SET-VALUED PREŠIĆ-ČIRIĆ TYPE CONTRACTION  
IN 0-COMPLETE PARTIAL METRIC SPACES

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Abstract. The purpose of this paper is to introduce the set-valued Prešić-Čirić type contraction in 0-complete partial metric spaces and to prove some coincidence and common fixed point theorems for such mappings in product spaces, in partial metric case. Results of this paper extend, generalize and unify several known results in metric and partial metric spaces. An example shows how the results of this paper can be used while the existing one cannot.

1. Introduction and preliminaries

There are a number of generalizations of Banach contraction principle. One such generalization is given by S.B. Prešić [28,29] in 1965. Prešić proved following theorem.

Theorem 1. Let \((X, d)\) be a complete metric space, \(k\) a positive integer and \(T : X^k \to X\) a mapping satisfying the following contractive type condition:

\[d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1})\] (1)

for every \(x_1, x_2, \ldots, x_{k+1} \in X\), where \(q_1, q_2, \ldots, q_k\) are nonnegative constants such that \(q_1 + q_2 + \cdots + q_k < 1\). Then there exists a unique point \(x \in X\) such that \(T(x, x, \ldots, x) = x\). Moreover, if \(x_1, x_2, \ldots, x_k\) are arbitrary points in \(X\) and for \(n \in \mathbb{N}\), \(x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})\), then the sequence \(\{x_n\}\) is convergent and \(\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)\).

Note that condition (1) in the case \(k = 1\) reduces to the well-known Banach contraction mapping principle. So, Theorem 1 is a generalization of the Banach fixed point theorem. Some generalizations and applications of Theorem 1 can be seen in [11,13,16,19,20,25–27,33,35,36,38–40].

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Inspired by the results in Theorem 1, Ćirić and Prešić [13] proved following theorem.

**Theorem 2.** Let \((X, d)\) be a complete metric space, \(k\) a positive integer and \(T : X^k \to X\) a mapping satisfying the following contractive type condition:

\[
d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\},
\]

where \(\lambda \in [0, 1)\) is a constant and \(x_1, x_2, \ldots, x_{k+1}\) are arbitrary points in \(X\). Then there exists a point \(x\) in \(X\) such that \(T(x, x, \ldots, x) = x\). Moreover, if \(x_1, x_2, \ldots, x_k\) are arbitrary points in \(X\) and for \(n \in \mathbb{N}\), \(x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})\), then the sequence \(\{x_n\}\) is convergent and \(\lim x_n = T(\lim x_n, \lim x_{n+1}, \ldots, \lim x_n)\). If in addition we suppose that on diagonal \(\Delta \subset X^k\), \(d(T(u, u, \ldots, u), T(v, v, \ldots, v)) < d(u, v)\) holds for \(u, v \in X\), with \(u \neq v\), then \(x\) is a unique fixed point satisfying \(x = T(x, x, \ldots, x)\).

Nadler [24] generalized the Banach contraction mapping principle to set-valued functions and proved the following fixed point theorem.

**Theorem 3.** Let \((X, d)\) be a complete metric space and let \(T\) be a mapping from \(X\) into \(CB(X)\) (here \(CB(X)\) denotes the set of all nonempty closed bounded subset of \(X\)) such that for all \(x, y \in X\),

\[
H(Tx, Ty) \leq \lambda d(x, y)
\]

where, \(0 \leq \lambda < 1\). Then \(T\) has a fixed point.

After the work of Nadler, several authors proved fixed point results for set-valued mappings (see, e.g., [5,6,8,10,12,15,23,39–41]).

Recently, in [39], the author introduced the notion of weak compatibility of set-valued Prešić type mappings with a single-valued mapping and proved some coincidence and common fixed point theorems for such mappings in product spaces. The following theorem was one of the main results of [39].

