SIGNED TOTAL DISTANCE $k$-DOMATIC NUMBERS OF GRAPHS

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Abstract. In this paper we initiate the study of signed total distance $k$-domatic numbers in graphs and we present its sharp upper bounds.

1. Introduction

In this paper, $k$ is a positive integer, and $G$ is a finite simple graph without isolated vertices and with vertex set $V = V(G)$ and edge set $E = E(G)$. For a vertex $v \in V(G)$, the open $k$-neighborhood $N_{k,G}(v)$ is the set $\{u \in V(G) \mid u \neq v \text{ and } d(u,v) \leq k\}$. The open $k$-neighborhood $N_{k,G}(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N_{k,G}(v)$. The $k$-degree of a vertex $v$ is defined as $\text{deg}_{k,G}(v) = |N_{k,G}(v)|$. The minimum and maximum $k$-degree of a graph $G$ are denoted by $\delta_k(G)$ and $\Delta_k(G)$, respectively. If $\delta_k(G) = \Delta_k(G)$, then the graph $G$ is called distance-$k$-regular. The $k$-th power $G^k$ of a graph $G$ is the graph with vertex set $V(G)$ where two vertices $u$ and $v$ are adjacent if and only if the distance $d(u,v)$ is at most $k$ in $G$.

Now we observe that $N_{k,G}(v) = N_{1,G^k}(v) = N_{G^k}(v)$, $\text{deg}_{k,G}(v) = \text{deg}_{1,G^k}(v) = \text{deg}_{G^k}(v)$, $\delta_k(G) = \delta_1(G^k) = \delta(G^k)$ and $\Delta_k(G) = \Delta_1(G^k) = \Delta(G^k)$. If $k = 1$, then we also write $\text{deg}_G(v)$, $N_G(v)$, $\delta(G)$ for $\text{deg}_{1,G}(v)$, $N_{1,G}(v)$, $\delta_1(G)$ etc. Consult [7] for the notation and terminology which are not defined here.

For a real-valued function $f : V(G) \rightarrow \mathbb{R}$, the weight of $f$ is $w(f) = \sum_{v \in V} f(v)$. For $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$. A signed total distance $k$-dominating function (STD$k$D function) is a function $f : V(G) \rightarrow \{-1,1\}$ satisfying $\sum_{u \in N_{k,G}(v)} f(u) \geq 1$ for every $v \in V(G)$. The minimum of the values of $\sum_{v \in V(G)} f(v)$ taken over all signed total distance $k$-dominating functions $f$ is called the signed total distance $k$-domination number and is denoted by $\gamma_{k,s}^-(G)$. Then the function assigning $+1$ to every vertex of $G$ is a STD$k$D function, called the function $\epsilon$, of weight $n$. Thus $\gamma_{k,s}^-(G) \leq n$ for every graph of order $n$. Moreover, the weight of every STD$k$D function different from $\epsilon$.

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is at most \( n - 2 \) and more generally, \( \gamma^t_{k,s}(G) \equiv n \) (mod 2). Hence \( \gamma^t_{k,s}(G) = n \) if and only if \( \epsilon \) is the unique STD\( k \)-function of \( G \). In the special case when \( k = 1 \), \( \gamma^t_{k,s}(G) \) is the signed total domination number \( \gamma^t_s(G) \) investigated in [8] and has been studied by several authors (see for example [2]). The signed total distance 2-domination number of graphs was introduced by Zelinka [9]. By these definitions, we easily obtain

\[ \gamma^t_{k,s}(G) = \gamma^t_s(G^k). \tag{1} \]

