ON THE INVERTIBILITY OF $AA^+ - A^+A$ IN A HILBERT SPACE

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Abstract. Let $H$ be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. In this paper, we study the class of operators $A \in B(H)$ with closed range such that $AA^+ - A^+A$ is invertible, where $A^+$ is the Moore-Penrose inverse of $A$. Also, we present new relations between $(AA^* + A^*A)^{-1}$ and $(A + A^*)^{-1}$. The present paper is an extension of results from [J. Benítez and V. Rakočević, Appl. Math. Comput. 217 (2010) 3493–3503] to infinite-dimensional Hilbert space.

1. Introduction

Let $H$ be a Hilbert space and $B(H)$ be the set of all bounded linear operators on $H$. Throughout this paper, the range, the null space and the adjoint of $A \in B(H)$ are denoted by $N(A)$, $R(A)$ and $A^*$, respectively. An operator $A \in B(H)$ is said to be positive if $(Ax, x) \geq 0$. An operator $P \in B(H)$ is said to be idempotent if $P^2 = P$. An orthogonal projection is a self-adjoint idempotent. Clearly, any orthogonal projection is positive. For $A \in B(H)$, if there exists an operator $A^+ \in B(H)$ satisfying the following four operator equations:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad AA^+ = (AA^+)^*, \quad A^+A = (A^+A)^*,$$

then $A^+$ is called the Moore-Penrose inverse (for short, MP inverse) of $A$. It is well known that $A$ has the MP inverse if and only if $R(A)$ is closed, the MP inverse of $A$ is unique [5]. It is easy to see that $R(A^+) = R(A^*)$, $AA^+$ is the orthogonal projection of $H$ onto $R(A)$ and that $A^+A$ is the orthogonal projection of $H$ onto $R(A^*)$. $A \in B(H)$ is said to be an EP operator, if $R(A)$ is closed and $AA^+ = A^+A$ (see [1,7]). If $A$ is an EP operator, then $AA^+ - A^+A$ is not invertible.

In this paper we study the class of operators $A \in B(H)$ with closed range, such that $AA^+ - A^+A$ is invertible. Since $AA^+$ and $A^+A$ are orthogonal projections, the question of invertibility of $AA^+ - A^+A$ is strongly related to the invertibility of the difference $P - Q$, where $P, Q$ are orthogonal projections on a Hilbert space.

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Buckholtz [3,4] has proved that the operator $P - Q$ is invertible if and only if $H$ is the direct sum $H = R(P) \oplus R(Q)$ of the ranges of $P$ and $Q$. In this case there exists a linear idempotent $M$ with range $R(P)$, kernel $R(Q)$ and $(P - Q)^{-1} = M + M^* - I$ (see [11,12,13] for further references).

Recently, J. Benítez and V. Rakocević (see [2]) obtained interesting results concerning the nonsingularity of $AA^+ - A^+ A$, where $A$ is a square matrix. Notice that in [2] the finite-dimensional methods are mostly based on the CS decomposition and on the rank of a complex matrix. In the present paper we extend results obtained in [2] to infinite-dimensional Hilbert space.

2. Preliminary results

In this section, we present some Lemmas, needed in the sequel.

**Lemma 2.1.** [9] Let $A$ and $B$ be in $B(H)$. Then the following statements hold:
(i) $R(A)$ is closed if and only if $R(A) = R(AA^*)$,
(ii) $R(A)$ is closed if and only if $R(A^*)$ is closed,
(iii) $R(A) = R(AA^*)^\frac{1}{2}$,
(iv) $R(A) + R(B) = R((AA^* + BB^*)^\frac{1}{2})$.

**Lemma 2.1.** [6,8] Let $A \in B(H)$ be a positive operator. Then the following statements hold:
(i) $R(A) \subseteq R(A^\frac{1}{2})$ and $\overline{R(A)} = \overline{R(A^\frac{1}{2})}$, where $\overline{K}$ denotes the closure of $K$,
(ii) $R(A)$ is closed if and only if $R(A) = R(A^\frac{1}{2})$,
(iii) $R(A) = H$ if and only if $A$ is invertible.

**Lemma 2.1.** [10] If $P \in B(H)$ is an idempotent and $\|P\| \leq 1$, then $P$ is an orthogonal projection.

3. Main results

In this section we find several equivalent conditions that ensure the invertibility of $AA^+ - A^+ A$, where $A \in B(H)$ has the closed range.

