ON RIGHT IDEALS AND DERIVATIONS
IN PRIME RINGS WITH ENGEL CONDITION

Basudeb Dhara and Deepankar Das

Abstract. Let $R$ be an associative ring with center $Z(R)$ and $d$ a nonzero derivation of $R$. The main object in this paper is to study the situation $[[d(x^n)x^s], [y, d(y)]_m]^n \in Z(R)$ for all $x, y$ in some appropriate subset of $R$, where $n \geq 0$, $s \geq 0$, $t \geq 0$, $m \geq 1$, $r \geq 1$ are fixed integers and $R$ is a prime or semiprime ring.

1. Introduction

Throughout this paper, unless specifically stated, $R$ denotes a prime ring with center $Z(R)$, with extended centroid $C$, and two-sided Martindale quotient ring $Q$. Given $x, y \in R$, we set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$ and inductively $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$. By $d$, we mean a derivation of $R$.

In [12], Herstein proved that if $\text{char} (R) \neq 2$ and a derivation $d$ is nonzero such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then $R$ is commutative. Chang and Lin [5] proved that if $\rho$ is a nonzero right ideal of $R$ such that $d(x^n)x^s$ for all $x \in \rho$, $n \geq 1$ a fixed integer, then $d(\rho)\rho = 0$. Recently, De Filippis [10] proved that if $\text{char} (R) \neq 2$ and $\rho$ a nonzero right ideal of $R$ such that $[d(x)x^n, d(y)] = 0$ for all $x, y \in \rho$, then either $R$ is commutative or $d(\rho)\rho = 0$. In another paper, De Filippis [11] proved that if $\text{char} (R) \neq 2$, $d$ is nonzero and $\rho$ is a nonzero right ideal of $R$ such that $[[d(x), x], [d(y), y]] = 0$ for all $x, y \in \rho$, then either $\rho \rho = 0$ or $d(\rho)\rho = 0$. In [8], the first author of this paper extended the result of De Filippis by considering Engel conditions. The result of [8] states that if $\text{char} (R) \neq 2$ and $\rho$ a non-zero right ideal of $R$ such that $[[d(x), x], [d(y), y]] = 0$ for all $x, y \in \rho$, where $n \geq 0, m \geq 0, t \geq 1$ are fixed integers and $[\rho, \rho] = 0$, then $d(\rho)\rho = 0$.

On the other hand, a well known result of Posner [22] states that if $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or $R$ is commutative. In [18], Lee considered any constant power values of $x$ and proved that if $R$ be a prime ring and $\lambda$ a nonzero left ideal of $R$ such that $[d(x^n), x^m] = 0$ for all $x \in \lambda$, then either $d = 0$
or $R$ is commutative. Lee and Shiue [20] proved that if $R$ is noncommutative and $\lambda$ a nonzero left ideal of $R$ then: (i) if $[d(x^n)x^r, x^s] = 0$ for all $x \in \lambda$, then $d = 0$, except when $R \cong M_2(GF(2))$; (ii) if $[x^n d(x^m), x^r] = 0$ for all $x \in \lambda$, then either $d = ad(b)$ with $\lambda b = 0$ for some $b \in Q$ or $\lambda \lambda, \lambda = 0$ and $d(\lambda) \subseteq \lambda C$.

From the results above, it is natural to consider the situation when $[d(x^n)x^r, x^s] = 0$ for all $x, y$ in some appropriate subset of $R$, where $n \geq 0, s \geq 0, t \geq 0, m \geq 1, r \geq 1$ are fixed integers. As a particular case, we obtain results, when $[x, d(x)] = 0$ for all $x$ in some right ideal of a prime ring $R$ or for all $x$ in a semiprime ring $R$.

Let $R$ be a prime ring and $Q$ its two-sided Martindale quotient ring. Then $Q$ is also a prime ring with center $C = Z(Q)$, a field, which is the extended centroid of $R$. It is well known that any derivation of $R$ can be uniquely extended to a derivation of $Q$, and hence any derivation of $R$ can be defined on the whole of $Q$. We refer to [2, 19] for more details.

Denote by $Q \ast_C C\{x, y, z\}$ the free product of the $C$-algebra $Q$ and $C\{x, y, z\}$, the free $C$-algebra in noncommuting indeterminates $x, y, z$.

