ON \((f,g)\)-DERIVATIONS OF \(B\)-ALGEBRAS

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Abstract. In this paper, as a generalization of derivation of a \(B\)-algebra, we introduce the notion of \(f\)-derivation and \((f,g)\)-derivation of a \(B\)-algebra. Also, some properties of \((f,g)\)-derivation of commutative \(B\)-algebra are investigated.

1. Introduction and preliminaries

Y. Imai and K. Iséki introduced two classes of abstract algebras: \(BCK\)-algebras and \(BCI\)-algebras [7, 8]. It is known that the class of \(BCK\)-algebras is a proper subclass of the class \(BCI\)-algebras. In [5, 6], Q. P. Hu and X. Li introduced a wide class of abstract algebras, \(BCH\)-algebras. They have shown that the class of \(BCI\)-algebras is a proper subclass of \(BCH\)-algebras. In [9], Y. B. Jun, E. H. Roh and H.S. Kim introduced the notion of \(BH\)-algebras, which is a generalization of \(BCH/BCI/BCK\)-algebras. Recently, J. Neggers and H. S. Kim introduced in [12] a new notion, called a \(B\)-algebra. This class of algebras is related to several classes of interest such as \(BCH/BCI/BCK\)-algebras. In [1], N. O. Al-Shehrie introduced the notion of derivation in \(B\)-algebras which is defined in a way similar to the notion in ring theory (see [2, 3, 10, 15]) and investigated some properties related to this concept.

In this paper, we introduce the notions of \(f\)-derivation and \((f,g)\)-derivation of a \(B\)-algebra and some related are explored. Also, using the concept of derivation of commutative \(B\)-algebra we investigate some of its properties.

We recall the notion of a \(B\)-algebra and review some properties which we will need in the next section.

A \(B\)-algebra [12] is a non-empty set \(X\) with a constant 0 and a binary operation \(*\) satisfying the following conditions, for all \(x, y, z \in X\): (B1) \(x \ast x = 0\); (B2) \(x \ast 0 = x\); (B3) \((x \ast y) \ast z = x \ast (y \ast (0 \ast y))\). A \(B\)-algebra \((X, \ast, 0)\) is said to be commutative [12] if \(x \ast (0 \ast y) = y \ast (0 \ast x)\), for all \(x, y \in X\).

In any \(B\)-algebra \(X\), the following properties are valid, for all \(x, y, z \in X\) [4, 12]: (1) \((x \ast y) \ast (0 \ast y) = x\); (2) \(x \ast (y \ast z) = (x \ast (0 \ast z)) \ast y\); (3) \(x \ast y = 0\) implies

\[2010\text{ AMS Subject Classification: } 06F35, 16B70, 16W25.\]

\textit{Keywords and phrases:} Commutative \(B\)-algebra; \(f\)-derivation; \((f,g)\)-derivation.
that \( x = y; \) (4) \( 0 \ast (0 \ast x) = x; \) (5) \( (x \ast z) \ast (y \ast z) = x \ast y; \) (6) \( 0 \ast (x \ast y) = y \ast x; \) (7) \( x \ast z = y \ast z \) implies that \( x = y \) (right cancelation law); (8) \( z \ast x = z \ast y \) implies that \( x = y \) (left cancelation law). Moreover, if \( X \) is a commutative \( B \)-algebra, according to [11] we have: (9) \( (0 \ast x) \ast (0 \ast y) = y \ast x; \) (10) \( (z \ast y) \ast (z \ast x) = x \ast y; \) (11) \( (x \ast y) \ast z = (x \ast z) \ast y; \) (12) \( (x \ast (x \ast y)) \ast y = 0; \) (13) \( (x \ast z) \ast (y \ast t) = (t \ast z) \ast (y \ast x). \)

For a \( B \)-algebra \( X \), one can define binary operation “\( \wedge \)" as \( x \wedge y = y \ast (y \ast x) \), for all \( x, y \in X \). If \( (X, \ast, 0) \) is a commutative \( B \)-algebra, then by (12) and (3), we get \( y \ast (y \ast x) = x \), for all \( x, y \in X \) that means \( x \wedge y = x \).

