Abstract. In this paper, we study the uniqueness of entire functions and prove two theorems which improve the result given by Fang [M.L. Fang, Entire functions and their derivatives share two finite sets, Bull. Malaysian Math. Sci. Soc. 24 (2001), 7–16].

1. Introduction, definitions and results

Let \( f \) and \( g \) be two nonconstant meromorphic functions defined in the open complex plane \( \mathbb{C} \). If for some \( a \in \mathbb{C} \cup \{\infty\} \), \( f \) and \( g \) have the same set of \( a \)-points with the same multiplicities then we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities). If we do not take the multiplicities into account, \( f \) and \( g \) are said to share the value \( a \) IM (ignoring multiplicities). We assume that the reader is familiar with the notations of Nevanlinna theory that can be found, for instance, in [3] or [6].

Let \( S \) be a set of distinct elements of \( \mathbb{C} \cup \{\infty\} \) and \( E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\} \), where each zero is counted according to its multiplicity. If we do not count the multiplicity the set \( \bigcup_{a \in S} \{z : f(z) - a = 0\} \) is denoted by \( \overline{E_f}(S) \). If \( E_f(S) = E_g(S) \) we say that \( f \) and \( g \) share the set \( S \) CM. On the other hand, if \( \overline{E_f}(S) = \overline{E_g}(S) \), we say that \( f \) and \( g \) share the set \( S \) IM. Let \( m \) be a positive integer or infinity and \( a \in \mathbb{C} \cup \{\infty\} \). We denote by \( E_m(a, f) \) the set of all \( a \)-points of \( f \) with multiplicities not exceeding \( m \), where an \( a \)-point is counted according to its multiplicity. For a set \( S \) of distinct elements of \( \mathbb{C} \) we define \( E_m(S, f) = \bigcup_{a \in S} E_m(a, f) \). If for some \( a \in \mathbb{C} \cup \{\infty\} \), \( E_\infty(a, f) = E_\infty(a, g) \), we say that \( f \) and \( g \) share the value \( a \) CM. We can define \( \overline{E_m}(a, f) \) and \( \overline{E_m}(S, f) \) similarly.

In 1977, Gross [2] posed the following question.

Question. Can one find two finite sets \( S_j (j = 1, 2) \) such that any two nonconstant entire functions \( f \) and \( g \) satisfying \( E_f(S_j) = E_g(S_j) \) for \( j = 1, 2 \) must be identical?

2010 AMS Subject Classification: 30D35
Keywords and phrases: Entire function; share set; uniqueness.
Yi [7] gave a positive answer to the question. He proved

**Theorem A.** [7] Let $f$ and $g$ be two nonconstant entire functions, $n \geq 5$ a positive integer, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a\}$, where $a \neq 0$ is a constant satisfying $a^{2n} \neq 1$. If $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

In 2001, Fang [1] investigated the question and proved the following theorems

**Theorem B.** [1] Let $f$ and $g$ be two nonconstant entire functions, $n \geq 5$, $k$ two positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a, b, c\}$, where $a, b, c$ are nonzero finite distinct constants. If $E_f(S_1) = E_g(S_1)$ and $E_f(S_2) = E_g(S_2)$, then one of the following cases must occur: (1) $f \equiv g$; (2) $g = \frac{1}{b}e^{cz-d}$, where $c, d, t$ are three constants satisfying $t^n = 1$ and $(-1)^k e^{2k} = a^2$; (3) $f = a^2$, $g = te^{-cz-d}$, where $c, d, t$ are three constants satisfying $t^n = 1$ and $(-1)^k e^{2k} = a^2$.

**Theorem C.** [1] Let $f$ and $g$ be two nonconstant entire functions, $n \geq 5$, $k$ two positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a, b, c\}$, where $a \neq 0, \infty$. If $E_f(S_1) = E_g(S_1)$ and $E_f(S_2) = E_g(S_2)$, then one of the following cases must occur: (1) $f \equiv g$; (2) $f = e^{cz-d}, g = te^{-cz-d}$, where $c, d, t$ are three constants satisfying $t^n = 1$ and $(-1)^k e^{2k} = a^2$.

In this paper, we consider the more general sets $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. We prove the following results which improve Theorem B, Theorem C and Theorem D.

**Theorem 1.** Let $n(\geq 5)$, $k, m$ be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. If two nonconstant entire functions $f$ and $g$ satisfy $E_3(S_1) = E_3(S_1, g)$, and $E_f(S_2) = E_g(S_2)$, then one of the following cases must occur: (1) $f \equiv g$, $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$, where $t$ is a constant satisfying $t^n = 1$; (2) $f(z) = e^{cz}$, $g(z) = \frac{1}{t}e^{cz}$, $\{a_1, a_2, \ldots, a_m\} = \{(-1)^k e^{2k} \frac{1}{a_1}, \ldots, \frac{1}{a_m}\}$, where $t, c, d$ are nonzero constants and $t^n = 1$.