A set \( \{f_1, f_2, \ldots, f_d\} \) of signed total distance \( k \)-dominating functions on \( G \) with the property that \( \sum_{i=1}^{d} f_i(v) \leq 1 \) for each \( v \in V(G) \), is called a signed total distance \( k \)-dominating family on \( G \). The maximum number of functions in a signed total distance \( k \)-dominating family on \( G \) is the signed total distance \( k \)-domatic number of \( G \), denoted by \( d^t_{k,s}(G) \). The signed total distance \( k \)-domatic number is well-defined and \( d^t_{k,s}(G) \geq 1 \) for all graphs \( G \), since the set consisting of any one STD\( k \)-function, for instance the function \( \epsilon \), forms a STD\( k \)-family of \( G \). A \( d^t_{k,s} \)-family of a graph \( G \) is a STD\( k \)-family containing \( d^t_{k,s}(G) \) STD\( k \)-functions. The signed total distance 1-domatic number \( d^t_{1,s}(G) \) is the usual signed total domatic number \( d^t_s(G) \) which was introduced by Henning in [3] and has been studied by several authors (see for example [5]). Obviously,

\[ d^t_{k,s}(G) = d^t_s(G^k). \tag{2} \]

**Observation 1.** Let \( G \) be a graph of order \( n \) without isolated vertices. If \( \gamma^t_{k,s}(G) = n \), then \( \epsilon \) is the unique STD\( k \)-function of \( G \) and so \( d^t_{k,s}(G) = 1 \).

We first study basic properties and sharp upper bounds for the signed total distance \( k \)-domatic number of a graph. Some of them generalize the result obtained for the signed total domatic number.

In this paper we make use of the following results.

**Proposition A.** Let \( G \) be a graph of order \( n \) and minimum degree \( \delta(G) \geq 1 \). Then \( \gamma^t_s(G) = n \) if and only if for each \( v \in V(G) \), there exists a vertex \( u \in N_G(v) \) such that \( \deg_G(u) = 1 \) or \( \deg_G(u) = 2 \).

**Proof.** Assume that \( \gamma^t_s(G) = n \) and there exists a vertex \( v \) every neighbor of which has degree at least 3. Then the function \( f \) that assigns to \( v \) the value 1 and to all other vertices the value 1 is a signed total dominating function of \( G \). This leads to the contradiction \( \gamma^t_s(G) \leq n - 2 \). Hence at least one neighbor of \( v \) is of degree 1 or 2. On the other hand, if every vertex of \( v \) has a neighbor of degree 1 or 2, then \( \epsilon \) is the unique signed total dominating function of \( G \), and so \( \gamma^t_s(G) = n \).

The special case of Proposition A that \( G \) is a tree can be found in [2], the proof is almost the same.

**Proposition B.** [3] The signed total domatic number \( d^t_s(G) \) of a graph \( G \), without isolated vertex, is an odd integer.

**Proposition C.** [3] If \( G \) is a graph without isolated vertices, then \( 1 \leq d^t_s(G) \leq \delta(G) \).
Proposition D. [4, 6] Let $G$ be a graph with $\delta(G) \geq 1$, and let $v$ be a vertex of even degree $\deg_G(v) = 2t$ with an integer $t \geq 1$. Then $d_s^t(G) \leq t$ when $t$ is odd and $d_s^t(G) \leq t - 1$ when $t$ is even.

Proposition E. [3] Let $k \geq 1$ be an integer, and let $K_n$ be the complete graph of order $n$. For $n \geq 2$, we have

$$\gamma_{k,s}(K_n) = \gamma_s^t(K_n) = \begin{cases} 3 & \text{if } n \equiv 1 \pmod{2} \\ 2 & \text{otherwise}. \end{cases} \quad (3)$$

Proposition F. [3] If $K_n$ is the complete graph of order $n \geq 2$, then

$$d_s^t(K_n) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor - \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{4} \rfloor & \text{if } n \text{ is odd}, \\ \frac{n}{2} - \lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is even}. \end{cases} \quad (4)$$

Since $N_{k,K_n}(v) = N_{K_n}(v)$ for each vertex $v \in V(K_n)$ and each positive integer $k$, each signed total dominating function of $K_n$ is a signed total distance $k$-dominating function of $K_n$ and vice versa. Using Proposition F, we obtain

$$d_{k,s}(K_n) = d_s^t(K_n) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor - \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{4} \rfloor & \text{if } n \text{ is odd}, \\ \frac{n}{2} - \lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is even}. \end{cases}$$

More generally, the following result is valid.