**Theorem 3.1.** If $A \in B(H)$ have closed range, then the following statements are equivalent:
(i) $AA^+ - A^+ A$ is invertible,
(ii) $R(A) \oplus R(A^*) = H$,
(iii) There exists a bounded linear idempotent $P$ with range $N(A^*)$ and kernel $N(A)$,
(iv) $AA^+ + A^+ A$ is invertible and $\|A(A^+)^2A\| < 1$,
(v) $AA^* + A^* A$ is invertible and $R(A) \cap R(A^*) = \{0\}$,
(vi) $AA^* - A^* A$ is invertible and $R(A) \cap R(A^*) = \{0\}$. 
Proof. Since $AA^+$ and $A^+A$ are orthogonal projections onto $R(A)$ and $R(A^+)$ respectively, then the equivalence of (i), (ii) and (iv) follows from [4].

(ii)$\Leftrightarrow$(iii). Assume first that $R(A) \oplus R(A^*) = H$. Then, there exists a bounded linear idempotent $M$ in $B(H)$ such that $R(M) = R(A)$ and $N(M) = R(A^*)$.

Let us define $P = I - M^*$. Then $P$ is an idempotent with range $N(M^*)$ and Kernel $R(M^*)$. By using relations $N(B^*) = R(B)^\perp$ and $R(B^*) = N(B)^\perp$, which are valid for closed range operators $B \in B(H)$, we get $R(P) = N(A^*)$ and $N(P) = N(A)$.

Conversely, if $P$ is an idempotent with range $N(A^*)$ and kernel $N(A)$, then $I - P^*$ is idempotent with range $R(A)$ and kernel $R(A^*)$. According to the space decomposition $H = R(I - P^*) \oplus N(I - P^*)$, we obtain (ii).

(ii)$\Leftrightarrow$(v). Using Lemma 2.1, we obtain $R((AA^* + A^*A)^{1/2}) = R(A) + R(A^*)$. Since $(AA^* + A^*A)^{1/2}$ is a positive operator, it follows from Lemma 2.2, that $R(A) + R(A^*) = H$ if and only if $(AA^* + A^*A)^{1/2}$ is invertible, so $AA^* + A^*A$ is invertible. Hence, (ii)$\Leftrightarrow$(v).

(v)$\Rightarrow$(vi). Assume that (v) holds. By the equivalence (v)$\Leftrightarrow$(iii), there exists an idempotent $P$ such that $R(P) = N(A^*)$ and $N(P) = N(A)$. This implies $A^*P = 0$ and $A(I - P) = 0$. Hence, $AP = A$ and $P^*A = 0$.

Then we easily obtain $(AA^* + A^*A)(I - 2P) = (AA^* - A^*A)$. Since $I - 2P$ is invertible (because $(I - 2P)^2 = I$), we get that $AA^* - A^*A$ is invertible.

(vi)$\Rightarrow$(v). Suppose that (vi) holds. From the invertibility of $AA^* - A^*A$, we deduce $H = R(AA^* - A^*A) = R(A) + R(A^*)$. According to Lemmas 2.1 and 2.2, $(AA^* + A^*A)^{1/2}$ is invertible. Hence $AA^* + A^*A$ is invertible. \hfill \blacksquare

Remark 3.2. If $P$ is the idempotent given by Theorem 3.1, then from the proof of (ii)$\Leftrightarrow$(iii), we deduce

\[ A^+AP = A^+A, \quad AA^+P = 0, \quad AA^+(I - P^*) = I - P^*, \quad A^+AP^* = P^*. \]

Using these results, we obtain

\[ (AA^+ - A^+A)(I - P - P^*) = I - P^* + P^* = I. \]

Taking the adjoint, we get

\[ (I - P - P^*)(AA^+ - A^+A) = I. \]

Hence, $(AA^+ - A^+A)^{-1} = I - P - P^*$.

From Theorem 3.1, we see that if $AA^+ - A^+A$ is invertible, then $AA^* + A^*A$ is invertible. In the following example we show that the converse is not true.