A mapping \( f \) of a \( B \)-algebra \( X \) in to itself is called an \emph{endomorphism} of \( X \) if \( f(x \ast y) = f(x) \ast f(y) \), for all \( x, y \in X \). Note that \( f(0) = 0 \).

Let \( (X, \ast, +, 0) \) be an algebra of type \( (2, 2, 0) \) satisfying \( B1, B2, B3 \) and \( B4 : x + y = x \ast (0 \ast y) \), for all \( x, y \in X \). Then, \( (X, \ast, 0) \) is a \( B \)-algebra. Conversely, if \( (X, \ast, 0) \) be a \( B \)-algebra and we define \( x + y \) by \( x \ast (0 \ast y) \), for all \( x, y \in X \), then \( (X, \ast, +, 0) \) obeys the equations \( B1 - B4 \) (see [15]).

\textbf{2. \((f, g)\)-derivation of \( B \)-algebras}

In this section, we introduce the notion of \( f \)-derivation and \((f, g)\)-derivation of \( B \)-algebras.

\textbf{Definition 1.} [1] Let \( X \) be a \( B \)-algebra. By a \emph{left-right derivation} (briefly, \((l, r)\)-derivation) of \( X \), a self map \( d \) of \( X \) satisfying the identity \( d(x \ast y) = (d(x) \ast y) \wedge (x \ast d(y)) \), for all \( x, y \in X \). If \( d \) satisfies the identity \( d(x \ast y) = (x \ast d(y)) \wedge (d(x) \ast y) \), for all \( x, y \in X \), then it is said that \( d \) is a \emph{right-left derivation} (briefly, \((r, l)\)-derivation) of \( X \). Moreover, if \( d \) is both an \((l, r)\)- and \((r, l)\)-derivation, it is said that \( d \) is a \emph{derivation}.

\textbf{Definition 2.} [1] A self map \( d \) of a \( B \)-algebra \( X \) is said to be \emph{regular} if \( d(0) = 0 \). If \( d(0) \neq 0 \), then \( d \) is called an \emph{irregular} map.

\textbf{Definition 3.} Let \( X \) be a \( B \)-algebra. A \emph{left-right \( f \)-derivation} (briefly, \((l, r)\)-\( f \)-derivation) of \( X \) is a self map \( d \) of \( X \) satisfying the identity \( d(x \ast y) = (d(x) \ast f(y)) \wedge (f(x) \ast d(y)) \), for all \( x, y \in X \), where \( f \) is an endomorphism of \( X \). If \( d \) satisfies the identity \( d(x \ast y) = (f(x) \ast d(y)) \wedge (d(x) \ast f(y)) \), for all \( x, y \in X \), then we say \( d \) is a \emph{right-left \( f \)-derivation} (briefly, \((r, l)\)-\( f \)-derivation) of \( X \). Moreover, if \( d \) is both an \((l, r)\)- and \((r, l)\)-\( f \)-derivation, we say \( d \) is an \emph{\( f \)-derivation}.

\textbf{Example 1.} Let \( X = \{0, 1, 2\} \) and the binary operation \( \ast \) is defined as follows:

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Then, \((X, * , 0)\) is a \(B\)-algebra (see [12]). Define the map \(d, f : X \rightarrow X\) by
\[
d(x) = f(x) = \begin{cases} 
0 & \text{if } x = 0 \\
2 & \text{if } x = 1 \\
1 & \text{if } x = 2.
\end{cases}
\]

Then, \(f\) is an endomorphism. It is easily to check that \(d\) is both \((l, r)\)- and \((r, l)\)-\(f\)-derivation of \(X\). So \(d\) is an \(f\)-derivation. Now, we define \(d' = 0\). Then, \(d'\) is not an \((l, r)\)-\(f\)-derivation, since \(d'(1 * 2) = 0\) but \((d'(1) * f(2)) \land (f(1) * d'(2)) = (0 * 1) \land (2 * 0) = 2\). Also, \(d'\) is not an \((r, l)\)-\(f\)-derivation, since \(d'(1 * 2) = 0\) but \((f(1) * d'(2)) \land (d'(1) * f(2)) = (2 * 0) \land (0 * 1) = 2\).