2. The case: $R$ a prime ring

We need the following lemma.

**Lemma 2.1.** Let $I$ be a nonzero right ideal of $R$ and $d$ a derivation of $R$. Then the following conditions are equivalent: (i) $d$ is an inner derivation induced by some $b \in Q$ such that $bI = 0$; (ii) $d(I)I = 0$.

For its proof we refer to [13] or [4, Lemma].

**Theorem 2.2.** Let $R$ be a prime ring of char $(R) \neq 2$ and $d$ a non-zero derivation of $R$ such that $[[d(x^n)x^r, x^s], [y, d(y)]_t] = 0$ for all $x, y \in R$, where $n, s, t \geq 0$ and $m, r \geq 1$ are fixed integers, then $R$ is commutative.

*Proof.* Assume that $R$ is noncommutative, otherwise we are done. Assume next that $d$ is $Q$-inner derivation i.e., $d(x) = [a, x]$ for all $x \in R$ and for some $a \in Q$. Then we have

$$[[ax^n, x^r]_{s+1}, [y, [a, y]]_t] = 0$$

for all $x, y \in R$. Since $d \neq 0$, $a \notin C$ and hence $R$ satisfies a nontrivial generalized polynomial identity (GPI). Since $Q$ and $R$ satisfy the same generalized polynomial identities with coefficients in $Q$ (see [7]), $[[ax^n, x^r]_{s+1}, [y, [a, y]]_t]$ is also satisfied by $Q$. Since $Q$ is prime, we may replace $R$ by $Q$ and then assume that $a \in R$ and $C = Z(R)$. In this case $R$ is centrally closed (i.e. $RC = R$) prime $C$-algebra [9]. Then by Martindale’s theorem [21], $R$ is a primitive ring. By Jacobson’s theorem [15, p. 75] $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over a division ring $D$. Since $R$ is noncommutative, $\dim_D V \geq 2$. We assume that for some $v \in V$, $\{av, v\}$ is linearly $D$-independent. If $a^2 v \notin \text{span}_D \{v, av\}$,
then \(\{v, av, a^2v\}\) is linearly \(D\)-independent. By density there exist \(x, y \in R\) such that

\[
\begin{align*}
  xv &= v, \quad xav = 0, \quad xa^2v = 0; \\
yv &= 0, \quad yav = v, \quad ya^2v = 0
\end{align*}
\]

for which we have \([a, y]v = -v, [a, y]av = av, [ax^n, x^r]_{s+1}v = av\) and hence

\[
[y, [a, y]]_tv = \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, y]^j y[a, y]^{t-j}v = 0
\]

and

\[
[y, [a, y]]_tv = \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, y]^j y[a, y]^{t-j}av = \sum_{j=0}^{t} \binom{t}{j} v = 2^t v.
\]

Thus

\[
0 = [(ax^n, x^r)_{s+1}, [y, [a, y]]]v = [ax^n, x^r]_{s+1}v - [y, [a, y]]_t [ax^n, x^r]_{s+1}v = 0 - 2^t v = -2^t v
\]

and hence

\[
0 = [(ax^n, x^r)_{s+1}, [y, [a, y]]]v = -2^t v
\]

which is a contradiction, since \(\text{char } (R) \neq 2\).

If \(a^2v \in \text{span}_D \{v, av\}\), then \(a^2v = av + \beta av\) for some \(\alpha, \beta \in D\). Then again by density there exist \(x, y \in R\) such that \(xv = v, xav = 0; yv = 0, yav = v\) for which we get \([a, y]v = -v, [a, y]^n av = av\) or \(a^2v = -\beta v\) according as \(n\) is even or odd, \([ax^n, x^r]_{s+1}v = av\) and hence \([y, [a, y]]_tv = \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, y]^j y[a, y]^{t-j}v = 0\) and \([y, [a, y]]_tv = \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, y]^j y[a, y]^{t-j}av = \sum_{j=0}^{t} \binom{t}{j} v = 2^t v\). Therefore,

\[
[(ax^n, x^r)_{s+1}, [y, [a, y]]]v = -2^t v
\]

