A NEW CLASS OF MEROMORPHIC FUNCTIONS RELATED TO CHO-KWON-SRIVASTAVA OPERATOR

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Abstract. In the present paper, we introduce a new class of meromorphic functions defined by means of the Hadamard product of Cho-Kwon-Srivastava operator and we define here a similar transformation by means of an operator introduced by Ghanim and Darus. We investigate a number of inclusion relationships of this class. We also derive some interesting properties of this class.

1. Introduction

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

which are analytic in the punctured unit disk $U = \{z : 0 < |z| < 1\}$. For $0 \leq \beta$, we denote by $S^*(\beta)$ and $k(\beta)$, the subclasses of $\Sigma$ consisting of all meromorphic functions which are, respectively, starlike of order $\beta$ and convex of order $\beta$ in $U$.

For functions $f_j(z)(j = 1; 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n,$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 \ast f_2) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n.$$

Let us define the function $\tilde{\phi}(\alpha, \beta; z)$ by

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \left| \frac{\alpha}{n+1} \right| z^n,$$

2010 AMS Subject Classification: 30C45, 30C50

Keywords and phrases: Subordination; meromorphic function; Cho-Kwon-Srivastava operator; Choi-Saigo-Srivastava operator; Hadamard product; integral operator.

The work presented here was partially supported by MOHE: UKM-ST-06-FRGS0244-2010.
for \( \beta \neq 0, -1, -2, \ldots \), and \( \alpha \in \mathbb{C}/\{0\} \), where \((\lambda)n = \lambda(\lambda+1)n+1\) is the Pochhammer symbol. We note that
\[
\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} F_1(1, \alpha, \beta; z)
\]
where
\[
F_1(b; \alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{(b)_n (\alpha)_n}{(\beta)_n} \frac{z^n}{n!}
\]
is the well-known Gaussian hypergeometric function.

Let us put
\[
q_{\lambda, \mu}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{\lambda}{n+1+\lambda} \right)^\mu \frac{z^n}{n!} \quad (\lambda > 0, \mu \geq 0).
\]
Corresponding to the functions \( \tilde{\phi}(\alpha, \beta; z) \) and \( q_{\lambda, \mu}(z) \), and using the Hadamard product for \( f(z) \in \Sigma \), we define a new linear operator \( L(\alpha, \beta, \lambda, \mu) \) on \( \Sigma \) by
\[
L(\alpha, \beta, \lambda, \mu) f(z) = f(z) \ast \tilde{\phi}(\alpha, \beta; z) \ast q_{\lambda, \mu}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \left( \frac{\lambda}{n+1+\lambda} \right)^\mu a_n z^n. \tag{1.5}
\]
The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [4,5], Liu [10], Liu and Srivastava [13–15], Cho and Kim [1].

For a function \( f \in L(\alpha, \beta, \lambda, \mu) \) we define
\[
I_{\mu,0}^{\alpha, \beta, \lambda} f(z) = L(\alpha, \beta, \lambda, \mu) f(z)
\]
and for \( k = 1, 2, 3, \ldots \),
\[
I_{\mu,k}^{\alpha, \beta, \lambda} f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{n^k}{n} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \left( \frac{\lambda}{n+1+\lambda} \right)^\mu a_n z^n. \tag{1.6}
\]
Note that if \( n = \beta, k = 0 \) the operator \( I_{\alpha,0, \lambda}^{0,0} \) have been introduced by N.E. Cho, O.S. Kwon and H.M. Srivastava [2] for \( \mu \in \mathbb{N}_0 = \mathbb{N} \cup 0 \). It was known that the definition of the operator \( I_{\alpha,0, \lambda}^{0,0} \) was motivated essentially by the Choi-Saigo-Srivastava operator [3] for analytic functions, which includes a simpler integral operator studied earlier by Noor [17] and others (cf. [11,12,18]). Note also the operator \( I_{\alpha, \beta}^{0,k} \) have been recently introduced and studied by Ghanim and Darus [6–8]. To our best knowledge, the recent work regarding operator \( I_{\alpha,0, \lambda}^{\mu, \beta} \) was charmingly studied by Piejko and Sokól [19]. Moreover, the operator \( I_{\alpha, \beta, \lambda}^{\alpha, \beta} \) was defined and studied by Ghanim and Darus [9]. In the same direction, we will study for the operator \( I_{\alpha, \beta, \lambda}^{\mu, \alpha, \beta} \) given in (1.6).