**Theorem 1.** Let \(d\) be an \((l, r)\)-\(f\)-derivation of \(B\)-algebra \(X\). Then, \(d(0) = d(x) * f(x)\), for all \(x \in X\).

**Proof.** For all \(x \in X\), we have:
\[
d(0) = d(x * x) = (d(x) * f(x)) \land (f(x) * d(x)) = (f(x) * d(x)) \land (d(x) * f(x))
\]
\[
= (0 * (d(x) * f(x))) = (d(x) * f(x)) = 0 * (f(x) * d(x)) = d(x) * f(x).
\]

**Theorem 2.** Let \(d\) be an \((r, l)\)-\(f\)-derivation of \(B\)-algebra \(X\). Then, \(d(0) = f(x) * d(x)\) and \(d(x) = d(x) \land f(x)\), for all \(x \in X\).

**Proof.** For all \(x \in X\), we have:
\[
d(0) = d(x * x) = (f(x) * d(x)) \land (d(x) * f(x)) = (f(x) * d(x)) \land (d(x) * f(x)) = (d(x) * f(x)) \land (d(x) * f(x)) = 0 * (d(x) * f(x)) = f(x) * d(x).
\]

Also, we have for all \(x \in X\),
\[
d(x) * 0 = d(x) = d(x * 0) = (f(x) * d(0)) \land (d(x) * f(0)) = d(x) \land (d(x) \land (f(x) \land d(0))) = d(x) \land (d(x) \land (f(x) \land d(x))).
\]

By (8) and (3), we get \(d(x) \land f(x) = d(x)\).

**Corollary 1.** Let \(d\) be an \((l, r)\)-\(f\)-derivation \((r, l)\)-\(f\)-derivation) of \(B\)-algebra \(X\). Then, (1) \(d\) is injective if and only if \(f\) be injective; (2) If \(d\) is regular, then \(d = f\); (3) If there is an element \(x_0 \in X\) such that \(d(x_0) = f(x_0)\), then \(d = f\).

**Proof.** Let \(d\) be an \((l, r)\)-\(f\)-derivation.
(1) Suppose that $d$ is injective and $f(x) = f(y)$, $x, y \in X$. Then, $d(0) = d(x) * f(x)$ and $d(0) = d(y) * f(y)$, by Theorem 1. So, $d(x) * f(x) = d(y) * f(y)$. Thus, $d(x) = d(y)$, by (7). Therefore, $x = y$, since $d$ is injective.

Conversely, suppose that $f$ is injective and $d(x) = d(y)$, $x, y \in X$. Then, $d(0) = d(x) * f(x)$ and $d(0) = d(y) * f(y)$, by Theorem 1. So, $d(x) * f(x) = d(y) * f(y)$. Thus $f(x) = f(y)$, by (8). Therefore $x = y$, since $f$ is injective.

(2) Suppose that $d$ is regular and $x \in X$. Then $0 = d(0) = d(x) * f(x)$, by Theorem 1. Hence, $d(x) = f(x)$, by (3).

(3) Suppose that there is an element $x_0 \in X$ such that $d(x_0) = f(x_0)$. Then, $d(x_0) * f(x_0) = 0$. So, $d(0) = 0$, by Theorem 1. Part (2) implies that $d = f$.

Similarly, when $d$ is an $(r, l)$- $f$-derivation, the proof follows by Theorem 2. $$

\textbf{Definition 4.} Let $X$ be a $B$-algebra. A \textit{left-right $(f, g)$-derivation} (briefly, $(l, r)$- $(f, g)$-derivation) of $X$ is a self map $d$ of $X$ satisfying the identity $d(x * y) = (d(x) * f(y)) \land (g(x) * d(y))$, for all $x, y \in X$, where $f, g$ are endomorphisms of $X$. If $d$ satisfies the identity $d(x * y) = (f(x) * d(y)) \land (d(x) * g(y))$, for all $x, y \in X$, then we say $d$ is a \textit{right-left $(f, g)$-derivation} (briefly, $(r, l)$- $(f, g)$-derivation) of $X$. Moreover, if $d$ is both an $(l, r)$- and $(r, l)$- $(f, g)$-derivation, then $d$ is a $(f, g)$-derivation.