**Theorem 4.** Let \((X, d)\) be any complete metric space, \(k\) a positive integer. Let \(f : X^k \to CB(X)\) and \(g : X \to X\) be two mappings such that \(g(X)\) is a closed subspace of \(X\) and \((f(x_1, x_2, \ldots, x_k)) \subset g(X)\) for all \(x_1, x_2, \ldots, x_k \in X\). Suppose that the following condition holds:

\[
H(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{k} \alpha_i d(gx_i, gx_{i+1}),
\]

for all \(x_1, x_2, \ldots, x_{k+1} \in X\), where \(\alpha_i\) are nonnegative constants such that \(\sum_{i=1}^{k} \alpha_i < 1\). Then \(f\) and \(g\) have a point of coincidence \(v \in X\).

The above theorem generalizes the results of Prešić and Nadler in product spaces in metric case. A generalization of the above theorem can be seen in [40].

On the other hand, Matthews [22] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, with
the interesting property of “non-zero self distance” in the space. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Subsequently, several authors (see, e.g., [1–4,6,7,9,14,17,18,30–32,34,37]) derived fixed point theorems in partial metric spaces. Romaguera [30] introduced the notion of 0-Cauchy sequence, 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

Recently, Aydi et al. [6] introduced the notion of partial Hausdorff metric and extended the Nadler’s theorem to partial metric spaces.

In the present paper, we prove some coincidence and common fixed point theorems for the mappings satisfying Prešić-Ćirić type contractive conditions (see [13]) in 0-complete partial metric spaces. Our results extend, generalize and unify the results of Matthews [22], Prešić [28], Ćirić and Prešić [13], Nadler [24] and recent results of Shukla et al. [39] and Aydi et al. [6] to 0-complete partial metric spaces.

Consistent with [4,6,18,22,30,32], the following definitions and results will be needed in the sequel.

**Definition 1.** A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that for all $x, y, z \in X$:

(P1) $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
(P2) $p(x, x) \leq p(x, y),$
(P3) $p(x, y) = p(y, x),$
(P4) $p(x, y) \leq p(x, z) + p(y, z) - p(z, z).$

A partial metric space is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$.

It is clear that, if $p(x, y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. Also every metric space is a partial metric space, with zero self distance.

**Example 1.** If $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x, y) = \max\{x, y\}$, for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a partial metric space.

Some more examples of partial metric space can be seen in [6,18,22].

Each partial metric on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has a base the family of open $p$-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

**Theorem 5.** [22] For each partial metric $p : X \times X \to \mathbb{R}^+$ the pair $(X, d)$ where, $d(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, is a metric space.

Here $(X, d)$ is called the induced metric space and $d$ is the induced metric. In further discussion until unless specified $(X, d)$ will represent the induced metric space.
Let \((X, p)\) be a partial metric space.

1. A sequence \(\{x_n\}\) in \((X, p)\) converges to a point \(x \in X\) if and only if \(p(x, x) = \lim_{n \to \infty} p(x_n, x)\).

2. A sequence \(\{x_n\}\) in \((X, p)\) is called Cauchy sequence if there exists (and is finite) \(\lim_{n,m \to \infty} p(x_n, x_m)\).

3. \((X, p)\) is said to be complete if every Cauchy sequence \(\{x_n\}\) in \(X\) converges with respect to \(\tau_p\) to a point \(x \in X\) such that \(p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)\).

4. A sequence \(\{x_n\}\) in \((X, p)\) is called 0-Cauchy sequence if \(\lim_{n,m \to \infty} p(x_n, x_m) = 0\). The space \((X, p)\) is said to be 0-complete if every 0-Cauchy sequence in \(X\) converges with respect to \(\tau_p\) to a point \(x \in X\) such that \(p(x, x) = 0\).

**Lemma 1.** [22,30,32] Let \((X, p)\) be a partial metric space and \(\{x_n\}\) be any sequence in \(X\).

(i) \(\{x_n\}\) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in metric space \((X, d)\).

(ii) \((X, p)\) is complete if and only if the metric space \((X, d)\) is complete. Furthermore, \(\lim_{n \to \infty} d(x_n, x) = 0\) if and only if \(p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m)\).

(iii) Every 0-Cauchy sequence in \((X, p)\) is Cauchy in \((X, d)\).

(iv) If \((X, p)\) is complete then it is 0-complete.