and hence

\[
0 = [(ax^n, x^r)_{s+1}, [y, [a, y]]]v = -2^t v
\]

which is a contradiction, since \(\text{char } (R) \neq 2\). Thus we conclude that \(v\) and \(av\) are linearly \(D\)-dependent for all \(v \in V\). Let \(av = \alpha_v v\) for all \(v \in V\), where \(\alpha_v \in D\). It is very easy to prove that \(\alpha_v\) is independent of choice of \(v \in V\). Hence \(av = \alpha v\) for all \(v \in V\), where \(\alpha \in D\) is fixed. Then for all \(r \in R\) and \(v \in V\), we have \([a, r]v = a(rv) - r(av) = a(rv) - r(\alpha v) = 0\) that is \([a, r]V = 0\). Since \(V\) is a left faithful irreducible \(R\)-modulo, \([a, r] = 0\) for all \(r \in R\), that is \(a \in Z(R)\). This leads \(d = 0\), a contradiction.

Assume next that \(d\) is not a \(Q\)-inner derivation in \(R\). By assumption, we have

\[
[(\sum_{i=0}^{r-1} x^id(x)x^{r-i-1})x^n, x^r]s, [y, d(y)]_t v = 0
\]
for all \(x, y \in R\). Then by Kharchenko’s theorem [16], we have
\[
[[((\sum_{i=0}^{r-1} x^i u x^{r-i-1}) x^n, x^r)_{s}, [y, v]_t]^m = 0
\]
for all \(x, u, v \in R\). This is a polynomial identity for \(R\) and hence there exists a field \(F\) such that \(R \subseteq M_k(F)\) with \(k > 1\) and \(M_k(F)\) satisfies the same polynomial identity [17, Lemma 1]. But by choosing \(u = e_{21}, v = e_{22}, x = e_{11}, y = e_{12}\), we get
\[
0 = [[[((\sum_{i=0}^{r-1} x^i u x^{r-i-1}) x^n, x^r)_{s}, [y, v]_t]^m = e_{22} + (-1)^m e_{11},
\]
a contradiction.

Our next theorem is to study the central case.

**Theorem 2.3.** Let \(R\) be a prime ring of char \((R) \neq 2\) and \(d\) a nonzero derivation of \(R\) such that \([[d(x^r)x^n, x^r]_s, [y, d(y)]_t] \in Z(R)\) for all \(x, y \in R\), where \(n, s, t \geq 0\) and \(r \geq 1\) are fixed integers, then \(R\) is commutative.

**Proof.** If \(R\) is commutative, we are done. So, let \(R\) be noncommutative. We have that \(R\) satisfies
\[
[[d(x^r)x^n, x^r]_s, [y, d(y)]_t] \in Z(R).
\]
(1)
If for all \(x, y \in R\), \([[d(x^r)x^n, x^r]_s, [y, d(y)]_t] = 0\), then we are done by Theorem 2.2.
So, let there exist \(x_1, x_2 \in R\), such that \(0 \neq [[d(x^r)x^n, x^r]_s, [x_2, d(x_2)]_t] \in Z(R)\).
Then (1) is a central differential identity for \(R\). It follows from [6, Theorem 1] that \(R\) is a prime PI-ring and so \(RC = Q\) is a finite-dimensional central simple \(C\)-algebra by Posner’s theorem for prime PI-ring.