Now, it follows from (1.5) and (1.6) that
\[
z \left( I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)' = a_{\alpha, \beta, \lambda}^{\mu, k} f(z) - (\alpha + 1) I_{\alpha, \beta, \lambda}^{\mu, k} f(z). \tag{1.7}
\]
Let $\Omega$ be the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in the open unit disk $U = U^* \cup \{0\}$. For functions $f$ and $g$ analytic in $U$, we say that $f$ is subordinate to $g$ and write $f \prec g$, if $g$ is univalent in $U$, $f(0) = g(0)$ and $f(U) \subset g(U)$.

**Definition 1.1.** A function $f \in \Sigma$ is said to be in the class $\Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho; h)$, if it satisfies the subordination condition

\[
(1 + \rho) z \left( I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right) + \rho z^2 \left( I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right)' < h(z) \quad (1.8)
\]

where $\rho$ is a real or complex number and $h(z) \in \Omega$.

Let $A$ be class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.9)
\]

which are analytic in $U$. A function $h(z) \in A$ is said to be in the class $S^*(a)$, if

\[
\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > a \quad (z \in U).
\]

For some $a (a < 1)$. When $0 < a < 1$, $S^*(a)$ is the class of starlike functions of order $a$ in $U$. A function $h(z) \in A$ is said to be prestarlike of order $a$ in $U$, if

\[
\frac{z}{(1-z)^{2(1-a)}} * f(z) \in S^*(a) \quad (a < 1)
\]

where the symbol $*$ means the familiar Hadamard product (or convolution) of two analytic functions in $U$. We denote this class by $R(a)$ (see [20,24]). A function $f(z) \in A$ is in the class $R(0)$, if and only if $f(z)$ is convex univalent in $U$ and

\[
R \left( \frac{1}{2} \right) = S^* \left( \frac{1}{2} \right)
\]

In this paper, we introduce and investigate various inclusion relationships and convolution properties of a certain class of meromorphic functions, which are defined in this paper by means of a linear operator.

**2. Preliminary results**

In order to prove our main results, we need the following lemmas.

**Lemma 2.1.** [16] Let $g(z)$ be analytic in $U$, and $h(z)$ be analytic and convex univalent in $U$ with $h(0) = g(0)$. If,

\[
g(z) + \frac{1}{m} z g'(z) \prec h(z) \quad (2.1)
\]

where $\operatorname{Re} m \geq 0$ and $m \neq 0$, then

\[
g(z) \prec \tilde{h}(z) = mz^{-m} \int_0^z t^{m-1} h(t) \, dt \prec h(z)
\]

and $\tilde{h}(z)$ is the best dominant of (2.1).
Lemma 2.2. [20] Let \( a < 1, f(z) \in S^*(a) \) and \( g(z) \in R(a) \). For any analytic function \( F(z) \) in \( U \), then

\[
\frac{g \ast (fF)}{g \ast f} (U) \subset \overline{co} (F(U)),
\]

where \( \overline{co} (F(U)) \) denotes the convex hull of \( F(U) \).

3. Main results

Theorem 3.1. For some real \( \rho \), let \( 0 \leq \rho_1 < \rho_2 \). Then

\[
\Sigma_{\alpha,\beta}^{\mu,k,\lambda} (\rho_2; h) \subset \Sigma_{\alpha,\beta}^{\mu,k,\lambda} (\rho_1; h)
\]

Proof. Let \( 0 \leq \rho_1 < \rho_2 \) and suppose that

\[
g(z) = z \left( I_{\alpha,\beta,\lambda}^{\mu,k,\lambda} f(z) \right) \quad (3.1)
\]

for \( f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda} (\rho_2; h) \). Then the function \( g(z) \) is analytic in \( U \) with \( g(0) = 1 \). Differentiating both sides of (3.1) with respect to \( z \) and using (1.7), we have

\[
(1 + \rho_2) z \left( I_{\alpha,\beta,\lambda}^{\mu,k,\lambda} f(z) \right) + \rho_2 z^2 \left( I_{\alpha,\beta,\lambda}^{\mu,k,\lambda} f(z) \right)' = g(z) + \rho_2 z g'(z) < h(z). \quad (3.2)
\]

Hence an application of Lemma 2.1 with \( m = \frac{\rho_1}{\rho_2} > 0 \) yields

\[
g(z) < h(z). \quad (3.3)
\]