It is clear that if the function $g$ is equal to the function $f$, then the $(f, g)$-derivation is $f$-derivation defined in Definition 3. Also, if we choose the functions $f$ and $g$ the identity functions, then the $(f, g)$-derivation that we define coincides with the derivation defined in Definition 1.

\textbf{Example 2.} Let $(X, \ast, 0)$, $d$ and $f$ are as Example 1. Define $g = I$, where $I$ is an identity function. It is easily checked that $d$ is an $(f, g)$-derivation. But $d$ is not an $(l, r)$- $(g, f)$-derivation, since $d(1 * 2) = 1$ but $(d(1) * g(2)) \land (f(1) * d(2)) = (2 * 2) \land (2 * 1) = 0$. Also, $d$ is not an $(r, l)$- $(g, f)$-derivation, since $d(1 * 2) = 1$ but $(g(1) * d(2)) \land (d(1) * f(2)) = (1 * 1) \land (2 * 1) = 0$.

\textbf{Example 3.} Let $X = \{0, 1, 2, 3\}$ and binary operation $\ast$ is defined as:

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Then, $(X, \ast, 0)$ is a $B$-algebra (see [1]). Define maps $d, f, g : X \rightarrow X$ by

\[ d(x) = f(x) = \begin{cases} 
0 & \text{if } x = 0 \\
2 & \text{if } x = 1 \\
1 & \text{if } x = 2 \\
3 & \text{if } x = 3 
\end{cases} \quad \text{and} \quad g(x) = \begin{cases} 
0 & \text{if } x = 0 \\
3 & \text{if } x = 1, 2, 3. 
\end{cases} \]

Then, $f$ and $g$ are endomorphisms. It is easily to check that $d$ is an $(f, g)$-derivation.
THEOREM 3. Let $d$ be a self map of a $B$-algebra $X$. Then, the following hold:

1) If $d$ is a regular $(l, r)$-$(f, g)$-derivation of $X$, then $d(x) = d(x) \wedge g(x)$, for all $x \in X$;

2) If $d$ is an $(r, l)$-$(f, g)$-derivation of $X$, then $d(x) = f(x) \wedge d(x)$, for all $x \in X$ if and only if $d$ is a regular.

Proof. (1) Suppose that $d$ is a regular $(l, r)$-$(f, g)$-derivation of $X$ and $x \in X$. Then, $d(x) = d(x + 0) = (d(x) * f(0)) \wedge (g(x) * d(0)) = d(x) \wedge g(x)$.

(2) Suppose that $d$ is an $(r, l)$-$(f, g)$-derivation of $X$. If $d(x) = f(x) \wedge d(x)$, for all $x \in X$, then $d(0) = f(0) \wedge d(0) = d(0) * (d(0) * 0) = d(0) * d(0) = 0$. Conversely, suppose that $d(0) = 0$. Then, $d(x) = d(x + 0) = (f(x) * d(0)) \wedge (d(x) * g(0)) = f(x) \wedge d(x)$, for all $x \in X$. ■

Now, we investigate $(f, g)$-derivation of commutative $B$-algebras.

THEOREM 4. Let $X$ be a commutative $B$-algebra. Then, for all $x, y \in X$,

1) If $d$ is an $(l, r)$-$(f, g)$-derivation of $X$, then $d(x * y) = d(x) * f(y)$. Moreover, $d(0) = d(x) * f(x)$;

2) If $d$ is an $(r, l)$-$(f, g)$-derivation of $X$, then $d(x * y) = f(x) * d(y)$. Moreover $d(0) = f(x) * d(x)$;

3) If $d$ is an $(l, r)$-$(f, g)$-derivation $((r, l)$-$(f, g)$-derivation), then Corollary 1 is valid.