Let \(d\) be an inner derivation of \(Q\) induced by \(a \in Q\). Since \(R\) and \(Q\) satisfy same GPIs [7], we have
\[
[[[a^n, x^r]_{s+1}, [y, [a, y]_t], z] = 0
\]
(2)
for all \(x, y \in Q\). Since there exist \(x_1, x_2 \in R\), such that \([[a^n, x^r]_{s+1}, [x_2, [a, x_2]_t]] \neq 0\), (2) is a nontrivial GPI for \(Q\). Since \(Q\) is a finite-dimensional central simple \(C\)-algebra, it follows from Lemma 2 in [17] that there exists a suitable field \(F\) such that \(Q \subseteq M_k(F)\), \(k > 1\), the ring of all \(k \times k\) matrices over \(F\), and moreover \(M_k(F)\) satisfies (2), that is,
\[
[[[a^n, x^r]_{s+1}, [y, [a, y]_t], z] = 0
\]
(3)
for all \(x, y, z \in M_k(F)\). Let \(e\) and \(f\) be any two orthogonal idempotent elements in \(M_k(F)\). Now, we replace \(x\) with \(e\), \(y\) with \(ef\) and \(z\) with \(ef\) in (3) and let \(Y = [[a^n, e]_{s+1}, [ef, [a, ef]]_t]\). Then we compute
\[
Ye = [[a^n, e]_{s+1}, [ef, [a, ef]]_t]e
= [a^n, e]_{s+1}[ef, [a, ef]]_t e - [ef, [a, ef]]_t[a^n, e]_{s+1} e
= [a^n, e]_{s+1} \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, ef]^j ef[a, ef]^{t-j} e
\]
in is a polynomial identity for have 0 = \left[\left[ \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, \text{exf}^j \text{exf} [a, \text{exf}^{t-j}] [ae^n, e]_{s+1} e \right. \\

= 0 - \sum_{j=0}^{t} (-1)^j \binom{t}{j} (-\text{exf}a)^j \text{exf} (\text{exf} [a, \text{exf}^{t-j}] ae \\

= -2^t (\text{exf}a)^{t+1} e. \quad (4)

fY = f[[ae^n, e]_{s+1}, [\text{exf}, [a, \text{exf}]]_1] \\

= f[ae^n, e]_{s+1} [\text{exf}, [a, \text{exf}]]_1 - f[\text{exf}, [a, \text{exf}]]_1 [ae^n, e]_{s+1} \\

= f[ae^n, e]_{s+1} \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, \text{exf}^j \text{exf} [a, \text{exf}^{t-j}] \\

- f \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, \text{exf}^j \text{exf} [a, \text{exf}^{t-j}] [ae^n, e]_{s+1} \\

= fae \sum_{j=0}^{t} (-1)^j \binom{t}{j} (-\text{exf}a)^j \text{exf} (\text{exf} [a, \text{exf}^{t-j}] - 0 \\

= 2^t (\text{exf} [a, \text{exf}])^{t+1} f. \quad (5)

Hence

\begin{align*}
0 &= \left[ [ae^n, e]_{s+1}, [\text{exf}, [a, \text{exf}]]_1, \text{exf} \right] \\
&= [Y, \text{exf}] \\
&= \{-2^t (\text{exf}a)^{t+1} \text{exf} - 2^t \text{exf} (\text{exf} [a, \text{exf}])^{t+1} f\} \\
&= -2^{t+1} (\text{exf}a)^{t+1} \text{exf}. \quad (6)
\end{align*}

Since char \((R) \neq 2\), this implies \((\text{exf} [a, \text{exf}])^{t+3} = 0\) for all \(x \in M_k(F)\). By Levitzki’s lemma [14, Lemma 1.1], \(\text{exf} [a, \text{exf}] = 0\) for all \(x \in M_k(F)\) and so \(\text{exf} = 0\). Since \(f\) and \(e\) are any two orthogonal idempotent elements in \(M_k(F)\), we have for any idempotent \(e\) in \(M_k(F)\), \((1 - e)ae = 0 = ea(1 - e)\) which implies \([a, e] = 0\). Since \(a\) commutes with all idempotents in \(M_k(F)\), \(a \in C\) and hence \(d = 0\).

If \(d\) is not \(Q\)-inner derivation of \(R\), then by Kharchenko’s Theorem [16], we have \(0 = \left[ [\sum_{i=0}^{r-1} x^i z x^{r-i-1} x^n, x^r]_s, [y, v]_t \right] = 0\) for all \(x, y, z, u, v \in R\). Since this is a polynomial identity for \(R\), there exists a field \(F\) such that \(R \subseteq M_k(F)\) with \(k > 1\) and \(R\) and \(M_k(F)\) satisfy the same polynomial identity [17, Lemma 1]. But by choosing \(u = e_{21}, v = e_{22}, x = e_{11}, y = e_{12}\), we get

\[ [[(\sum_{i=0}^{r-1} x^i z x^{r-i-1}) x^n, x^r]_s, [y, v]_t] = e_{22} - e_{11} \in Z(M_k(F)), \]