Example 3.3. Consider the real Hilbert space $\ell_2$ and let $A \in B(\ell_2)$ be the left shift, i.e. $A(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$, then $A^*(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ and $A^+ = A^*$. In this case $AA^+ = I$ and $A^+A(x_1, x_2, \ldots) = (0, x_2, x_3, \ldots)$. Then, $AA^* + A^*A$ is invertible and $AA^+ - A^+A$ is not injective. Hence $AA^+ - A^+A$ is not invertible.
Theorem 3.4. Let \( A \in B(H) \) have closed range, then the following statements are equivalent:

(i) \( AA^+ - A^+A \) is invertible,

(ii) \( A + A^* \) is invertible and there exists an idempotent \( P \in B(H) \) such that \( AP = A \) and \( P^*A = 0 \),

(iii) \( A - A^* \) is invertible and there exists an idempotent \( P \in B(H) \) such that \( AP = A \) and \( P^*A = 0 \),

(iv) \( A + A^* \) is invertible, \( A(A + A^*)^{-1}A = A \) and \( A^*(A + A^*)^{-1}A = 0 \),

(v) \( A - A^* \) is invertible, \( A(A - A^*)^{-1}A = A \) and \( A^*(A - A^*)^{-1}A = 0 \).

Proof. (i)⇒(ii). Since \( AA^+ - A^+A \) is invertible, by the proof of Theorem 3.1,(v), there exist an idempotent \( P \in B(H) \), such that \( AP = A \) and \( P^*A = 0 \). Then it is easy to check that

\[
(A + A^*)(I - P - P^*)(A + A^*) = A^*A - AA^*.
\]

We conclude that \( R(A^*A - AA^*) \subset R(A + A^*) \) and \( N(A + A^*) \subset N(A^*A - AA^*) \).

(ii)⇒(i). If \( A + A^* \) is invertible, we easily obtain \( R(A) + R(A^*) = H \). On the other hand, if \( P \in B(H) \) is an idempotent, such that \( AP = A \) and \( P^*A = 0 \), then \( R(A^*) \subset R(P^*) \) and \( R(A) \subset N(P^*) \). Since \( R(P^*) \cap N(P^*) = \{0\} \), we obtain \( R(A) \cap R(A^*) = \{0\} \). Consequently, \( R(A) \oplus R(A^*) = H \). Thus, by Theorem 3.1, \( AA^+ - A^+A \) is invertible.

(ii)⇔(iii). Suppose that \( P \in B(H) \) is idempotent such that \( AP = A \) and \( P^*A = 0 \), then \( (A + A^*)(2P - I) = A - A^* \). Since \( 2P - I \) is invertible, then \( A + A^* \) is invertible if and only if \( A - A^* \) is invertible. Hence, (ii)⇔(iii).

(ii)⇒(iv). From \( AP = A \) and \( P^*A = 0 \), we have \( (A + A^*)P = A \). Since \( A + A^* \) is invertible, then \( P = (A + A^*)^{-1}A \). This implies \( A(A + A^*)^{-1}A = AP = A \) and \( A^*(A + A^*)^{-1}A = A^*P = 0 \).

(iv)⇒(ii). Let us define \( P = (A + A^*)^{-1}A \). From \( A(A + A^*)^{-1}A = A \) and \( A^*(A + A^*)^{-1}A = 0 \), we easily obtain \( P^2 = P, AP = A \) and \( A^*P = 0 \).

The proof of (iii)⇔(v) works in the same way as in (ii)⇔(iv).

Remark 3.5. The existence of the idempotent \( P \), such that \( AP = A \) and \( P^*A = 0 \) is necessary for the invertibility of \( AA^+ - A^+A \); for example, let \( A \in B(H) \) be self-adjoint invertible, then \( A + A^* \) is invertible, but \( AA^+ - A^+A = 0 \).

Corollary 3.6. Let \( A \in B(H) \) have closed range. If \( AA^+ - A^+A \) is invertible, then the idempotent \( P \) given by Theorem 3.4 is unique and \( R(P) = N(A^*) \) and \( N(P) = N(A) \).

Proof. Let \( P \) be the idempotent given in Theorem 3.4. From the proof of Theorem 3.4, (ii)⇒(iv), we get \( P = (A + A^*)^{-1}A \). This proves the uniqueness of the idempotent \( P \) and the equality \( N(P) = N(A) \).
Now, we prove that \( R(P) = N(A^*) \). From \( A^*P = (P^*A)^* = 0 \), we get the inclusion \( R(P) \subseteq N(A^*) \). To prove the reverse inclusion we first, observe that

\[
\]

Hence, we get

\[
\]

From \( I - AA^+ = P(I - AA^+) \), we obtain \( R(I - AA^+) \subseteq R(P) \). Since \( R(I - AA^+) = N(AA^+) = N(A^*) \), Then \( N(A^*) \subseteq R(P) \). Consequently, \( R(P) = N(A^*) \). ■

**Corollary 3.7.** Let \( A \in B(H) \) have closed range. If \( AA^+ - A^+A \) is invertible. Then

(i) \((AA^+ - A^+A)^{-1} = (A + A^*)^{-1} (A^* A - AA^*) (A + A^*)^{-1}\),

(ii) \((AA^+ - A^+A)^{-1} = (A - A^*)^{-1} (AA^+ - A^*A) (A - A^*)^{-1}\).