Proof. The proof is clear. ■

THEOREM 5. Let $X$ be a commutative $B$-algebra and $f, g$ be endomorphisms. If $d = f$, then $d$ is an $(f, g)$-derivation.

Proof. Suppose that $x, y \in X$. Then, $d(x * y) = f(x * y) = f(x) * f(y) = d(x) * f(y) = (d(x) * f(y)) \wedge (g(x) * d(y))$. So, $d$ is an $(r, l)$-$(f, g)$-derivation. Also, we have $d(x * y) = f(x * y) = f(x) * f(y) = f(x) * d(y) = (f(x) * d(y)) \wedge (d(x) * g(y))$. Hence, $d$ is an $(l, r)$-$(f, g)$-derivation. Therefore, $d$ is an $(f, g)$-derivation. ■

THEOREM 6. Let $d$ be an $(l, r)$-$(f, g)$-derivation $((r, l)$-$(f, g)$-derivation) of a commutative $B$-algebra $X$. Then, for all $x, y \in X$, (1) $d(x) = d(0) + f(x)$; (2) $d(x + y) = d(x) + d(y) - d(0)$; (3) $d(x) * d(y) = f(x) * f(y)$.

Proof. (1) Suppose that $d$ is an $(l, r)$-$(f, g)$-derivation of $X$ and $x \in X$. Then, by Theorem 4(1), we have $d(x) = d(0 * (0 + x)) = d(0) * f(0 + x) = d(0) * (0 + f(x)) = d(0) + f(x)$. Now, suppose that $d$ is an $(r, l)$-$(f, g)$-derivation of $X$ and $x \in X$. Then, by Theorem 4(2), we get $d(x) = d(x * 0) = f(x) * d(0)$ and $d(0) = f(0) * d(0)$. So, $d(x) = f(x) * (0 * d(0)) = d(0) * (0 * f(x)) = d(0) + f(x)$.

(2) By (1), for all $x, y \in X$, $d(x + y) = d(0) + f(x + y) = d(0) + f(x) + d(0) + f(y) - d(0) = d(x) + d(y) - d(0)$.

(3) Suppose that $d$ is an $(l, r)$-$(f, g)$-derivation of $X$. Then, by Theorem 4, $d(x) * f(x) = d(0) = d(y) * f(y)$, for all $x, y \in X$. Thus, $(d(y) * f(y)) * (d(x) * f(x)) = d(y) * f(x) * d(y) * f(y) = f(x) * f(y) * (0 * f(x) * f(y)) = f(x) * f(y) * d(0)$.
Now (1) and (2) follow by Theorem 7. Also, we get, induction we get, and of derivations of $d$ such that $d \circ X^4$, for all derivation) of derivation of $d$.

$\ast$ $\ast$

Let $(X, \ast, 0)$ be an $B$-algebra and $x \in X$. Define $x^n := x^{n-1} \ast (0 \ast x)$ ($n \geq 1$) and $x^0 := 0$. Note that $x^1 = x^0 \ast (0 \ast x) = 0 \ast (0 \ast x) = x$ [12].

**Theorem 7.** [12] Let $(X, \ast, 0)$ be a $B$-algebra. Then, for all $x \in X$,

\[
x^m \ast x^n = \begin{cases} x^{m-n} & \text{if } m \geq n \\ 0 \ast x^{n-m} & \text{if } m < n. \end{cases}
\]

**Theorem 8.** Let $(X, \ast, 0)$ be a commutative $B$-algebra $X$. For all $x \in X$,

1. If $d$ be an $(l, r)$-$(f, g)$-derivation, then

\[
d(x^m \ast x^n) = \begin{cases} d(0) \ast (0 \ast f(x))^{m-n} & \text{if } m \geq n \\ d(0) \ast f(x)^{n-m} & \text{if } m < n. \end{cases}
\]