Observation 2. Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n$ without isolated vertices. If $\text{diam}(G) \leq k$, then $\gamma_{k,s}(G) = \gamma_s^t(K_n)$ and $d_{k,s}(G) = d_s^t(K_n)$.

The next result is immediate by Observation 2, Propositions E and F.

Corollary 3. If $k \geq 2$ is an integer, and $G$ is a graph of order $n$ with $\text{diam}(G) = 2$ and $\delta(G) \geq 1$, then

$$\gamma_{k,s}(G) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even}, \end{cases}$$

and

$$d_{k,s}(G) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor - \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{4} \rfloor & \text{if } n \text{ is odd}, \\ \frac{n}{2} - \lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is even}. \end{cases}$$

Corollary 4. Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n$ with $\delta(G) \geq 1$. If $\text{diam}(G) \neq 3$, then $\gamma_{k,s}(G) = \gamma_s^t(K_n)$ and $d_{k,s}(G) = d_s^t(K_n)$ or $\gamma_{k,s}(\overline{G}) = \gamma_s^t(K_n)$ and $d_{k,s}(\overline{G}) = d_s^t(K_n)$.

Proof. If $\text{diam}(G) \leq 2$, then it follows from Observation 2 that $\gamma_{k,s}(G) = \gamma_s^t(K_n)$ and $d_{k,s}(G) = d_s^t(K_n)$. If $\text{diam}(G) \geq 3$, then the hypothesis $\text{diam}(G) \neq 3$ implies that $\text{diam}(G) \geq 4$. Now, according to a result of Bondy and Murty [1, page 14], we deduce that $\text{diam}(\overline{G}) \leq 2$. Applying again Observation 2, we obtain $\gamma_{k,s}(\overline{G}) = \gamma_s^t(K_n)$ and $d_{k,s}(\overline{G}) = d_s^t(K_n)$. \qed
Corollary 5. If \( k \geq 3 \) is an integer and \( G \) a graph of order \( n \) with \( \delta(G) \geq 1 \), then \( \gamma_{k,s}^t(G) = \gamma_s^k(K_n) \) and \( d_{k,s}^t(G) = d_s^k(K_n) \) or \( \gamma_{k,s}^t(\overline{G}) = \gamma_s^k(K_n) \) and \( d_{k,s}^t(\overline{G}) = d_s^k(K_n) \).

Proposition 6. Let \( k \geq 1 \) be an integer, and let \( G \) be a graph of order \( n \) and minimum degree \( \delta(G) \geq 1 \).

If \( k = 1 \), then \( \gamma^t_{k,s}(G) = n \) if and only if for each \( v \in V(G) \), there exists a vertex \( u \in N_G(v) \) such that \( \deg_G(u) = 1 \) or \( \deg_G(u) = 2 \).

If \( k \geq 2 \), then \( \gamma^t_{k,s}(G) = n \) if and only if all components of \( G \) are of order 2 or 3.

Proof. In the case \( k = 1 \), Proposition A implies the desired result.

Assume now that \( k \geq 2 \). If all components of \( G \) are of order 2 or 3, then it is easy to see that \( \epsilon \) is the unique STDkD function of \( G \) and thus \( \gamma^t_{k,s}(G) = n \).