Noting that \( 0 \leq \frac{\rho_1}{\rho_2} < 1 \) and that \( h(z) \) is convex univalent in \( U \), it follows from (3.1)–(3.3) that

\[
(1 + \rho_1) z \left( I_{\alpha,\beta,\lambda}^{\mu,k,\lambda} f(z) \right) + \rho_1 z^2 \left( I_{\alpha,\beta,\lambda}^{\mu,k,\lambda} f(z) \right)' = \frac{\rho_1}{\rho_2} \left( (1 + \rho_2) z \left( I_{\alpha,\beta,\lambda}^{\mu,k,\lambda} f(z) \right) + \rho_2 z^2 \left( I_{\alpha,\beta,\lambda}^{\mu,k,\lambda} f(z) \right)' \right) + \left( 1 - \frac{\rho_1}{\rho_2} \right) g(z) < h(z).
\]

Thus, \( f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda} (\rho_1; h) \) and the proof of Theorem 3.1 is complete.■

Theorem 3.2. Let,

\[
\text{Re} \left\{ z \tilde{\phi} (\alpha_1, \alpha_2; z) \right\} > \frac{1}{2} \quad (z \in U; \alpha_2 \notin \{0, -1, -2, \ldots \}),
\]

where \( \tilde{\phi} (\alpha_1, \alpha_2; z) \) is defined as in (1.4). Then,

\[
\Sigma_{\alpha_2,\beta}^{\mu,k,\lambda} (\rho; h) \subset \Sigma_{\alpha_1,\beta}^{\mu,k,\lambda} (\rho; h).
\]

Proof. For \( f(z) \in \Sigma \) it is easy to verify that

\[
z \left( I_{\alpha_1,\beta,\lambda}^{\mu,k,\lambda} f(z) \right) = \left( z \tilde{\phi} (\alpha_1, \alpha_2; z) \ast \left( z I_{\alpha_2,\beta,\lambda}^{\mu,k,\lambda} f(z) \right) \right) \quad (3.5)
\]
and
\[ z^2 \left( I_{\alpha_1, \beta, \lambda}^{(k)} f(z) \right)' = \left( z \tilde{\phi}(\alpha_1, \alpha_2; z) \ast z^2 \left( I_{\alpha_2, \beta, \lambda}^{(k)} f(z) \right)' \right). \] (3.6)

Let \( f(z) \in \Sigma_{\alpha_2, \beta}^{(k, \lambda)} (\rho; h) \). Then from (3.5) and (3.6), we deduce that
\[ (1 + \rho) z \left( I_{\alpha_1, \beta, \lambda}^{(k)} f(z) \right) + \rho z^2 \left( I_{\alpha_1, \beta, \lambda}^{(k, \lambda)} f(z) \right)' = \left( z \tilde{\phi}(\alpha_1, \alpha_2; z) \right) * \Psi(z) \] (3.7)
and
\[ \Psi(z) = (1 + \rho) z \left( I_{\alpha_1, \beta, \lambda}^{(k)} f(z) \right) + \rho z^2 \left( I_{\alpha_1, \beta, \lambda}^{(k, \lambda)} f(z) \right)' < h(z) \] (3.8)

In view of (3.4), the function \( z \tilde{\phi}(\alpha_1, \alpha_2; z) \) has the Herglotz representation
\[ z \tilde{\phi}(\alpha_1, \alpha_2; z) = \int_{|x|=1} \frac{dm(x)}{1 - \bar{z}x} (z \in U), \] (3.9)
where \( m(x) \) is a probability measure defined on the unit circle \( |x| = 1 \) and \( \int_{|x|=1} d\tilde{m}(x) = 1 \).

Since \( h(z) \) is convex univalent in \( U \), it follows from (3.7)–(3.9) that
\[ (1 + \rho) z \left( I_{\alpha_1, \beta, \lambda}^{(k)} f(z) \right) + \rho z^2 \left( I_{\alpha_1, \beta, \lambda}^{(k, \lambda)} f(z) \right)' = \int_{|x|=1} \Psi(\bar{z}x) d\tilde{m}(x) < h(z). \]

This shows that \( f(z) \in \Sigma_{\alpha_2, \beta}^{(k, \lambda)} (\rho; h) \) and the theorem is proved. \( \blacksquare \)

**Theorem 3.3** Let \( 0 < \alpha_1 < \alpha_2 \). Then
\[ \Sigma_{\alpha_2, \beta}^{(k, \lambda)} (\rho; h) \subset \Sigma_{\alpha_1, \beta}^{(k, \lambda)} (\rho; h). \]

