and $\phi_v$ are principal directions on $M$. In this case

$$D_r f(p) = \{ f_s(Y_p) \in T_{f(p)}M_r \mid \varepsilon_1 k_1(1 + rk_1)y_1^2 + \varepsilon_2 k_2(1 + rk_2)y_2^2 = \pm 1 \}.$$ 

Hence the point $f(p)$ of $M_r$ is:

(a) an elliptic point, if $\varepsilon_1 \varepsilon_2 k_1 k_2 (1 + rk_1)(1 + rk_2) > 0$,
(b) a hyperbolic point, if $\varepsilon_1 \varepsilon_2 k_1 k_2 (1 + rk_1)(1 + rk_2) < 0$,
(c) a parabolic point, if $k_1 k_2 (1 + rk_1)(1 + rk_2) = 0$.

REFERENCES


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