**Proof.** Let \( P \) be the idempotent given by Theorem 2.1.

(i) From the proof of Theorem 3.4, (i)⇒(ii), we get

\[
(A + A^*)(I - P - P^*)(A + A^*) = A^* A - AA^*. 
\]

Using the equality \( I - P - P^* = (AA^+ - A^+A)^{-1} \) and the invertibility of \( A + A^* \) (guaranteed by Theorem 3.4), we deduce the equality (i).

(ii) From \( AP = A \) and \( P^*A = 0 \), we get

\[
(A - A^*)(I - P - P^*)(A - A^*) = AA^* - A^* A. 
\]

The rest of the proof of (ii) is similar to the proof of (i). ■

**Theorem 3.8.** Let \( A \in B(H) \) have closed range, then the following statements are equivalent:

(i) \( AA^+ - A^+A \) is invertible,

(ii) \( AA^+ + A^+A \) is invertible and \( A^* A (AA^+ + A^+A)^{-1} A^* A = A^* A \),

(iii) \( AA^+ - A^+A \) is invertible and \( A^* A (AA^+ + A^+A)^{-1} A^* A = A^* A \).

**Proof.** (i)⇒(ii). Assume that (i) holds. Using Theorems 3.1 and 3.4, we get \( AA^+ + A^+A \) is invertible and there exists an idempotent \( P \in B(H) \), such that \( AP = A \) and \( P^*A = 0 \). Then \((AA^* + A^*A)P = A^* A \), which implies \( P = (AA^* + A^*A)^{-1} A^* A \). Hence \( A^* A (AA^* + A^*A)^{-1} A^* A = A^* A \).

(ii)⇒(i). Assume that (ii) holds. Let \( P = (AA^* + A^*A)^{-1} A^* A \). From hypotheses, it is easy to get \( P^2 = P \) and \( N(P) = N(A^* A) \). Since \( N(A^* A) = N(A) \) (see Lemma 2.1), then \( N(P) = N(A) \).

On the other hand, we have:

\[
\]
Hence, \( AA^*P = 0 \). So \( R(P) \subset N(A^*) \). Since \( R(P) + N(P) = H \), we get \( N(A) + N(A^*) = H \). This implies \( N(A) \perp N(A^*) = \{0\} \). Therefore \( R(A) \cap R(A^*) = \{0\} \).

Using Theorem 3.1 (v), we obtain \( AA^* - A^+A \) is invertible.

(i)\( \Leftrightarrow \) (iii). This is similar as (i)\( \Rightarrow \) (ii) and (ii)\( \Rightarrow \) (i).

From the above proof and Theorem 3.4, we obtain the following corollary.

**Corollary 3.9.** Let \( A \in B(H) \) with closed range, such that \( AA^* - A^+A \) is invertible. If \( P \) is the idempotent given by Theorem 3.1, then

(i) \( P = (AA^* + A^*A)^{-1}A^*A = (A^*A - AA^*)^{-1}A^*A = (A + A^*)^{-1}A = (A - A^*)^{-1}A \),

(ii) \( A(AA^* + A^*A)^{-1}A^*A = A(A^*A - AA^*)^{-1}A^*A = A \),

(iii) \( A^*(AA^* + A^*A)^{-1}A^*A = A^*(A^*A - AA^*)^{-1}A^*A = 0 \).

As we have seen in Theorem 3.1, \( AA^* - A^+A \) is invertible if and only if \( R(A) \oplus R(A^*) = H \). But what happens if \( H \) is the orthogonal direct sum \( R(A) \oplus R(A^*) = H \) of the ranges of \( A \) and \( A^* \)?

In the next result we study the class of operators \( A \) with closed range such that \( R(A)^{\perp} = R(A^*) \).