2. If $d$ be an $(r, l)$-$(f, g)$-derivation, then

\[
d(x^m \ast x^n) = \begin{cases} f(x)^{m-n-1} \ast (0 \ast d(x)) & \text{if } m \geq n \\ (0 \ast d(x)) \ast f(x)^{n-m-1} & \text{if } m < n. \end{cases}
\]

**Proof.** It is clear that $(x \ast y^n) \ast y = x \ast y^{n+1}$, for all $x, y \in X$ and $n \geq 1$. So by induction we get, $d(x^n) = d(0) \ast (0 \ast f(x))^n$, where $d$ is an $(l, r)$-$(f, g)$-derivation. Also, we have $d(x^n) = f(x)^{n-1} \ast (0 \ast d(x))$, where $d$ is an $(r, l)$-$(f, g)$-derivation. Now (1) and (2) follow by Theorem 7.

**Theorem 9.** Let $X$ be a commutative $B$-algebra and $f, g$ be endomorphisms such that $f \circ f = f$. Also, let $d$ and $d'$ be $(l, r)$-$(f, g)$-derivations $(r, l)$-$(f, g)$-derivations) of $X$. Then, $d \circ d'$ is also an $(l, r)$-$(f, g)$-derivation $(r, l)$-$(f, g)$-derivation) of $X$.

**Proof.** Let $d$ and $d'$ are the $(l, r)$-$(f, g)$-derivations of $X$. Then, by Theorem 4, for all $x, y \in X$, we have $(d \circ d')(x \ast y) = f(d'(x) \ast f(y)) = f(d'(x)) \ast f(f(y)) = f(d'(x) \ast f(y)) = (d \circ d'(x) \ast f(y)) \ast (g(x) \ast d \circ d'(y)).$ Thus, $d \circ d'$ is a $(l, r)$-$(f, g)$-derivation of $X$. Now, suppose that $d, d'$ are $(r, l)$-$(f, g)$-derivations of $X$. Similarly, we can prove $d \circ d'$ is a $(r, l)$-$(f, g)$-derivation of $X$.

**Theorem 10.** Let $X$ be a commutative $B$-algebra, $d$ and $d'$ be $(f, g)$-derivations of $X$ such that $f \circ d = d \circ f$, $d' \circ f = f \circ d'$. Then, $d \circ d' = d' \circ d$.

**Proof.** Since $d'$ is an $(l, r)$-$(f, g)$-derivation and $d$ is an $(r, l)$-$(f, g)$-derivation of $X$, for all $x, y \in X$,

\[
(d \circ d')(x \ast y) = d((d'(x) \ast f(y)) \ast (g(x) \ast d'(y))) = d(d'(x) \ast f(y)) = (d \circ d')(x) \ast d \circ f(y).
\]
Also, since \(d'\) is an \((l, r)-(f, g)\)-derivation and \(d\) is an \((r, l)-(f, g)\)-derivation of \(X\), for all \(x, y \in X\), we have
\[
(d' \circ d)(x \ast y) = d'((f(x) * d(y)) \land (d(x) * g(y))) = d'(f(x) * d(y)) = d' \circ f(x) * f \circ d(y) = f \circ d'(x) * d \circ f(y).
\]
By the relations (1) and (2), we have \(d \circ d'(x \ast y) = d' \circ d(x \ast y)\), for all \(x, y \in X\). By putting \(y = 0\), we get \(d \circ d'(x) = d' \circ d(x)\), for all \(x \in X\).

Let \(X\) be a \(B\)-algebra and \(d, d'\) be two self maps of \(X\). We define \(d \bullet d' : X \longrightarrow X\) as follows: \((d \bullet d')(x) = d(x) \ast d'(x)\), for all \(x \in X\).

**Theorem 11.** Let \(X\) be a commutative \(B\)-algebra and \(d, d'\) be \((f, g)\)-derivations of \(X\). Then, (1) \((f \circ d') \bullet (d \circ f) = (d \circ f) \bullet (f \circ d')\); (2) \((d \circ d') \bullet (f \circ f) = (f \circ f) \bullet (d \circ d')\).