a contradiction, since char \((F) \neq 2\). ■

**Theorem 2.4.** Let \(R\) be a prime ring of char \((R) \neq 2\), \(d\) a nonzero derivation of \(R\) and \(I\) a nonzero right ideal of \(R\) such that \([d(x^r) x^n, x^r]_s, [y, d(y)]_t \in Z(R)\) for all \(x, y \in I\), where \(n \geq 0, s \geq 0, t \geq 0, r \geq 1\) are fixed integers. If \([I, I] \neq 0\), then \(d = ad(b)\) with \(bI = 0\) for some \(b \in Q\).
We begin with the following lemma.

**Lemma 2.5.** If \( d(I) \neq 0 \) and \([d(x^r)x^n, x^r]_s, [y, d(y)]_t] \in Z(R) \) for all \( x, y \in I \), then \( R \) satisfies a non-trivial generalized polynomial identity (GPI).

**Proof.** Suppose on the contrary that \( R \) does not satisfy any non-trivial GPI. We may assume that \( R \) is noncommutative, otherwise \( R \) satisfies trivially a non-trivial GPI.

**Case I.** Suppose that \( d \) is a \( Q \)-inner derivation induced by an element \( a \in Q \). Then for any \( u \in I \)

\[
[[a(ux)^n, (ux)^r]_{s+1}, [uy, [a, uy]]_t, uz]
\]

is a GPI for \( R \), so it is the zero element in \( Q \ast_C C \{x, y, z\} \). Expanding this we get,

\[
\left\{ \sum_{j=0}^{s+1} (-1)^j \binom{s+1}{j} (ux)^r_j a(ux)^n(uz)^r_t \right\} [uy, [a, uy]]_t
\]

\[
- \sum_{j=0}^{n} (-1)^j \binom{n}{j} (uy - uya)^j [uy, [a, uy]]_t [a(ux)^n, (ux)^r]_{s+1} uz
\]

\[
- uz[[a(ux)^n, (ux)^r]_{s+1}, [uy, [a, uy]]_t] = 0. \tag{7}
\]

If \( au \) and \( u \) are linearly \( C \)-independent for some \( u \in I \) then

\[
a(ux)^n(ux)^r_{s+1}[uy, [a, uy]]_t uz
\]

\[
- auy \sum_{j=1}^{n} (-1)^j \binom{n}{j} (ay - uya)^j [uy, [a, uy]]_t [a(ux)^n, (ux)^r]_{s+1} uz = 0. \tag{8}
\]

This implies

\[
a(ux)^n(ux)^r_{s+1}[uy, [a, uy]]_t uz = 0 \tag{9}
\]

in \( Q \ast_C C \{x, y, z\} \). Expanding this we write

\[
a(ux)^n(ux)^r_{s+1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (ay - uya)^j [uy, [a, uy]]_t uz = 0.
\]

Again, since \( au \) and \( u \) are linearly \( C \)-independent, in the above expression we see that \( a(ux)^n(ux)^r_{s+1} uy(ay) uz \) appears nontrivially, a contradiction. Thus for any \( u \in I \), \( au \) and \( u \) are \( C \)-dependent. Then \( (a - \alpha)I = 0 \) for some \( \alpha \in C \). Replacing \( a \) with \( a - \alpha \), we may assume that \( aI = 0 \). But then by Lemma 2.1, \( d(I)I = 0 \), contradiction.

**Case II.** Suppose that \( d \) is not a \( Q \)-inner derivation of \( R \). If for all \( u \in I \), \( d(u) \in uC \), then \( [d(u), u] = 0 \) which implies \( R \) to be commutative (see [3]), a contradiction. Therefore there exists \( u \in I \) such that \( d(u) \notin uC \) i.e., \( u \) and \( d(u) \) are linearly \( C \)-independent.

