COUPLED FIXED POINT THEOREMS IN $G_b$-METRIC SPACES

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Abstract. T. G. Bhaskar and V. Lakshmikantham [Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379–1393], V. Lakshmikantham and Lj. B. Ćirić [Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009) 4341–4349] introduced the concept of a coupled coincidence point of a mapping $F$ from $X \times X$ into $X$ and a mapping $g$ from $X$ into $X$. In this paper we prove a coupled coincidence fixed point theorem in the setting of a generalized $b$-metric space. Three examples are presented to verify the effectiveness and applicability of our main result.

1. Introduction

Mustafa and Sims [25] introduced a new notion of generalized metric space called a $G$-metric space. Mustafa, Sims and others studied fixed point theorems for mappings satisfying different contractive conditions [1, 2, 6, 10, 11, 19, 22, 23, 25, 27, 28, 32, 35, 36, 39]. Abbas and Rhoades [1] obtained some common fixed point theorems for non-commuting maps without continuity satisfying different contractive conditions in the setting of generalized metric spaces. Lakshmikantham et al. in [7, 21] introduced the concept of a coupled coincidence point for a mapping $F$ from $X \times X$ into $X$ and a mapping $g$ from $X$ into $X$, and studied coupled fixed point theorems in partially ordered metric spaces. In [33], Sedghi et al. proved a coupled fixed point theorem for contractive mappings in complete fuzzy metric spaces. On the other hand, the concept of $b$-metric space was introduced by Czerwik in [13]. After that, several interesting results for the existence of fixed point for single-valued and multivalued operators in $b$-metric spaces have been obtained [3, 5, 8, 9, 12, 14, 15, 16, 18, 20, 30, 31, 34, 37, 38]. Pacurar [29] proved some results on sequences of almost contractions and fixed points in $b$-metric spaces. Recently, Hussain and Shah [17] obtained results on KKM mappings in cone $b$-metric spaces.

Aghajani et al., in a submitted paper [4], extended the notion of $G$-metric space to the concept of $G_b$-metric space. Very recently, Mustafa et al. [24] have obtained
some coupled coincidence point theorems for nonlinear \((\psi, \varphi)\)-weakly contractive mappings in partially ordered \(G_b\)-metric spaces.

In this paper, we prove a coupled coincidence fixed point theorem in the setting of a generalized \(b\)-metric space. First, we present some basic properties of \(G_b\)-metric spaces.

Following is the definition of generalized \(b\)-metric spaces or \(G_b\)-metric spaces.

**Definition 1.1.** [24] Let \(X\) be a nonempty set and \(s \geq 1\) be a given real number. Suppose that a mapping \(G : X \times X \times X \to \mathbb{R}^+\) satisfies:

\[(G_1)\]  \(G(x, y, z) = 0\) if \(x = y = z\),

\[(G_2)\]  \(0 < G(x, y)\) for all \(x, y \in X\) with \(x \neq y\),

\[(G_3)\]  \(G(x, x, y) \leq G(x, y, z)\) for all \(x, y, z \in X\) with \(y \neq z\),

\[(G_4)\]  \(G(x, y, z) = G(p\{x, y, z\})\), where \(p\) is a permutation of \(x, y, z\) (symmetry),

\[(G_5)\]  \(G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))\) for all \(x, y, z, a \in X\) (rectangle inequality).

Then \(G\) is called a generalized \(b\)-metric and the pair \((X, G)\) is called a generalized \(b\)-metric space or \(G_b\)-metric space.

It should be noted that the class of \(G_b\)-metric spaces is effectively larger than that of \(G\)-metric spaces given in [25]. Indeed, each \(G\)-metric space is a \(G_b\)-metric space with \(s = 1\). The following example shows that a \(G_b\)-metric on \(X\) need not be a \(G\)-metric on \(X\).

**Example 1.1.** [24] Let \((X, G)\) be a \(G\)-metric space, and \(G_s(x, y, z) = G^p(x, y, z)\), where \(p > 1\) is a real number. Note that \(G_s\) is a \(G_b\)-metric with \(s = 2^{p-1}\). In [24], it is proved that \((X, G_s)\) is not necessarily a \(G\)-metric space.

**Example 1.2.** [24] Let \(X = \mathbb{R}\) and \(d(x, y) = |x - y|^2\). We know that \((X, d)\) is a \(b\)-metric space with \(s = 2\). Let \(G(x, y, z) = d(x, y) + d(y, z) + d(z, x)\), then \((X, G)\) is not a \(G_b\)-metric space.