**Theorem 2.** Let $n(\geq 5)$, $k, m$ be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. If two nonconstant entire functions $f$ and $g$ satisfy $E_2(S_1, f) = E_2(S_1, g)$, and $E_f(S_2) = E_g(S_2)$, then one of the following cases must occur: (1) $f \equiv g$, $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$, where $t$ is a constant satisfying $t^n = 1$; (2) $f(z) = e^{cz}$, $g(z) = \frac{1}{t}e^{cz}$, $\{a_1, a_2, \ldots, a_m\} = \{(-1)^k e^{2k} \frac{1}{a_1}, \ldots, \frac{1}{a_m}\}$, where $t, c, d$ are nonzero constants and $t^n = 1$.
2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by $H$ the following function:

$$H = \left( \frac{F''}{F'} - \frac{2F'}{F} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G} \right).$$

**Lemma 1.** [5] Let $f$ be a nonconstant meromorphic function, and let $a_0, a_1, a_2, \ldots, a_n$ be finite complex numbers, $a_n \neq 0$. Then

$$T(r, a_n f^n + \cdots + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f).$$

**Lemma 2.** [4] Let $F, G$ be two nonconstant meromorphic functions such that $E_3(1, F) = E_3(1, G)$, then one of the following cases holds: (1) $T(r, F) + T(r, G) \leq 2\{N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G)\} + S(r, F) + S(r, G)$; (2) $F \equiv G$; (3) $FG \equiv 1$.

**Lemma 3.** [9] Let $F$ and $G$ be two nonconstant meromorphic functions and $E_2(1, F) = E_2(1, G)$. If $H \equiv 0$, then

$$T(r, F) + T(r, G) \leq 2N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, G)$$

$$+ N_3(r, \frac{1}{F-1}) + S(r, F) + S(r, G).$$

**Lemma 4.** [8] Let $H$ be defined as above. If $H \equiv 0$ and

$$\limsup_{r \to \infty} \frac{N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + N(r, F) + N(r, G)}{T(r)} < 1, \quad r \in I,$$

where $I$ is a set with infinite linear measure and $T(r) = \max\{T(r, F), T(r, G)\}$, then $FG \equiv 1$ or $F \equiv G$.

**Lemma 5.** [3] Let $f$ be a nonconstant meromorphic function, $n$ be a positive integer, and let $\Psi$ be a function of the form $\Psi = f^n + Q$, where $Q$ is a differential polynomial of $f$ with degree $\leq n - 1$. If

$$N(r, f) + N(r, \frac{1}{\Psi}) = S(r, f),$$

then $\Psi = (f + \alpha)^n$, where $\alpha$ is a meromorphic function with $T(r, \alpha) = S(r, f)$, determined by the term of degree $n - 1$ in $Q$. 
3. Proof of Theorem 1

Set \( F = f^n, G = g^n \). By Lemma 1, we have

\[
T(r, F) = nT(r, f) + S(r, f), \quad T(r, G) = nT(r, g) + S(r, g). \tag{1}
\]

From \( E_3(S_1, f) = E_3(S_1, g) \), we deduce \( E_3(1, F) = E_3(1, G) \). Then \( F \) and \( G \) satisfy the condition of Lemma 2. We assume Case (1) in Lemma 2 holds, that is,

\[
T(r, F) + T(r, G) \leq 2\{N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G})\} + S(r, F) + S(r, G)
\]

\[
\leq 4T(r, f) + 4T(r, g) + S(r, f) + S(r, g) \tag{2}
\]

Combining (1) and (2) together we have

\[
(n - 4)T(r, f) + (n - 4)T(r, g) \leq S(r, f) + S(r, g), \tag{3}
\]

which contradicts \( n \geq 5 \). Thus by Lemma 2, we have \( FG \equiv 1 \) or \( F \equiv G \), that is \( f = tg \) or \( fg = t \) where \( t \) is a constant and \( t^n = 1 \). Next we consider the following two cases:

Case 1. \( f = tg \). Then \( f^{(k)} = t g^{(k)} \). By \( E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2) \), we get \( \{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\} \).