By our assumption we have that \( R \) satisfies

\[
[[[d((ux)^r)](ux)^n, (ux)^r]_s, [d(uy), uy]_t, uz] = 0
\]
that is
\[
\left[\left(\sum_{i=0}^{r-1}(ux)^i(d(u)x + ud(x))(ux)^{r-1-i}(ux)^n, (ux)^r\right), [uy, d(u)y + ud(y)]_s], uz\right] = 0.
\]

By Kharchenko's theorem [16],
\[
\left[\left(\sum_{i=0}^{r-1}(ux)^i(d(u)x + ux_1)(ux)^{n+r-1-i}, (ux)^r\right), [uy, d(u)y + uy_1]_s], uz\right] = 0 \quad (10)
\]
for all \(x, y, z, x_1, y_1 \in R\). In particular, for \(x_1 = y_1 = 0\),
\[
\left[\left(\sum_{i=0}^{r-1}(ux)^i(d(u)x)(ux)^{n+r-1-i}, (ux)^r\right), [uy, d(u)y]_s], uz\right] = 0 \quad (11)
\]
which is a non-trivial GPI for \(R\), because \(u\) and \(d(u)\) are linearly \(C\)-independent, a contradiction. \(\blacksquare\)

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. If \(d(I)I = 0\), then by Lemma 2.1 we obtain our conclusion. So, let \(d(I)I \neq 0\). By Lemma 2.5, \(R\) is a GPI-ring, so is \(Q\) [7]. By [21], \(Q\) is a primitive ring with \(H = Soc(Q) \neq 0\). Moreover, we may assume that \([HI, IH]I\) is a \(Q\)-ring, otherwise by \([7]\), \([IQ, IQ]I\) is a \(Q\)-ring, which is a contradiction. We may also assume that \(d(IH)I\) is a \(Q\)-ring, otherwise by Lemma 2.1, \(d\) is an inner derivation induced by an element \(b \in Q\) such that \(bIH = 0\) that is \(bI = 0\), implying \(d(I)I = 0\), a contradiction.

Let \(a \in IH\). Since \(H\) is a regular ring, there exists \(e^2 = e \in H\) such that \(eH = aH\). Then \(e \in IH\) and \(a = ce\). By our assumption and by [12, Theorem 2], we may also assume that \(\left[\left(d(x^n)x^n, [y, d(y)]_s\right), z\right]\) is an identity for \(IQ\). In particular, \(\left[\left(d(x^n)x^n, [y, d(y)]_s\right), z\right]\) is an identity for \(IH\) and so for \(eH\). Replacing \(x\) with \(e\) and \(y\) with \(ey(1-e)\) and \(z\) with \(ey(1-e)\), it follows that, for all \(y \in H\),
\[
0 = \left[\left(d(e)e^n, [ey(1-e), d(ey(1-e))]_s\right), ey(1-e)\right]. \quad (12)
\]
Let \(V = \left[\left(d(e)e^n, [ey(1-e), d(ey(1-e))]_s\right), ey(1-e)\right]\). We have the facts that for any idempotent \(e\), \(d(x(1-e))e = -e(1-e)d(e)\), \(e(1-e)d(ex) = (1-e)d(e)ex\) and \(ed(e)e = 0\) and hence we compute
\[
Ve = \left[\left(d(e)e^n, [ey(1-e), d(ey(1-e))]_s\right), ey(1-e)\right]e \\
= \left[\left(d(e)e^n, [ey(1-e), d(ey(1-e))]_s\right), ey(1-e)\right]e \\
= \left[\left(d(e)e^n, [ey(1-e), d(ey(1-e))]_s\right), ey(1-e)\right]e \\
- \sum_{j=0}^{t} (1)^j \left(\frac{t}{j}\right) \left[\left(d(e)e^n, [ey(1-e), d(ey(1-e))]_s\right), ey(1-e)\right]e \\
= 0 - \sum_{j=0}^{t} (1)^j \left(\frac{t}{j}\right) \left[\left(d(e)e^n, [ey(1-e), d(ey(1-e))]_s\right), ey(1-e)\right]e \\
= -2^t (ey(1-e)d(e))^{t+1}e \quad (13)
\]
and
\[(1-e)V = (1-e)[e(d)e^n, e]_s, [e(y(1-e), d(ey(1-e)))]_t \]
\[= (1-e)d(e)[e(y(1-e), d(ey(1-e)))]_s, \]
\[- (1-e)[e(y(1-e), d(ey(1-e)))]_t [d(e)e^n, e]_s \]
\[= (1-e)d(e)e \sum_{j=0}^{t} (-1)^j \left( \frac{t}{j} \right) d(ey(1-e))^j ey(1-e)d(ey(1-e))^{t-j} \]
\[- (1-e) \sum_{j=0}^{t} (-1)^j \left( \frac{t}{j} \right) d(ey(1-e))^j ey(1-e)d(ey(1-e))^{t-j}[d(e)e^n, e]_s \]
\[= (1-e)d(e)e \sum_{j=0}^{t} (-1)^j \left( \frac{t}{j} \right) (-ey(1-e)d(e))^j ey(1-e)(d(e)ey(1-e))^{t-j} - 0 \]
\[= 2^t((1-e)d(e)ey)^{t+1}(1-e). \] (14)