Conversely, assume that \( \gamma^t_{k,s}(G) = n \). Suppose to the contrary that \( G \) has a component \( G_1 \) of order \( n(G_1) \geq 4 \). If \( \text{diam}(G_1) \geq 3 \), then assume that \( x_1 x_2 \ldots x_m \) is a longest path in \( G_1 \). It is straightforward to verify that the function \( f : V(G) \rightarrow \{-1, 1\} \) defined by \( f(x_1) = -1 \) and \( f(x) = 1 \) otherwise is a signed total distance \( k \)-dominating function of \( G \) which is a contradiction. If \( \text{diam}(G_1) \leq 2 \), then Proposition E, Observation 2 and Corollary 3 show that \( \gamma^t_{k,s}(G_1) \leq 3 < 4 \leq n(G_1) \) and consequently \( \gamma^t_{k,s}(G) < n \). This contradiction completes the proof. ■

2. Basic properties of the signed total distance \( k \)-domatic number

In this section we present basic properties of \( d_{k,s}^t(G) \) and sharp bounds on the signed total distance \( k \)-domatic number of a graph.

Proposition 7. Let \( G \) be a graph with \( \delta(G) \geq 1 \). The signed total distance \( k \)-domatic number of \( G \) is an odd integer.

Proof. According to the identity (2), we have \( d_{k,s}^t(G) = d_s^k(G^k) \). In view of Proposition B, \( d_s^k(G^k) \) is odd and thus \( d_{k,s}^t(G) \) is odd, and the proof is complete. ■

Theorem 8. If \( G \) is a graph with \( \delta(G) \geq 1 \), then

\[
1 \leq d_{k,s}^t(G) \leq \delta_k(G).
\]

Moreover, if \( d_{k,s}^t(G) = \delta_k(G) \), then for each function of any \( d_{k,s}^t \)-family \( \{f_1, f_2, \ldots, f_d\} \) and for all vertices \( v \) of minimum \( k \)-degree \( \delta_k(G) \), \( \sum_{u \in N_{k,G}(v)} f_i(u) = 1 \) and \( \sum_{i=1}^d f_i(u) = 1 \) for every \( u \in N_{k,G}(v) \).

Proof. Let \( \{f_1, f_2, \ldots, f_d\} \) be an STDkD family of \( G \) such that \( d = d_{k,s}^t(G) \), and let \( v \) be a vertex of minimum \( k \)-degree \( \delta_k(G) \). Then \( |N_{k,G}(v)| = \delta_k(G) \) and

\[
1 \leq d = \sum_{i=1}^d 1 \leq \sum_{i=1}^d \sum_{u \in N_{k,G}(v)} f_i(u) \]
\[
= \sum_{u \in N_{k,G}(v)} \sum_{i=1}^d f_i(u) \leq \sum_{u \in N_{k,G}(v)} 1 = \delta_k(G).
\]

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If $d_{k,s}^t(G) = \delta_k(G)$, then the two inequalities occurring in the proof become equalities, which gives the two properties given in the statement. ■

**Theorem 9.** Let $k \geq 1$ be an integer, and let $G$ be a graph with $\delta(G) \geq 1$. If $G$ contains a vertex $v$ of even $k$-degree $\deg_{k,G}(v) = 2t$ with an integer $t \geq 1$, then $d_{k,s}^t(G) \leq t$ when $t$ is odd and $d_{k,s}^t(G) \leq t - 1$ when $t$ is even.

*Proof.* Since $\deg_{k,G}(v) = \deg_G(v) = 2t$, Proposition D and (2) imply that $d_{k,s}^t(G) = d_k^t(G^k) \leq t$ when $t$ is odd and $d_{k,s}^t(G) = d_k^t(G^k) \leq t - 1$ when $t$ is even. ■

Restricting our attention to graphs $G$ of even minimum $k$-degree, Theorem 9 leads to a considerable improvement of the upper bound of $d_{k,s}^t(G)$ given in Theorem 8.

**Corollary 10.** If $k \geq 1$ is an integer, and $G$ is a graph of even minimum $k$-degree $\delta_k(G) \geq 1$, then $d_{k,s}^t(G) \leq \delta_k(G)/2$ when $\delta_k(G) \equiv 2 \pmod{4}$ and $d_{k,s}^t(G) \leq \delta_k(G)/2 - 1$ when $\delta_k(G) \equiv 0 \pmod{4}$.