The converse assertions of (iii) and (iv) do not hold. Indeed the partial metric space \((\mathbb{Q} \cap [0, \infty), p)\), where \(\mathbb{Q}\) denotes the set of rational numbers and the partial metric \(p\) is given by \(p(x, y) = \max\{x, y\}\), provides an easy example of a 0-complete partial metric space which is not complete. It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

Let \((X, p)\) be a partial metric space. Let \(CB^p(X)\) be the family of all nonempty, closed and bounded subsets of the partial metric space \((X, p)\), induced by the partial metric \(p\). Note that closedness is taken in the sense of \((X, \tau_p)\) (\(\tau_p\) is the topology induced by \(p\)) and boundedness is given as follows: \(A\) is a bounded subset in \((X, p)\) if there exist \(x_0 \in X\) and \(M \geq 0\) such that for all \(a \in A\), we have \(a \in B_p(x_0, M)\), that is, \(p(x_0, a) < p(a, a) + M\).

For \(A, B \in CB^p(X)\) and \(x \in X\), define
\[
p(x, A) = \inf\{p(x, a) : a \in A\}, \quad \delta_p(A, B) = \sup\{p(a, B) : a \in A\}.
\]

**Lemma 2.** [4] Let \((X, p)\) be a partial metric space, \(A \subset X\). Then \(a \in \overline{A}\) if and only if \(p(a, A) = p(a, a)\).

**Proposition 1.** [6] Let \((X, p)\) be a partial metric space. For any \(A, B, C \in CB^p(X)\), we have the following:

(i) \(\delta_p(A, A) = \sup\{p(a, a) : a \in A\}\);

(ii) \(\delta_p(A, A) \leq \delta_p(A, B)\);

(iii) \(\delta_p(A, B) = 0\) implies that \(A \subseteq B\);

(iv) \(\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)\).
Let \((X, p)\) be a partial metric spaces. For \(A, B \in CB^p(X)\), define

\[
H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.
\]

**Proposition 2.** [6] Let \((X, p)\) be a partial metric space. For \(A, B, C \in CB^p(X)\), we have

1. \(H_p(A, A) \leq H_p(A, B)\);
2. \(H_p(A, B) = H_p(B, A)\);
3. \(H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)\).

**Corollary 1.** [6] Let \((X, p)\) be a partial metric space. For \(A, B \in CB^p(X)\) the following holds

\[
H_p(A, B) = 0 \quad \text{implies that} \quad A = B.
\]

In view of Proposition 2 and Corollary 1, we call the mapping \(H_p : CB^p(X) \times CB^p(X) \to [0, \infty)\), a partial Hausdorff metric induced by \(p\).

**Lemma 3.** [6] Let \((X, p)\) be a partial metric space and \(A, B \in CB^p(X)\) and \(h > 1\). For any \(a \in A\) there exists \(b = b(a) \in B\) such that \(p(a, b) \leq hH_p(A, B)\).

**Definition 2.** [39] Let \(X\) be a nonempty set, \(k\) a positive integer, \(f : X^k \to 2^X\) and \(g : X \to X\) be mappings.

1. If \(x \in f(x, \ldots, x)\), then \(x \in X\) is called a fixed point of \(f\).
2. An element \(x \in X\) said to be a coincidence point of \(f\) and \(g\) if \(gx \in f(x, \ldots, x)\).
3. If \(w = gx \in f(x, \ldots, x)\), then \(w\) is called a point of coincidence of \(f\) and \(g\).
4. If \(x \in f(x, \ldots, x)\), then \(x\) is called a common fixed point of \(f\) and \(g\).
5. Mappings \(f\) and \(g\) are said to be commuting if \(g(f(x, \ldots, x)) = f(gx, \ldots, gx)\) for all \(x \in X\).
6. Mappings \(f\) and \(g\) are said to be weakly compatible if \(gx \in f(x, \ldots, x)\) implies \(g(f(x, \ldots, x)) \subseteq f(gx, \ldots, gx)\).