Thus (12) gives
\[0 = [V, ey(1-e)] \]
\[= Vey(1-e) - ey(1-e)V \]
\[= -2^t(ey(1-e)d(e))^{t+1} ey(1-e) - 2^t ey((1-e)d(e)ey)^{t+1}(1-e) \]
\[= -2^{t+1}(ey(1-e)d(e))^{t+1} ey(1-e). \] (15)

Multiplying on the left by \((1-e)d(e)\) and on the right by \(d(e)ey\) and using char \((R) \neq 2\), the above equation gives \((1-e)d(e)ey)^{t+2} = 0\) for all \(y \in H\). By Levitzki’s lemma [14, Lemma 1.1], \((1-e)d(e)eH = 0\). By primeness of \(H\), \((1-e)d(e)e = 0\). This implies \((1-e)d(e) = (1-e)d(e^2) = (1-e)d(e)e = 0\). Thus \(d(e) = ed(e) \in eH \subseteq IH\). Now \(d(a) = d(ea) = d(e)ea + ed(ea) \in IH\). Hence, \(d(IH) \subseteq IH\). Since \(d(l_H(IH)) \subseteq l_H(IH)\) holds, \(d\) naturally induces a derivation \(\delta\) on the prime ring \(\overline{TH} = \frac{IH}{l_H(IH)}\) defined by \(\delta(\overline{x}) = \overline{d(x)}\) for \(x \in IH\), where \(l_H(IH)\) denotes the left annihilator of \(IH\) in \(H\). Thus by assumption we have
\[\delta(\overline{[\pi^n, \pi^r]_s, [\gamma, \delta(\overline{\pi})]_t, \overline{\pi}} = 0\]
for all \(\pi, \gamma, \overline{\pi} \in \overline{IH}\). By Theorem 2.3, we have either \(\delta = 0\) or \(\overline{TH}\) is commutative. Therefore, we have that either \(d(IH)IH = 0\) or \([IH, IH]IH = 0\). In both cases, we have contradictions. This completes the proof of the theorem. \(\blacksquare\)

**Corollary 2.6.** Let \(R\) be a prime ring of char \((R) \neq 2\), \(d\) a nonzero derivation of \(R\) and \(I\) a nonzero right ideal of \(R\) such that \([d(x^n)x^r, x^r]_s = 0\) for all \(x \in I\), where \(n \geq 0, s \geq 0, r \geq 1\) are fixed integers. If \([I, I]I \neq 0\), then \(d(I)I = 0\).

**Corollary 2.7.** Let \(R\) be a prime ring of char \((R) \neq 2\), \(d\) a nonzero derivation of \(R\) and \(I\) a nonzero right ideal of \(R\) such that \([x, d(x)]_t = 0\) for all \(x \in I\), where \(t \geq 1\) is a fixed integer. If \([I, I]I \neq 0\), then \(d(I)I = 0\).
3. The case: \( R \) a semiprime ring

In this section we extend Theorems 2.2 and 2.3 to the case of semiprime ring. Let \( R \) be a semiprime ring and \( U \) be its right Utumi quotient ring. The center of \( U \) is called extended centroid of \( R \) and is denoted by \( C \). It is well known fact that any derivation of a semiprime ring \( R \) can be uniquely extended to a derivation of its right Utumi quotient ring \( U \) and so any derivation of \( R \) can be defined on the whole of \( U \) [19, Lemma 2]. Let \( M(C) \) be the set of all maximal ideals of \( C \). Now by the standard theory of orthogonal completions for semiprime rings (see [19, p. 31-32]), we have the following lemma.