However, \(G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}\) is a \(G_b\)-metric on \(\mathbb{R}\) with \(s = 2\). Similarly, if \(d(x, y) = |x - y|^p\) is selected with \(p \geq 1\), then \(G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}\) is a \(G_b\)-metric on \(\mathbb{R}\) with \(s = 2^{p-1}\).

Now we present some definitions and propositions in \(G_b\)-metric spaces.

**Definition 1.2.** [24] A \(G_b\)-metric \(G\) is said to be symmetric if \(G(x, y, y) = G(y, x, x)\) for all \(x, y \in X\).

**Definition 1.3.** [24] Let \((X, G)\) be a \(G_b\)-metric space. Then, for \(x_0 \in X\), \(r > 0\), the \(G_b\)-ball with center \(x_0\) and radius \(r\) is

\[B_G(x_0, r) = \{y \in X \mid G(x_0, y, y) < r\}.\]

**Definition 1.4.** [24] Let \(X\) be a \(G_b\)-metric space and let \(d_G(x, y) = G(x, y, y) + G(x, x, y)\). Then \(d_G\) defines a \(b\)-metric on \(X\), which is called the \(b\)-metric associated with \(G\).
Proposition 1.2. [24] Let $X$ be a $G_b$-metric space. For any $x_0 \in X$ and $r > 0$, if $y \in B_G(x_0, r)$ then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.

From the above proposition the family of all $G_b$-balls

$$\Lambda = \{B_G(x, r) \mid x \in X, r > 0\}$$

is a base of a topology $\tau(G)$ on $X$, which is called the $G_b$-metric topology.

Definition 1.5. [24] Let $X$ be a $G_b$-metric space. A sequence $(x_n)$ in $X$ is said to be:

1. $G_b$-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that, for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;
2. $G_b$-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that, for all $m, n \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

Using the above definitions, one can easily prove the following proposition.

Proposition 1.4. [24] Let $X$ be a $G_b$-metric space and $(x_n)$ be a sequence in $X$. Then the following are equivalent:

1. the sequence $(x_n)$ is $G_b$-Cauchy;
2. for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq n_0$.

Definition 1.6. [24] A $G_b$-metric space $X$ is called complete if every $G_b$-Cauchy sequence is $G_b$-convergent in $X$.

Mustafa and Sims proved that each $G$-metric function $G(x, y, z)$ is jointly continuous in all three of its variables (see [26, Proposition 8]). But in general a $G_b$-metric function $G(x, y, z)$ for $s > 1$ is not jointly continuous in all three of its variables. Now we recall an example of a discontinuous $G_b$-metric.

Example 1.3. [24] Let $X = \mathbb{N} \cup \{\infty\}$ and let $D : X \times X \to \mathbb{R}^+$ be defined by

$$D(m, n) = \begin{cases} 0, & \text{if } m = n, \\ \frac{1}{m} - \frac{1}{n}, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \\ \text{or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$D(m, p) \leq \frac{5}{2}(D(m, n) + D(n, p)).$$

Thus, $(X, D)$ is a $b$-metric space with $s = \frac{5}{2}$ (see [16, Example 2]). Let $G(x, y, z) = \max\{D(x, y), D(y, z), D(z, x)\}$. It is easy to see that $G$ is a $G_b$-metric with $s = \frac{5}{2}$. In [24], it is proved that $G(x, y, z)$ is not a continuous function.
**Definition 1.7.** Let \((X, G)\) and \((X', G')\) be \(G_b\)-metric spaces, and let \(f : X \to X'\) be a mapping. Then \(f\) is said to be continuous at a point \(a \in X\) if and only if for every \(\varepsilon > 0\), there is \(\delta > 0\) such that \(x, y \in X\) and \(G(a, x, y) < \delta\) implies \(G'(f(a), f(x), f(y)) < \varepsilon\). A function \(f\) is continuous at \(X\) if and only if it is continuous at all \(a \in X\).

**Definition 1.8.** [7] Let \(X\) be a nonempty set. An element \((x, y) \in X \times X\) is called a coupled fixed point of a mapping \(F : X \times X \to X\) if \(F(x, y) = x\) and \(F(y, x) = y\).

**Definition 1.9.** [21] Let \(X\) be a nonempty set. An element \((x, y) \in X \times X\) is called a coupled coincidence point of mappings \(F : X \times X \to X\) and \(g : X \to X\) if \(F(x, y) = gx\) and \(F(y, x) = gy\).