**2. Main results**

**Theorem 6.** Let \((X, p)\) be a 0-complete partial metric space, \(k\) a positive integer. Let \(f : X^k \to CB^p(X)\) and \(g : X \to X\) be two mappings such that \(g(X)\) is a closed subspace of \(X\) and \(f(x_1, x_2, \ldots, x_k) \subset g(X)\) for all \(x_1, x_2, \ldots, x_k \in X\). Suppose following condition holds:

\[
H_p(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \lambda \max\{p(gx_i, gx_{i+1}), 1 \leq i \leq k\} \tag{2}
\]

for all \(x_1, x_2, \ldots, x_{k+1} \in X\), where \(\lambda \in [0, 1)\). Then \(f\) and \(g\) have a point of coincidence \(v \in X\).

**Proof.** We define a sequence \(\{y_n\} = \{gx_n\}\) in \(X\) as follows: let \(x_1, x_2, \ldots, x_k \in X\) be arbitrary and \(y_n = gx_n\) for \(n = 1, 2, \ldots, k\). As \(f(x_1, \ldots, x_k) \subset CB^p(X)\) and \(f(x_1, \ldots, x_k) \subset g(X)\), we can assume \(y_{k+1} = gx_{k+1} \in f(x_1, \ldots, x_k)\), for
some \(x_{k+1} \in X\) also, \(\lambda < 1\) so using Lemma 3 with \(h = 1/\sqrt{\lambda}\), there exists \(y_{k+2} = gx_{k+2} \in f(x_2, \ldots, x_{k+1})\) such that
\[
p(y_{k+1}, y_{k+2}) = p(gx_{k+1}, gx_{k+2}) \leq \frac{1}{\sqrt{\lambda}} H_p(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) \leq \sqrt{\lambda} \max\{p(gx_i, gx_{i+1}), 1 \leq i \leq k\} = \sqrt{\lambda} \max\{p(y_i, y_{i+1}), 1 \leq i \leq k\}.
\]
Similarly, there exists \(y_{k+3} = gx_{k+3} \in f(x_3, \ldots, x_{k+2})\) such that
\[
p(y_{k+2}, y_{k+3}) = p(gx_{k+2}, gx_{k+3}) \leq \frac{1}{\sqrt{\lambda}} H_p(f(x_2, \ldots, x_{k+1}), f(x_3, \ldots, x_{k+2})) \leq \sqrt{\lambda} \max\{p(y_i, y_{i+1}), 2 \leq i \leq k+1\}.
\]
Continuing this procedure we obtain a sequence \(\{y_n\}\) such that \(y_n = gx_n\) for \(n = 1, 2, \ldots, k\) and \(y_{n+k} = gx_{n+k} \in f(x_n, \ldots, x_{n+k-1})\) for \(n = 1, 2, \ldots\) with
\[
p(y_{n+k}, y_{n+k+1}) \leq \sqrt{\lambda} \max\{p(y_i, y_{i+1}), n \leq i \leq n+k-1\}. \tag{3}
\]
for all \(n \in \mathbb{N}\).

Set \(p_n = p(gx_n, gx_{n+1}) = p(y_n, y_{n+1})\) for all \(n \in \mathbb{N}\) and
\[
\mu = \max\left\{\frac{p(gx_1, gx_2)}{\delta^1}, \frac{p(gx_2, gx_3)}{\delta^2}, \ldots, \frac{p(gx_k, gx_{k+1})}{\delta^k}\right\} = \max\{\frac{p_1}{\delta^1}, \frac{p_2}{\delta^2}, \ldots, \frac{p_k}{\delta^k}\}
\]
where \(\delta = \lambda^{1/2k}\). By the method of mathematical induction we shall prove that
\[
p_n \leq \mu \delta^n \text{ for all } n \in \mathbb{N}. \tag{4}
\]
By the definition of \(\mu\) it is clear that (4) is true for \(n = 1, 2, \ldots, k\). Let the \(k\) inequalities \(p_n \leq \mu \delta^n, p_{n+1} \leq \mu \delta^{n+1}, \ldots, p_{n+k-1} \leq \mu \delta^{n+k-1}\) be the induction hypothesis. Using (3) we obtain
\[
p_{n+k} = p(y_{n+k}, y_{n+k+1}) \leq \sqrt{\lambda} \max\{p(y_i, y_{i+1}), n \leq i \leq n+k-1\} = \sqrt{\lambda} \max\{p_n, p_{n+1}, \ldots, p_{n+k-1}\} \leq \sqrt{\lambda} \max\{\mu \delta^n, \mu \delta^{n+1}, \ldots, \mu \delta^{n+k-1}\} = \sqrt{\lambda} \mu \delta^n \quad \text{ (as } \delta = \lambda^{1/2k} < 1) = \mu \delta^{n+k}.
\]
Thus, inductive proof of (4) is complete. Now we shall show that the sequence \(\{y_n\} = \{gx_n\}\) is a Cauchy sequence in \(g(X)\). Let \(m, n \in \mathbb{N}\) with \(m > n\), then using (4) we obtain
\[
p(y_n, y_m) \leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m)
\]
for all $x_n \in X$ set-valued Prešić-Ćirić type contraction. Above theorem, we obtain a generalization of the result of [35] in metric spaces. We shall show that $u, v \in X$ such that $v = gu$ and $v, v \in (\cdot \cdot \cdot , u)$ and using (2) in the above inequality we obtain