**Theorem 3.10.** Let \( A \in B(H) \) have closed range, then the following statements are equivalent:

(i) \( R(A)^{\perp} \oplus R(A^*) = H \),

(ii) \( AA^* + A^+A = I \),

(iii) \( (AA^* - A^+A)^2 = I \),

(iv) \( A + A^* \) is invertible and there exists a unique orthogonal projection \( P \) such that \( AP = A \) and \( PA = 0 \),

(v) \( A - A^* \) is invertible and there exists a unique orthogonal projection \( P \) such that \( AP = A \) and \( PA = 0 \).

**Proof.** (i)\( \Leftrightarrow \) (ii). It is well know that \( R(A)^{\perp} \oplus R(A^*) = H \) if and only if \( R(A)^{\perp} = R(A^*) \). Since \( AA^* \) and \( A^+A \) are orthogonal projections onto \( R(A) \) and \( R(A^*) \) respectively, then \( R(A)^{\perp} = R(A^*) \) if and only if \( A^+A = I - AA^* \). So that \( AA^* + A^+A = I \). Hence, (i)\( \Leftrightarrow \) (ii).

(ii)\( \Leftrightarrow \) (iii). Let us first define the orthogonal projections \( P_1 = AA^* \) and \( P_2 = A^+A \). If \( P_1 + P_2 = I \), then \( P_1P_2 = P_1(I - P_1) = 0 \) and \( P_2P_1 = P_2(I - P_2) = 0 \). Hence \( (P_1 - P_2)^2 = P_1 + P_2 = I \).

Conversely, if \( (P_1 - P_2)^2 = I \), then \( P_1 + P_2 = P_1P_2 - P_2P_1 = I \). Multiplying the previous equality by \( P_1 \) from the left side, we get \( P_1P_2P_1 = 0 \). So that \( (P_2P_1)^*P_2P_1 = 0 \). This is equivalent to \( P_1P_2 = P_2P_1 = 0 \). Thus \( P_1 + P_2 = I \).

(iii)\( \Leftrightarrow \) (iv). Assume that (iii) holds. Then \( AA^* - A^+A \) is invertible and \( (AA^* - A^+A)^{-1} = AA^* - A^+A \). By theorem 3.4, \( A + A^* \) is invertible and there exists a unique idempotent \( P \in B(H) \) such that \( AP = A \) and \( P^*A = 0 \). It follows from Remark 3.2, that \( I - P - P^* = AA^* - A^+A \). Multiplying the previous
equality by $P^*$ from the left side, we get $P^*P = P^*A^+A = (A^+AP)^* = A^+A$. Hence, $\|P\| = (\|P^*P\|)^{\frac{1}{2}} = 1$. According to Lemma 2.3, we conclude that $P$ is an orthogonal projection ($P = P^*$), which satisfies $AP = A$ and $PA = 0$.

(iv)$\Rightarrow$(i). Suppose that (iv) holds. From the invertibility of $A + A^*$, we deduce that $R(A) + R(A^*) = H$.

Now, we prove that $R(A) \perp R(A^*)$. From $AP = A$ and $PA = 0$, we get $A^2 = 0$. So $R(A) \subset N(A)$. Since $N(A) = R(A^*)^\perp$, we conclude that $R(A) \perp R(A^*)$.

(iv)$\Leftrightarrow$(v). This equivalence can be proved in a similar way as (ii)$\Leftrightarrow$(iii), Theorem 3.4.

**Corollary 3.11.** Let $A \in B(H)$ have closed range. If any item in Theorem 3.10, is satisfied, then

(i) $A^+ = (A + A^*)^{-1}A(A + A^*)^{-1}$,

(ii) $A^+ = (A - A^*)^{-1}A(A - A^*)^{-1}$,

(iii) $A^+ + (A^*)^* = (A + A^*)^{-1}$,

(iv) $A^+ - (A^*)^* = (A - A^*)^{-1}$,

(v) $A^+ = \frac{1}{2}[(A + A^*)^{-1} + (A - A^*)^{-1}]$.

**Proof.** (i). By the proof of Theorem 3.10, (iv)$\Rightarrow$(i), we get $A^2 = 0$. Then $(A + A^*)A^+A = A$.

Since $A + A^*$ is invertible, then it is easy to check that

$$A^+ = (A^+A)^+(A + A^*)^{-1} = A^+A(A + A^*)^{-1}$$

By using $A^+A = (A + A^*)^{-1}A$, we obtain (i).

The proof of (ii) is similar to that of (i).

The proof of the remaining statements follows immediately from (i) and (ii) of this corollary.

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**References**


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