**Proof.** Define,
\[ g(z) = z + \sum_{n=1}^{\infty} \left\langle \frac{\alpha_1}{\alpha_2}, \frac{n+1}{n+1} \right\rangle |z|^{n+1} (z \in U; 0 < \alpha_1 < \alpha_2). \]

Then,
\[ z^2 \tilde{\phi}(\alpha_1, \alpha_2; z) = g(z) \in A \] (3.10)
where \( \tilde{\phi}(\alpha_1, \alpha_2; z) \) is defined as in (1.4), and
\[ \frac{z}{(1-z)^{\alpha_2}} * g(z) = \frac{z}{(1-z)^{\alpha_1}}. \] (3.11)

By (3.11), we see that
\[ \frac{z}{(1-z)^{\alpha_2}} * g(z) \in S^* \left( 1 - \frac{\alpha_1}{2} \right) \subset S^* \left( 1 - \frac{\alpha_2}{2} \right) \]
for \( 0 < \alpha_1 < \alpha_2 \), which implies that
\[ g(z) \in R \left( 1 - \frac{\alpha_2}{2} \right) \] (3.12)
Let \( f(z) \in \Sigma^\mu_{\alpha,\beta} (\rho; h) \). Then we deduce from (3.7), (3.8) and (3.10) that

\[
(1 + \rho) z \left( I_{\alpha_1,\beta,\lambda}^\mu f(z) \right) + \rho z^2 \left( I_{\alpha_1,\beta,\lambda}^\mu f(z) \right)' = \frac{g(z)}{z} \ast \Psi(z) = \frac{g(z) * (z \Psi(z))}{g(z) * z},
\]

where

\[
\Psi(z) = (1 + \rho) z \left( I_{\alpha_2,\beta,\lambda}^\mu f(z) \right) + \rho z^2 \left( I_{\alpha_2,\beta,\lambda}^\mu f(z) \right)' < h(z).
\]

Since \( z \) belongs to \( S^* \left( 1 - \frac{\alpha z}{2} \right) \) and \( h(z) \) is convex univalent in \( U \), it follows from (3.12)–(3.14) and Lemma 2.2 that

\[
(1 + \rho) z \left( I_{\alpha_1,\beta,\lambda}^\mu f(z) \right) + \rho z^2 \left( I_{\alpha_1,\beta,\lambda}^\mu f(z) \right)' < h(z)
\]

Thus, \( f(z) \in \Sigma^\mu_{\alpha_1,\beta} (\rho; h) \) and the proof is completed. ■

As a special case of Theorem 3.3, we have

\[
\Sigma^\mu_{\alpha+1,\beta} (\rho; h) \subset \Sigma^\mu_{\alpha,\beta} (\rho; h) \quad (\alpha > 0)
\]

In Theorem 3.4 below we give a generalization of the above result.

**Theorem 3.4** Let \( \text{Re} \alpha \geq 0 \) and \( \alpha \neq 0 \). Then,

\[
\Sigma^\mu_{\alpha+1,\beta} (\rho; h) \subset \Sigma^\mu_{\alpha,\beta} \left( \rho; \tilde{h} \right),
\]

where

\[
\tilde{h}(z) = \alpha z^{-\alpha} \int_0^z t^{\alpha-1} h(t) \, dt < h(z).
\]

**Proof.** Let us define

\[
g(z) = (1 + \rho) z \left( I_{\alpha_1,\beta,\lambda}^\mu f(z) \right) + \rho z^2 \left( I_{\alpha_1,\beta,\lambda}^\mu f(z) \right)'
\]

for \( f(z) \in \Sigma \). Then (1.7) and (3.15) lead to

\[
g(z) = \alpha \rho \left( I_{\alpha_1+1,\beta,\lambda}^\mu f(z) \right) + (1 - \alpha \rho) \left( I_{\alpha_1,\beta,\lambda}^\mu f(z) \right).
\]

Differentiating both sides of (3.16) and using (1.7), we obtain the following

\[
g'(z) - \frac{g(z)}{z} = \alpha \rho z \left( I_{\alpha_1+1,\beta,\lambda}^\mu f(z) \right)' \\
+ (1 - \alpha \rho) \left[ \alpha \left( I_{\alpha_1+1,\beta,\lambda}^\mu f(z) \right) - (1 + \alpha) \left( I_{\alpha_1,\beta,\lambda}^\mu f(z) \right) \right].
\]