**Theorem 11.** Let $G$ be a graph of order $n$ with signed total distance $k$-domination number $\gamma_{k,s}^t(G)$ and signed total distance $k$-domatic number $d_{k,s}^t(G)$. Then

$$\gamma_{k,s}^t(G) \cdot d_{k,s}^t(G) \leq n.$$  

Moreover, if $\gamma_{k,s}^t(G) \cdot d_{k,s}^t(G) = n$, then for each STD$kD$ family $\{f_1, f_2, \ldots, f_d\}$ on $G$ with $d = d_{k,s}^t(G)$, each function $f_i$ is a $\gamma_{k,s}^t$-function and $\sum_{i=1}^d f_i(v) = 1$ for all $v \in V$.

*Proof.* Let $\{f_1, f_2, \ldots, f_d\}$ be a STD$kD$ family on $G$ such that $d = d_{k,s}^t(G)$ and let $v \in V$. Then

$$d \cdot \gamma_{k,s}^t(G) = \sum_{i=1}^d \gamma_{k,s}^t(G) \leq \sum_{i=1}^d \sum_{v \in V} f_i(v)$$

$$= \sum_{v \in V} \sum_{i=1}^d f_i(v) \leq \sum_{v \in V} 1 = n.$$  

If $\gamma_{k,s}^t(G) \cdot d_{k,s}^t(G) = n$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{k,s}^t$-family $\{f_1, f_2, \ldots, f_d\}$ on $G$ and for each $i$, $\sum_{v \in V} f_i(v) = \gamma_{k,s}^t(G)$, thus each function $f_i$ is a $\gamma_{k,s}^t$-function, and $\sum_{i=1}^d f_i(v) = 1$ for all $v$. ■

The next corollary is a consequence of Theorem 11 and Proposition 7, and it improves Observation 1.

**Corollary 12.** If $\gamma_{k,s}^t(G) > \frac{n}{3}$, then $d_{k,s}^t(G) = 1$.

The upper bound on the product $\gamma_{k,s}^t(G) \cdot d_{k,s}^t(G)$ leads to a bound on the sum.

**Theorem 13.** If $G$ is a graph of order $n$ with minimum degree $\delta(G) \geq 1$, then

$$\gamma_{k,s}^t(G) + d_{k,s}^t(G) \leq n + 1,$$

with equality if and only if $d_{k,s}^t(G) = 1$ and $\gamma_{k,s}^t(G) = n$.  

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Proof. According to Theorem 11, we obtain
\[ \gamma_{k,s}^t(G) + d_{k,s}^t(G) \leq \frac{n}{d_{k,s}^t(G)} + d_{k,s}^t(G). \] (6)
In view of Theorem 8, we have \( 1 \leq d_{k,s}^t(G) \leq n \). Using these inequalities, and the fact that the function \( g(x) = x + n/x \) is decreasing for \( 1 \leq x \leq \sqrt{n} \) and increasing for \( \sqrt{n} \leq x \leq n \) inequality (6) leads to
\[ \gamma_{k,s}^t(G) + d_{k,s}^t(G) \leq \frac{n}{d_{k,s}^t(G)} + d_{k,s}^t(G) \leq \max\{g(1), g(n)\} = n + 1, \]
and the desired bound is proved.

If \( d_{k,s}^t(G) = 1 \) and \( \gamma_{k,s}^t(G) = n \), then obviously \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1 \).