**Definition 1.10.** [21] Let \(X\) be a nonempty set. Then we say that the mappings \(F : X \times X \to X\) and \(g : X \to X\) are commutative if \(gF(x, y) = F(gx, gy)\).

**2. Common fixed point results**

Let \(\Phi\) denote the class of all functions \(\phi : [0, \infty) \to [0, \infty)\) such that \(\phi\) is increasing, continuous, \(\phi(t) < \frac{t}{2}\) for all \(t > 0\) and \(\phi(0) = 0\). It is easy to see that for every \(\phi \in \Phi\) we can choose a \(0 < k < \frac{1}{2}\) such that \(\phi(t) \leq kt\).

We start our work by proving the following two crucial lemmas.

**Lemma 2.1.** Let \((X, G)\) be a \(G_b\)-metric space with \(s \geq 1\), and suppose that \((x_n)\) is \(G_b\)-convergent to \(x\). Then we have

\[
\frac{1}{s} G(x, y, y) \leq \liminf_{n \to \infty} G(x_n, y, y) \leq \limsup_{n \to \infty} G(x_n, y, y) \leq s G(x, y, y).
\]

In particular, if \(x = y\), then we have \(\lim_{n \to \infty} G(x_n, y, y) = 0\).

**Proof.** Using the rectangle inequality in \((X, G)\), it is easy to see that

\[
G(x_n, y, y) \leq s G(x_n, x, x) + s G(x, y, y),
\]

and

\[
\frac{1}{s} G(x, y, y) \leq G(x, x_n, x_n) + G(x_n, y, y).
\]

Taking the upper limit as \(n \to \infty\) in the first inequality and the lower limit as \(n \to \infty\) in the second inequality we obtain the desired result. \(\blacksquare\)

**Lemma 2.2.** Let \((X, G)\) be a \(G_b\)-metric space and let \(F : X \times X \to X\) and \(g : X \to X\) be two mappings such that

\[
G(F(x, y), F(u, v), F(z, w)) \leq \phi(G(gx, gu, gz)) + G(gy, gv, gw)
\]

for some \(\phi \in \Phi\) and for all \(x, y, z, u, v \in X\). Assume that \((x, y)\) is a coupled coincidence point of the mappings \(F\) and \(g\). Then

\[
F(x, y) = gx = gy = F(y, x).
\]
Proof. Since \((x, y)\) is a coupled coincidence point of the mappings \(F\) and \(g\), we have \(gx = F(x, y)\) and \(gy = F(y, x)\). Assume \(gx \neq gy\). Then by (1), we get
\[
G(gx, gy, gy) = G(F(x, y), F(y, x), F(x, y)) \leq \phi(G(gx, gy, gy) + G(gy, gx, gx)).
\]
Also by (1), we have
\[
G(gy, gx, gx) = G(F(y, x), F(x, y), F(x, y)) \leq \phi(G(gy, gx, gx) + G(gx, gy, gy)).
\]
Therefore
\[
G(gx, gy, gy) + G(gy, gx, gx) \leq 2\phi(G(gx, gy, gy) + G(gy, gx, gx)).
\]
Since \(\phi(t) < \frac{1}{4}\), we get
\[
G(gx, gy, gy) + G(gy, gx, gx) < G(gx, gy, gy) + G(gy, gx, gx),
\]
which is a contradiction. So \(gx = gy\), and hence \(F(x, y) = gx = gy = F(y, x)\). 

The following is the main result of this section.

**Theorem 2.1.** Let \((X, G)\) be a complete \(G_b\)-metric space. Let \(F : X \times X \to X\) and \(g : X \to X\) be two mappings such that
\[
G(F(x, y), F(u, v), F(z, w)) \leq \frac{1}{s^2}\phi(G(gx, gu, gz) + G(gy, gv, gw)) \tag{2}
\]
for some \(\phi \in \Phi\) and all \(x, y, z, w, u, v \in X\). Assume that \(F\) and \(g\) satisfy the following conditions:

1. \(F(X \times X) \subseteq g(X)\),
2. \(g(X)\) is complete, and
3. \(g\) is continuous and commutes with \(F\).

Then there is a unique \(x\) in \(X\) such that \(gx = F(x, x) = x\).