**Lemma 3.1**. [1, Lemma 1 and Theorem 1] Let \( R \) be a 2-torsion free semiprime ring and \( P \) a maximal ideal of \( C \). Then \( P \) is a prime ideal of \( U \) invariant under all derivations of \( U \). Moreover, \( \bigcap \{ P \in M(C) \mid P \in U \} = 0 \).

**Theorem 3.2**. Let \( R \) be a 2-torsion free semiprime ring, \( d \) a non-zero derivation of \( R \) such that \([d(x^n)x^r], [y, d(y)]\] \( = 0 \) for all \( x, y \in R \), where \( n, s, t \geq 0 \) and \( m, r \geq 1 \) are fixed integers. Then \( d \) maps \( R \) into its centre.

**Proof.** By assumption and by [19, Theorem 3], we can write \([d(x^n)x^r], [y, d(y)]\] \( = 0 \) for all \( x, y \in U \). Note that \( U \) is also a 2-torsion free semiprime ring. Let \( P \in M(C) \) such that \( U/PU \) is 2-torsion free. Then by Lemma 3.1, \( PU \) is a prime ideal of \( U \) invariant under \( d \). Set \( U = U/PU \). Then derivation \( d \) canonically induces a derivation \( \overline{d} \) on \( U \) defined by \( \overline{d}(x) = \overline{d(x)} \) for all \( x \in U \). Therefore, \([\overline{d}(x^n)x^r], [y, \overline{d}(y)]\] \( = 0 \) for all \( x, y \in U \). By Theorem 2.2, either \( \overline{d} = 0 \) or \( \overline{U}, \overline{U} = 0 \) i.e., \( \overline{d(U)} \subseteq PU \) or \( \overline{U}, \overline{U} \subseteq PU \). In any case \( d(U)\] \( \subseteq PU \) for any \( P \in M(C) \). By Lemma 3.1, \( \bigcap \{ P \mid P \in M(C) \} = 0 \). Thus \( d(U)\] \( = 0 \). Without loss of generality, we have \( d(R) = 0 \). This implies \( d(R) = 0 \) and so \( [R, d(R)] = 0 \). Since \( R \) is semiprime, we have \( [R, d(R)] = 0 \), that is, \( d(R) \subseteq Z(R) \), as desired.

By a similar proof, Theorem 2.3 can be extended to semiprime ring as follows:

**Theorem 3.3**. Let \( R \) be a 2-torsion free semiprime ring, \( d \) a non-zero derivation of \( R \) such that \([d(x^n)x^r], [y, d(y)]\] \( \in Z(R) \) for all \( x, y \in R \), where \( n, s, t \geq 0 \) and \( r \geq 1 \) are fixed integers. Then \( d \) maps \( R \) into its centre.

**Corollary 3.4**. Let \( R \) be a 2-torsion free semiprime ring, \( d \) a non-zero derivation of \( R \) such that \([d(x^n)x^r], [y, d(y)]\] \( = 0 \) for all \( x \in R \), where \( n, s \geq 0 \) and \( r \geq 1 \) are fixed integers. Then \( d \) maps \( R \) into its centre.

**Corollary 3.5**. Let \( R \) be a 2-torsion free semiprime ring, \( d \) a non-zero derivation of \( R \) such that \([d(x^n)x^r], [y, d(y)]\] \( = 0 \) for all \( x \in R \), where \( t \geq 0 \) is a fixed integer. Then \( d \) maps \( R \) into its centre.

**Acknowledgement.** The authors would like to thank the referee for providing very helpful comments and suggestions. The second author is grateful to
University Grants Commission of India for financial support under grant No.F.PSW-168/11-12(ERO).

REFERENCES


(received 16.07.2011; in revised form 29.03.2012; available online 01.05.2012)

Department of Mathematics, Belda College, Belda, Paschim Medinipur-721424 (W.B.), India
E-mail: basudhara@yahoo.com

Department of Mathematics, Haldia Government College, Haldia, Purba Medinipur-721657 (W.B.), India
E-mail: deepankaralg@gmail.com