Conversely, assume that \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1 \). In view of Theorem 8, we observe that \( d_{k,s}^t(G) \leq \delta_k(G) \leq n - 1 \). If \( n = 2 \), then we deduce that \( d_{k,s}^t(G) = 1 \). If we assume in the case \( n \geq 3 \) that \( 2 \leq d_{k,s}^t(G) \), then we obtain as above that
\[ \gamma_{k,s}^t(G) + d_{k,s}^t(G) \leq \frac{n}{d_{k,s}^t(G)} + d_{k,s}^t(G) \leq \max\{g(2), g(n-1)\} \]
\[ = \max\left\{\frac{n}{2} + 2, \frac{n}{n-1} + n-1\right\} < n + 1, \]
a contradiction to the assumption \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1 \). It follows that \( d_{k,s}^t(G) = 1 \) in each case and hence \( \gamma_{k,s}^t(G) = n \). This completes the proof. \[ \blacksquare \]

**Corollary 14.** Let \( k \geq 1 \) be an integer, and let \( G \) be a graph of order \( n \) and minimum degree \( \delta(G) \geq 1 \).

If \( k = 1 \), then \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1 \) if and only if for each \( v \in V(G) \), there exists a vertex \( u \in N_G(v) \) such that \( \deg_G(u) = 1 \) or \( \deg_G(u) = 2 \).

If \( k \geq 2 \), then \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1 \) if and only if all components of \( G \) are of order 2 or 3.

**Proof.** If \( k = 1 \) and for each \( v \in V(G) \), there exists a vertex \( u \in N_G(v) \) such that \( \deg_G(u) = 1 \) or \( \deg_G(u) = 2 \), then Proposition A yields \( \gamma_{k,s}^t(G) = n \). Thus, by Observation 1, \( d_{k,s}^t(G) = 1 \) and so \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1 \). If \( k \geq 2 \) and all components of \( G \) are of order 2 or 3, then it follows from Proposition 6 that \( \gamma_{k,s}^t(G) = n \) and therefore \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1 \).

Conversely, assume that \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1 \). Theorem 13 implies that \( d_{k,s}^t(G) = 1 \) and hence \( \gamma_{k,s}^t(G) = n \). Now Proposition 6 leads to the desired result, and the proof is complete. \[ \blacksquare \]

If \( 2 \leq d_{k,s}^t(G) \), then Theorem 13 shows that \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) \leq n \). In the next corollary we will improve this bound slightly.

**Corollary 15.** Let \( G \) be a graph of order \( n \geq 3 \) with \( \delta(G) \geq 1 \). If \( 2 \leq d_{k,s}^t(G) \), then
\[ \gamma_{k,s}^t(G) + d_{k,s}^t(G) \leq n - 1. \]
Proof. Since \( d_{k,s}^t(G) \geq 2 \), Theorem 13 implies that \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) \leq n \). Now suppose to the contrary that \( \gamma_{k,s}^t(G) + d_{k,s}^t(G) = n \). It follows from Theorem 7 that \( d_{k,s}^t(G) \) is odd, a contradiction to the fact that, as seen in the introduction, \( \gamma_{k,s}^t(G) \equiv n \pmod{2} \).

Corollary 16. Let \( G \) be a graph of order \( n \) with \( \delta(G) \geq 1 \), and let \( k \geq 1 \) be an integer. If \( \min\{\gamma_{k,s}^t(G), d_{k,s}^t(G)\} \geq a \), with \( 2 \leq a \leq \sqrt{n} \), then
\[
\gamma_{k,s}^t(G) + d_{k,s}^t(G) \leq a + \frac{n}{a}.
\]

Proof. Since \( \min\{\gamma_{k,s}^t(G), d_{k,s}^t(G)\} \geq a \geq 2 \), it follows from Theorem 11 that \( a \leq d_{k,s}^t(G) \leq \frac{n}{a} \). Applying the inequality (6), we obtain
\[
\gamma_{k,s}^t(G) + d_{k,s}^t(G) \leq d_{k,s}^t(G) + \frac{n}{d_{k,s}^t(G)}.
\]
The bound results from the facts that the function \( g(x) = x + n/x \) is decreasing for \( 1 \leq x \leq \sqrt{n} \) and increasing for \( \sqrt{n} \leq x \leq n \).

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