$$
- \left[ p(y_{n+1}, y_{n+1}) + p(y_{n+2}, y_{n+2}) + \cdots + p(y_{m-1}, y_{m-1}) \right] \\
\leq p_n + p_{n+1} + \cdots + p_{m-1} \\
\leq \mu \delta^n + \mu \delta^{n+1} + \cdots + \mu \delta^{m-1} \\
\leq \mu \delta^n [1 + \delta + \delta^2 + \cdots] = \frac{\mu \delta^n}{1 - \delta}.
$$

As $\delta = \lambda^{1/2k} < 1$, therefore $\frac{\mu \delta^n}{1 - \delta} \to 0$ as $n \to \infty$. So, it follows from above inequality that

$$
\lim_{n,m \to \infty} p(y_n, y_m) = 0.
$$

Therefore $\{y_n\} = \{gx_n\}$ is a 0-Cauchy sequence in $g(X)$. As $g(X)$ is closed, there exists $u, v \in X$ such that $v = gu$ and

$$
\lim_{n \to \infty} p(y_n, v) = \lim_{n,m \to \infty} p(y_n, y_m) = p(gu, gu) = p(v, v) = 0. \tag{5}
$$

We shall show that $u$ is a coincidence point of $f$ and $g$.

Note that, $gx_{n+k} = y_{n+k} \in f(x_n, x_{n+1}, \ldots, x_{n+k-1})$ so, for any $n \in \mathbb{N}$ we have

$$
p(v, f(u, \ldots, u)) \leq p(v, y_{n+k}) + p(y_{n+k}, f(u, \ldots, u)) \\
\leq p(v, y_{n+k}) + H_p(f(x_n, \ldots, x_{n+k-1}), f(u, \ldots, u)) \\
\leq p(v, y_{n+k}) + H_p(f(x_n, \ldots, x_{n+k-1}), f(x_{n+1}, \ldots, x_{n+k-1}, u)) \\
+ H_p(f(x_{n+1}, \ldots, x_{n+k-1}, u), f(x_{n+2}, \ldots, x_{n+k-1}, u, u)) \\
+ \cdots + H_p(f(x_{n+k-1}, u, \ldots, u), f(u, \ldots, u)),
$$

and using (2) in the above inequality we obtain

$$
p(v, f(u, \ldots, u)) \leq p(v, y_{n+k}) + \lambda \max\{p_n, \ldots, p_{n+k-2}, p(gx_{n+k-1}, gu)\} \\
+ \lambda \max\{p_n+1, \ldots, p_{n+k-2}, p(gx_{n+k-1}, gu), p(gu, gu)\} \\
+ \cdots + \lambda \max\{p(gx_{n+k-1}, gu), p(gu, gu), \ldots, p(gu, gu)\} \\
=p(v, y_{n+k}) + \lambda \max\{p_n, \ldots, p_{n+k-2}, p(y_{n+k-1}, v)\} \\
+ \lambda \max\{p_n+1, \ldots, p_{n+k-2}, p(y_{n+k-1}, v), p(v, v)\} \\
+ \cdots + \lambda \max\{p(y_{n+k-1}, v), p(v, v), \ldots, p(v, v)\}.
$$

In view of (5), it follows from the above inequality that $p(v, f(u, \ldots, u)) = 0 = p(v, v)$. As $f(u, \ldots, u) \in CB^p(X)$, by Lemma 2 we have $v = gu \in f(u, \ldots, u)$ i.e. $u$ is a coincidence point and $v$ is a point of coincidence of $f$ and $g$.