**Proof.** Let \(x_0, y_0 \in X\). Since \(F(X \times X) \subseteq g(X)\), we can choose \(x_1, y_1 \in X\) such that \(gx_1 = F(x_0, y_0)\) and \(gy_1 = F(y_0, x_0)\). Again since \(F(X \times X) \subseteq g(X)\), we can choose \(x_2, y_2 \in X\) such that \(gx_2 = F(x_1, y_1)\) and \(gy_2 = F(y_1, x_1)\). Continuing this process, we can construct two sequences \((x_n)\) and \((y_n)\) in \(X\) such that \(gx_{n+1} = F(x_n, y_n)\) and \(gy_{n+1} = F(y_n, x_n)\). For \(n \in \mathbb{N} \cup \{0\}\), by (2) we have
\[
G(gx_{n-1}, gx_n, gx_n) = G(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}))
\]
\[
\leq \frac{1}{s^2}\phi(G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})).
\]
Similarly, by (2) we have
\[
G(gy_{n-1}, gy_n, gy_n) = G(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}))
\]
\[
\leq \frac{1}{s^2}\phi(G(gy_{n-2}, gy_{n-1}, gy_{n-1}) + G(gx_{n-2}, gx_{n-1}, gx_{n-1})).
\]
Hence, we have that
\[ a_n := G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n) \]
\[ \leq \frac{2}{s^2} \phi(G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})) \]
\[ = \frac{2}{s^2} \phi(a_{n-1}) \]
holds for all \( n \in \mathbb{N} \). Thus, we get a \( k, 0 < k < \frac{1}{2} \) such that
\[ a_n \leq \frac{2}{s^2} \phi(a_{n-1}) \leq \frac{2k}{s} a_{n-1} \leq \frac{2k}{s} a_{n-1} = qa_{n-1}, \]
for \( q = \frac{2k}{s} \). Hence we have
\[ a_n \leq \frac{2k}{s} a_{n-1} \leq \cdots \leq (\frac{2k}{s})^n a_0. \]
Let \( m, n \in \mathbb{N} \) with \( m > n \). By Axiom \( G_b5 \) of definition of \( G_b \)-metric spaces, we have
\[ G(gx_{n-1}, gx_m, gx_m) + G(gy_{n-1}, gy_m, gy_m) \]
\[ \leq s(G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_m, gx_m)) \]
\[ + s(G(gy_{n-1}, gy_n, gy_n) + G(gy_n, gy_m, gy_m)) \]
\[ = s(G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)) \]
\[ + s(G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m)) \]
\[ \leq \]
\[ \vdots \]
\[ \leq s a_n + s^2 a_{n+1} + s^3 a_{n+2} + \cdots + s^{m-n} a_{m-1} + s^{m-n} a_m \]
\[ \leq sq^n a_0 + s^2 q^{n+1} a_0 + \cdots + s^{m-n} q^{m-1} a_0 + s^{m-n} q^m a_0 \]
\[ \leq sq^n a_0 (1 + sq + s^2 q^2 + \cdots) \]
\[ \leq sq^n a_0 \frac{1}{1 - sq} \to 0, \]
since \( sq = 2k < 1 \). Thus \((gx_n)\) and \((gy_n)\) are \( G_b \)-Cauchy in \( g(X) \). Since \( g(X) \) is complete, we get \((gx_n)\) and \((gy_n)\) are \( G_b \)-convergent to some \( x \in X \) and \( y \in X \) respectively. Since \( g \) is continuous, we have that \((ggx_n)\) is \( G_b \)-convergent to \( gx \) and \((ggyn)\) is \( G_b \)-convergent to \( gy \). Also, since \( g \) and \( F \) commute, we have
\[ ggx_{n+1} = g(F(x, y)) = F(gx_n, gy_n), \]
and
\[ ggy_{n+1} = g(F(y, x)) = F(gy_n, gx_n). \]
Thus
\[ G(ggx_{n+1}, F(x, y), F(x, y)) = G(F(gx_n, gy_n), F(x, y), F(x, y)) \]
\[ \leq \frac{1}{s^2} \phi(G(ggx_n, gx, gx) + G(ggy_n, gy, gy)). \]
Letting \( n \to \infty \), and using Lemma 2.1, we get that
\[
\frac{1}{s}G(gx, F(x, y), F(x, y)) \leq \limsup_{n \to \infty} G(F(gx_n, gy_n), F(x, y), F(x, y)) \\
\leq \limsup_{n \to \infty} \frac{1}{s^2} \phi(G(gx, gx, gx) + G(gy, gy, gy)) \\
\leq \frac{1}{s^2} \phi(s(G(gx, gx, gx) + G(gy, gy, gy))) = 0.
\]
Hence, \( gx = F(x, y) \). Similarly, we may show that \( gy = F(y, x) \). By Lemma 2.2, \((x, y)\) is a coupled fixed point of the mappings \( F \) and \( g \), i.e.,
\[
gx = F(x, y) = F(y, x) = gy.
\]
Thus, using Lemma 2.1 we have
\[
\frac{1}{s}G(x, gx, gx) \leq \limsup_{n \to \infty} G(gx, gx, gx) \\
= \limsup_{n \to \infty} G(F(x_n, y_n), F(x, y), F(x, y)) \\
\leq \limsup_{n \to \infty} \frac{1}{s^2} \phi(G(gx, gx, gx) + G(gy, gy, gy)) \\
\leq \frac{1}{s^2} \phi(s(G(gx, gx, gx) + G(gy, gy, gy))).
\]
Hence, we get
\[
G(x, gx, gx) \leq \frac{1}{s} \phi(s(G(x, gx, gx) + G(y, gy, gy))).
\]
Similarly, we may show that
\[
G(y, gy, gy) \leq \frac{1}{s} \phi(s(G(x, gx, gx) + G(y, gy, gy))).
\]
Thus,
\[
G(x, gx, gx) + G(y, gy, gy) \leq \frac{2}{s} \phi(s(G(x, gx, gx) + G(y, gy, gy))) \\
\leq 2kG(x, gx, gx) + G(y, gy, gy).
\]
Since \( 2k < 1 \), the last inequality happens only if \( G(x, gx, gx) = 0 \) and \( G(y, gy, gy) = 0 \). Hence \( x = gx \) and \( y = gy \). Thus we get
\[
gx = F(x, x) = x.
\]
To prove the uniqueness, let \( z \in X \) with \( z \neq x \) such that
\[
z = gz = F(z, z).
\]
Then
\[
G(x, z, z) = G(F(x, x), F(z, z), F(z, z)) \leq \frac{1}{s^2} \phi(2G(gz, gz)) \\
< \frac{1}{s^2} 2kG(x, z, z) \leq 2kG(x, z, z).
\]
Since $2k < 1$, we get $G(x, z, z) < G(x, z, z)$, which is a contradiction. Thus, $F$ and $g$ have a unique common fixed point. ■