By (3.16) and (3.17), we get

\[
g'(z) - \frac{\alpha g(z)}{z} = \alpha \rho z \left( I_{\alpha_1+1,\beta,\lambda}^\mu f(z) \right)' + \alpha (1 + \rho) \left( I_{\alpha_1+1,\beta,\lambda}^\mu f(z) \right),
\]

that is,

\[
g(z) + \frac{z g'(z)}{\alpha} = (1 + \rho) z \left( I_{\alpha_1+1,\beta,\lambda}^\mu f(z) \right) + \rho z^2 \left( I_{\alpha_1+1,\beta,\lambda}^\mu f(z) \right)'.
\]
If $f \in \Sigma_{\alpha+1, \beta}^\mu (\rho; h)$, then it follows from (3.18) that
\[
g(z) + \frac{zg'(z)}{\alpha} \prec h(z) \quad \text{(Re} \alpha \geq 0, \alpha \neq 0)\).
\]

Hence an application of Lemma 2.1 yields
\[
g(z) \prec \tilde{h}(z) = az^{-\alpha} \int_0^z t^{\alpha-1} h(t) \, dt \prec h(z),
\]
which shows that
\[
f(z) \in \Sigma_{\alpha, \beta}^{\mu, k, \lambda} (\rho; \tilde{h}) \subset \Sigma_{\alpha, \beta}^{\mu, k, \lambda} (\rho; h)
\]

**Theorem 3.5** Let $\rho > 0, \delta > 0$ and $f(z) \in \Sigma_{\alpha, \beta}^{\mu, k, \lambda} (\rho; \delta h + 1 - \delta)$. If $\delta \leq \delta_0$, where
\[
\delta_0 = \frac{1}{2} \left( 1 - \frac{1}{\rho} \int_0^1 \frac{u^{\frac{1}{\rho}-1}}{1+u} \, du \right)^{-1}
\]
then $f(z) \in \Sigma_{\alpha, \beta}^{\mu, k, \lambda} (\rho; h)$.

**Proof.** Let us define
\[
g(z) = z \left( I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)
\]
for $f(z) \in \Sigma_{\alpha, \beta}^{\mu, k, \lambda} (\rho; \delta h + 1 - \delta)$. Then we have
\[
g(z) + \rho z g'(z) = (1 + \rho) z \left( I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right) + \rho z^2 \left( I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)' \prec \delta (h(z) - 1) + 1
\]

Hence an application of Lemma 2.1 yields
\[
g(z) \prec \frac{\delta}{\rho} z^{-\frac{1}{\rho}} \int_0^z t^{\frac{1}{\rho}-1} h(t) \, dt + 1 - \delta = (h * \Psi)(z),
\]
where
\[
\Psi(z) = \frac{\delta}{\rho} z^{-\frac{1}{\rho}} \int_0^z t^{\frac{1}{\rho}-1} \frac{1}{1-t} \, dt + 1 - \delta
\]
If $0 < \delta \leq \delta_0$, where $\delta_0 > 1$ is given by (3.19), then it follows from (3.22) that
\[
\text{Re} \Psi(z) = \frac{\delta}{\rho} \int_0^1 \frac{u^{\frac{1}{\rho}-1}}{1-u} \, du + 1 - \delta > \frac{\delta}{\rho} \int_0^1 \frac{u^{\frac{1}{\rho}-1}}{1+u} \, du + 1 - \delta \geq \frac{1}{2}
\]
(z $\in U$). Now, by using the Herglotz representation for $\Psi(z)$, from (3.20) and (3.21) we get
\[
z \left( I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right) \prec (h * \Psi)(z) \prec h(z)
\]
because $h(z)$ is convex univalent in $U$. This shows that $f(z) \in \Sigma (\alpha, \beta, k, \rho; h)$. For $h(z) = \frac{1}{1-z}$ and $f(z) \in \Sigma$ defined by
\[
z \left( I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right) = \frac{\delta}{\rho} z^{-\frac{1}{\rho}} \int_0^z t^{\frac{1}{\rho}-1} \, dt + 1 - \delta,
\]
it is easy to verify that
\[(1 + \rho) z \left( I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right) + \rho z^2 \left( I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right)' = \delta (h(z) - 1) + 1\]