**Remark 1.** If we take $p = d$, i.e., if we replace partial metric by metric in the above theorem, we obtain a generalization of the result of [35] in metric spaces.

Taking $g = I_X$ in Theorem 6, we obtain the following fixed point result for set-valued Prešić-Ćirić type contraction.

**Corollary 2.** Let $(X, p)$ be a 0-complete metric space, $k$ a positive integer. Let $f : X^k \to CB^p(X)$ be a set-valued Prešić-Ćirić type contraction, i.e., let it satisfy the following contractive type condition

$$
H_p(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \lambda \max\{p(x_i, x_{i+1}), 1 \leq i \leq k\} \tag{6}
$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$, where $\lambda \in [0, 1)$. Then $f$ has a fixed point $v \in X$. 

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Remark 2. The above corollary is a set-valued version and generalization of the result of Prešić and Ćirić [13] for set-valued mappings in 0-complete partial metric spaces. Note that for $k = 1$ the above corollary reduces to the of result of Aydi et al. (see Theorem 3.2 of [6]), therefore it is a generalization of the result of Aydi et al. Also, it generalizes the result of Prešić (Theorem 1) for set-valued mappings.

The following theorem provides some sufficient conditions for the uniqueness of point of coincidence of mappings $f$ and $g$.

Theorem 7. Let $(X, p)$ be a 0-complete partial metric space, $k$ a positive integer. Let $f : X^k \to CB^p(X)$ and $g : X \to X$ be two mappings such that, all the conditions of Theorem 6 are satisfied and for any coincidence point $u$ of $f$ and $g$ we have $f(u, \ldots, u) = \{gu\}$. If

(i) on the diagonal $\triangle \subset X^k$,

$$H_p(f(x, \ldots, x), f(y, \ldots, y)) < p(gx, gy)$$

holds for all $x, y \in X$ with $x \neq y$, or

(ii) in condition (2) the constant $\lambda \in (0, \frac{1}{k})$.

Then, there exists a unique point of coincidence of $f$ and $g$. Suppose in addition that $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.

Proof. The existence of coincidence point $u$ and point of coincidence $v = gu$ follows from Theorem 6.

First, suppose that (i) is satisfied. We shall show that the point of coincidence $v$ is unique. If $v'$ is another point of coincidence with coincidence point $u'$ of $f$ and $g$, then $f(u', \ldots, u') = \{gu'\} = \{v'\}$ and we have

$$p(v, v') = H_p(\{v\}, \{v'\}) = H_p(f(u, \ldots, u), f(u', \ldots, u')) < p(gu, gu') = p(v, v'),$$

a contradiction. So, the point of coincidence of $f$ and $g$ is unique.

Suppose (ii) is satisfied, then using (2) we obtain

$$p(v, v') = H_p(\{v\}, \{v'\}) = H_p(f(u, \ldots, u), f(u', \ldots, u')) \leq \lambda \max\{p(gu, gu), \ldots, p(gu, gu), p(gu, gu')\} + \lambda \max\{p(gu, gu), \ldots, p(gu, gu), p(gu', gu')\} + \ldots + \lambda \max\{p(gu, gu'), p(gu', gu') \ldots, p(gu', gu')\} = k\lambda p(gu, gu') = k\lambda p(v, v') < p(v, v'),$$

again a contradiction. So, the point of coincidence of $f$ and $g$ is unique.
Suppose that \( f \) and \( g \) are weakly compatible. Then we have
\[ g(f(u, \ldots, u)) \subseteq f(g(u, \ldots, u)) = f(v, \ldots, v) \]
i.e. \( \{gv\} \subseteq f(v, \ldots, v) \).
Therefore \( gv \in f(v, \ldots, v) \), which shows that \( gv \) is another point of coincidence of \( f \) and \( g \) and by uniqueness we have \( v = gv \in f(v, \ldots, v) \). Thus \( v \) is a unique common fixed point of \( f \) and \( g \). \( \blacksquare \)

The following is a simple example of set-valued Prešić-Ćirić contraction which illustrate the case when the results of this paper can be used while the existing one cannot.