Case 2. \( fg = t \). Then there exists an entire function \( h \) such that \( f = e^h \) and \( g = te^{-h} \). Therefore

\[
f^{(i)} = \alpha_i f, g^{(i)} = \beta_i g, i = 1, 2, \ldots, \tag{4}
\]

where \( \alpha_1 = h' \), \( \beta_1 = -h' \), and \( \alpha_i, \beta_i \) satisfy the following recurrence formulas, respectively.

\[
\alpha_{i+1} = \alpha_i' + \alpha_i^2, \beta_{i+1} = \beta_i' + \beta_i^2, i = 1, 2, \ldots \tag{5}
\]

Without loss of the generality, we assume that \( a_1 \) is not an exceptional value of \( f^{(k)} \). Suppose \( f^{(k)}(z_0) = a_1 \). Then \( \frac{t}{a_1}(\alpha_k(z_0))\beta_k(z_0) = g^{(k)}(z_0) \in S_2 \). Therefore,

\[
\prod_{j=1}^{m} \left( \frac{t}{a_1} \alpha_k(z_0)\beta_k(z_0) - a_j \right) = 0. \tag{6}
\]

Note that \( \overline{N}(r, 1/(f^{(k)} - a_1)) \neq S(r, f) \). We get

\[
\prod_{j=1}^{m} \left( \frac{t}{a_1} \alpha_k\beta_k - a_j \right) = 0, \tag{7}
\]

which implies that \( \alpha_k\beta_k \) is a nonzero constant. And thus \( \alpha_k \) and \( \beta_k \) have no zeros.

The recurrence formulas in (5) show that

\[
\alpha_k = \alpha_1^k + P(\alpha_1), \quad \beta_k = \beta_1^k + Q(\beta_1), \tag{8}
\]

where \( P(\alpha_1) \) is a differential polynomial in \( \alpha_1 \) of degree \( k - 1 \), and \( Q(\beta_1) \) is a differential polynomial in \( \beta_1 \) of degree \( k - 1 \). If \( \alpha_1 \) and \( \beta_1 \) are not constants, then by Lemma 5, we have

\[
\alpha_k = \left( \alpha_1 + \frac{\gamma_1}{k} \right)^k, \quad \beta_k = \left( \beta_1 + \frac{\gamma_2}{k} \right)^k, \tag{9}
\]
where $\gamma_1, \gamma_2$ are small functions of $\alpha_1$ and $\beta_1$, respectively. Note that $\alpha_1 = -\beta_1 = h'$. We conclude that $\alpha_k\beta_k$ can not be constant, which is a contradiction. Hence one of $\alpha_1$ and $\beta_1$ is constant. Thus $h$ is a linear function. Therefore, $f(z) = de^{cz}$ and $g(z) = \frac{e^c}{d}e^{-cz}$, where $c, d$ are nonzero constants. Now from $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, we get $\{a_1, a_2, \ldots, a_m\} = (-1)^k e^{2\pi k}\{\frac{1}{a_1}, \ldots, \frac{1}{a_m}\}$, which completes the proof of Theorem 1. ■

4. Proof of Theorem 2

Set $F = f^n$, $G = g^n$. From $E_3(S_1, f) = E_3(S_1, g)$, we deduce $E_3(1, F) = E_3(1, G)$. By Lemma 1, we have

$$T(r, F) = nT(r, f) + S(r, f), \quad T(r, G) = nT(r, g) + S(r, g). \quad (10)$$

Assume $H \neq 0$. By Lemma 3, we have

$$T(r, F) + T(r, G) \leq 2 \left( N_2(1, F') + N_2(1, F) + N_2(1, G) \right)$$

$$+ N(3) \left( r, \frac{1}{F - 1} \right) + N(3) \left( r, \frac{1}{G - 1} \right) + S(r, F) + S(r, G). \quad (11)$$

Obviously we have

$$N(3) \left( r, \frac{1}{F - 1} \right) \leq \frac{1}{2} N \left( r, \frac{F'}{F} \right) = \frac{1}{2} N \left( r, \frac{F'}{F} \right) + S(r, f)$$

$$\leq \frac{1}{2} N \left( r, \frac{1}{F} \right) + S(r, f) \leq \frac{1}{2} T(r, f) + S(r, f). \quad (12)$$

Similarly we have

$$N(3) \left( r, \frac{1}{G - 1} \right) \leq \frac{1}{2} T(r, g) + S(r, g). \quad (13)$$

Combining (10), (11), (12) and (13) together we have

$$(n - \frac{9}{2}) T(r, f) + (n - \frac{9}{2}) T(r, g) \leq S(r, f) + S(r, g), \quad (14)$$

which contradicts $n \geq 5$. Thus $H \equiv 0$. By Lemma 4, we have $FG \equiv 1$ or $F \equiv G$, that is $f = tg$ or $fg = t$ where $t$ is a constant and $t^n = 1$. Proceeding as in the proof of Theorem 1, we get the conclusion of Theorem 2. This completes the proof of Theorem 2. ■

5. Some Remarks

From Theorem 2, we know Theorem 1 still holds if we replace $E_3(S_1, f) = E_3(S_1, g)$ by $E_3(S_1, f) = E_3(S_1, g)$. But we do not know whether Theorem 1 and 2 still hold for $n < 5$. We intend to study the question in future work.

Acknowledgement. The author is grateful to the referee for a number of helpful suggestions to improve the paper.
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(received 20.10.2011; in revised form 13.04.2012; available online 10.09.2012)

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