**Proof.** (1) Since \(d\) is an \((r, l)-(f, g)\)-derivation and \(d'\) is an \((l, r)-(f, g)\)-derivation of \(X\), then for all \(x, y \in X\), \((d \circ d')(x \ast y) = d((d'(x) \ast f(y)) \land (g(x) \ast d'(y))) = d(d'(x) \ast f(y)) = f((f(d'(x)) \ast d(y)) \land (d'(x) \ast g(y))) = (f \circ d'(x) \ast d(y)) \land (d \circ f(y))\).

Also, \(d\) is a \((r, l)-(f, g)\)-derivation and \(d'\) is a \((r, l)-(f, g)\)-derivation of \(X\). Hence, for all \(x, y \in X\), \((d \circ d')(x \ast y) = d((f(x) \ast d'(y)) \land (d(x) \ast g(y))) = f((f(x) \ast d'(y)) \land (d(x) \ast g(y))) = (d \circ f(x) \ast (f \circ d'(y))) \land (d \circ f(x) \ast (f \circ d'(y))) = d \circ f(x) \ast (d \circ f(x) \ast (f \circ d'(y)))\). Now, we obtain \((f \circ d'(x)) \ast (d \circ f(y)) = (d \circ f(x)) \ast (d \circ f(y))\), for all \(x, y \in X\). By putting \(x = y\), we have \((f \circ d'(x)) \ast (d \circ f(x)) = (d \circ f(x)) \ast (d \circ f'(x))\). So, \((d \circ d'(x)) \ast (d \circ f(x)) = (d \circ f(x)) \ast (d \circ f'(x))\), for all \(x \in X\).

(2) The proof is similar to the proof of (1). 

Let \(\text{Der}(X)\) denotes the set of all \((f, g)\)-derivations on \(X\). Let \(d, d' \in \text{Der}(X)\). Define the binary operation \(\land\) as follows: \((d \land d')(x) = d(x) \ast d'(x)\), for all \(x \in X\).

**Theorem 12.** If \(X\) is a commutative \(B\)-algebra, then \((\text{Der}(X), \land)\) is a semigroup.

**Proof.** Suppose that \(d, d'\) are \((l, r)-(f, g)\)-derivations of \(X\). We prove \(d \land d'\) is also an \((l, r)-(f, g)\)-derivation. For all \(x, y \in X\), \((d \land d')(x \ast y) = d(x \ast y) \land d'(x \ast y) = f(x) \ast d(y) = f((x) \ast d'(y)) = f((x) \ast d(y)) = f((x) \ast (d(y) \land d'(y))) = ((d \land d'(y))(y) \ast (d \land d')(y)) \ast g(y)\). So, \(d \land d'\) is an \((l, r)-(f, g)\)-derivation of \(X\). Therefore, \(d \land d' \in \text{Der}(X)\). Let \(d, d', d'' \in \text{Der}(X)\). We prove \(d \land (d' \land d'') = (d \land d') \land d''\) (associative property). If \(x, y \in X\), then \((d \land (d' \land d''))(x \ast y) = d(x \ast y) \land (d' \land d'')(x \ast y) = d(x \ast y) \ast (d' \land d'')(x \ast y) = d(x \ast y) \ast (d' \land d')(x \ast y) = d(x \ast y) \ast (d' \land d')(x \ast y) = d(x \ast y) \ast (d' \land d')(x \ast y)\). Also, we have \(((d \land d')(x \ast y) = (d \land d')(x \ast y) \land (d' \land d')(x \ast y) = (d \land d'')(x \ast y)\). This shows that \((d \land (d' \land d''))(x \ast y) = (d \land d')(x \ast y) \land (d' \land d'')(x \ast y)\), for all \(x, y \in X\). By putting \(y = 0\), we obtain \(d \land (d' \land d'') = (d \land d') \land d''\). Therefore, \((\text{Der}(X), \land)\) is a semigroup.
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(received 11.05.2012; in revised form 23.09.2012; available online 01.11.2012)

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