**Example 2.** Let \( X = \mathbb{Q} \cap [0, 1] \) be endowed with the partial metric \( p : X \times X \to \mathbb{R}^+ \) defined by
\[ p(x, y) = |x - y| + \max\{x, y\} \quad \text{for all} \quad x, y \in X. \]
First, we shall show that the space \((X, p)\) is \(0\)-complete. If \( \{x_n\} \) is any \(0\)-Cauchy sequence in \( X \), then \( \lim_{n,m \to \infty} p(x_n, x_m) = 0 \), i.e.,
\[ \lim_{n,m \to \infty} [\max\{x_n - x_m, 0\} + \max\{x_n, x_m\}] = 0. \] (7)
Note that the partial metric space \((X, p_1)\) is \(0\)-complete, where \( p_1(x, y) = \max\{x, y\} \) for all \( x, y \in X \) (see [30]). Therefore it follows from (7) that \( \lim_{n \to \infty} p(x_n, 0) = 0 = p_1(0, 0) \) and \( \lim_{n \to \infty} |x_n - 0| = 0 \). So we have \( \lim_{n \to \infty} p(x_n, 0) = 0 = p(0, 0) \).
As \( 0 \in X \), the space \((X, p)\) is \(0\)-complete.

Note that, the metric \( d \) induced by \( p \) is given by
\[ d(x, y) = 2|x - y| + 2\max\{x, y\} - x - y = 3|x - y| \quad \text{for all} \quad x, y \in X, \]
and the metric space \((X, d)\) is not complete, and so the partial metric space \((X, p)\) is not complete. Note that, if \( x \in X \) then the singleton subset \( \{x\} \) of \( X \) is a closed subset with respect to \( p \). Indeed, for any \( y \in X \), we have
\[ y \in \{x\} \iff p(y, y) = p(y, \{x\}) \]
\[ \iff p(y, y) = p(y, x) \]
\[ \iff y = |y - x| + \max\{y, x\} \]
\[ \iff y = x. \]
Thus \( \{x\} \) is closed. Now, for \( k = 2 \), define a mapping \( T : X^2 \to X \) by
\[ T(x, y) = \begin{cases} 
0, & \text{if } x = y = 1; \\
\frac{x+y}{10}, & \text{otherwise,}
\end{cases} \]
and a mapping \( f : X^2 \to CBp(X) \) by
\[ f(x, y) = \{T(x, y)\} \cup \{0\} \quad \text{for all} \quad x, y \in [0, 1]. \]
We shall show that \( f \) satisfies condition (6) of Corollary 2 with \( \lambda \in \left[\frac{2}{5}, 1\right) \).
If \( x_1, x_2, x_3 \in [0, 1] \) with \( x_1 \leq x_2 \leq x_3 \) then
\[ H_p(f(x_1, x_2), f(x_2, x_3)) = H_p(\{\frac{x_1 + x_2}{10}, 0\}, \{\frac{x_2 + x_3}{10}, 0\}) \]
with λ and the referees for their valuable comments and suggestions on this paper.

Partial metric spaces is wider than that in metric spaces. Thus we can say that the class of Prešić-Ćirić contractions in

\[ f(x) = \begin{cases} 
 2x_3 + x_2 - x_1, & 2x_1 + 2x_2 \\
 2x_3 + x_2 - x_1, & 2x_2 + 2x_3 
\end{cases} \]

is not a Prešić-Ćirić contraction in both the spaces (\(X, d_u\)) and (\(X, d\)). Indeed, at \(x_1 = x_2 = 1, x_3 = \frac{9}{10}\) the mapping \(f\) fails to be a Prešić-Ćirić contraction in these metric spaces. Thus we can say that the class of Prešić-Ćirić contractions in partial metric spaces is wider than that in metric spaces.

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