Thus, \( f(z) \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k} (\rho; \delta h + 1 - \delta). \) Also, for \( \delta > \delta_0, \) we have
\[
\text{Re } z \left( I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right) \to \frac{\delta}{\rho} \int_{0}^{1} u^{\frac{1}{\rho} - 1} \frac{1}{1 + u} du + 1 - \delta < \frac{1}{2} (z \to -1),
\]

which implies that \( f(z) \notin \Sigma_{\alpha,\beta,\lambda}^{\mu,k} (\rho; h). \) ■

4. Convolution properties

**Theorem 4.1.** Let \( f(z) \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k} (\rho; h), g(z) \in \Sigma \) and \( \text{Re } (zg(z)) > \frac{1}{2} (z \in U). \)

Then,
\[ (f * g)(z) \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k} (\rho; h) \]

**Proof.** For \( f(z) \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k} (\rho; h) \) and \( g \in \Sigma, \) we have
\[
(1 + \rho) z \left( I_{\alpha,\beta,\lambda}^{\mu,k} (f * g)(z) \right) + \rho z^2 \left( I_{\alpha,\beta,\lambda}^{\mu,k} (f * g)(z) \right)' = (1 + \rho) z g(z) * z \left( I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right) + \rho z g(z) * z^2 \left( I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right)' = zg(z) * \Psi(z) \]

where
\[
\Psi(z) = (1 + \rho) z \left( I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right) + \rho z^2 \left( I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right)' < h(z) \quad (4.2)
\]

The remaining part of the proof of Theorem 4.1 is similar to that of Theorem 3.2 and hence we omit it. ■

**Corollary 4.1.** Let \( f(z) \in \Sigma_{\alpha,\beta,\lambda}^{\mu,k} (\rho; h) \) be given by (1.1) and let
\[
\omega_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} a_n z^{n-1} (m \in \mathbb{N} \setminus \{1\}).
\]

Then the function
\[
\sigma_m(z) = \int_{0}^{1} t \omega_m(tz) \, dt
\]
is also in the class \( \Sigma_{\alpha,\beta}^{\mu,k} (\rho; h). \)

**Proof.** We have
\[
\sigma_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} \frac{a_n}{n + 1} z^{n-1} = (f * g_m)(z) \quad (m \in \mathbb{N} \setminus \{1\}), \quad (4.3)
\]
where
\[
f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{n-1} \in \Sigma (\alpha, \beta, k, \rho; h)\]
and
\[ g_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} \frac{z^n}{n+1} \in \Sigma. \]

Also, for \( m \in N \setminus \{1\} \), it is known from \([21]\) that
\[
\Re \{ z g_m(z) \} = \Re \left\{ 1 + \sum_{n=1}^{m-1} \frac{z^n}{n+1} \right\} > \frac{1}{2} \quad (z \in U). \tag{4.4}
\]

In view of (4.3) and (4.4), an application of Theorem 4.1 leads to \( \sigma_m(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h). \)

**Theorem 4.2.** Let \( f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h) \), \( g(z) \in \Sigma \) and \( z^2 g(z) \in R(a) \) (a < 1).

Then,
\[
(f * g)(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h).
\]

**Proof.** For \( f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h) \) and \( g(z) \in \Sigma \), from (4.1) (used in the proof of Theorem 4.1), we can write
\[
(1 + \rho) z \left( I_{\alpha,\beta,\lambda}^{\mu,k} (f * g)(z) \right) + \rho z^2 \left( I_{\alpha,\beta,\lambda}^{\mu,k} (f * g)(z) \right)'
= \frac{z^2 g(z) * z \Psi(z)}{z^2 g(z) * z} (z \in U), \tag{4.5}
\]
where \( \Psi(z) \) is defined as in (4.2).

Since \( h(z) \) is convex univalent in \( U \), \( \Psi(z) \prec h(z) \), \( z^2 g(z) \in R(a) \) and \( z \in S^*(a) \) (a < 1), the desired result follows from (4.5) and Lemma 2.2.

Taking \( a = 0 \) and \( a = \frac{1}{2} \), Theorem 4.2 reduces to the following.

**Corollary 4.2.** Let \( f(z) \in \Sigma(\alpha,\beta,k,\rho;h) \) and let \( g(z) \in \Sigma \) satisfy either of the following conditions
(i) \( z^2 g(z) \) is convex univalent in \( U \) or
(ii) \( z^2 g(z) \in S^*(\frac{1}{2}) \).

Then, \( (f * g)(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h) \).

**REFERENCES**


(received 19.11.2011; in revised form 24.04.2012; available online 10.09.2012)

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