**Corollary 2.1.** Let $(X, G)$ be a $G_b$-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that

$$G(F(x, y), F(u, v), F(u, v)) \leq \frac{k}{s^2}(G(gx, gu, gu) + G(gy, gv, gv))$$

for all $x, y, u, v \in X$. Assume $F$ and $g$ satisfy the following conditions:
1. $F(X \times X) \subseteq g(X)$,
2. $g(X)$ is complete, and
3. $g$ is continuous and commutes with $F$.

If $k \in (0, \frac{1}{2})$, then there is a unique $x \in X$ such that $gx = F(x, x) = x$.

**Proof.** Follows from Theorem 2.1 by taking $z = u, v = w$ and $\phi(t) = kt$. ■

**Corollary 2.2.** Let $(X, G)$ be a complete $G_b$-metric space. Let $F : X \times X \to X$ be a mapping such that

$$G(F(x, y), F(u, v), F(u, v)) \leq \frac{k}{s^2}(G(x, u, u) + G(y, v, v))$$

for all $x, y, u, v \in X$. If $k \in [0, \frac{1}{2})$, then there is a unique $x \in X$ such that $F(x, x) = x$.

**Remark 2.1.** Since every $G_b$-metric is a $G$-metric when $s = 1$, so our results can be viewed as generalizations and extensions of corresponding results in [35] and several other comparable results.

Now, we introduce some examples for Theorem 2.1.

**Example 2.1.** Let $X = [0, 1]$. Define $G : X \times X \times X \to \mathbb{R}^+$ by

$$G(x, y, z) = (|x - y| + |x - z| + |y - z|)^2$$

for all $x, y, z \in X$. Then $(X, G)$ is a complete $G_b$-metric space with $s = 2$, according to Example 1.1. Define a map $F : X \times X \to X$ by $F(x, y) = \frac{x}{128} + \frac{y}{256}$ for $x, y \in X$. Also, define $g : X \to X$ by $g(x) = \frac{x}{4}$ for $x \in X$ and $\phi(t) = \frac{1}{4}$ for $t \in \mathbb{R}^+$. We have
that

\[ G(F(x, y), F(u, v), F(z, w)) \]
\[ = (|F(x, y) - F(u, v)|^2 + |F(u, v) - F(z, w)|^2 + |F(z, w) - F(x, y)|^2) \]
\[ = \left( \frac{x}{128} + \frac{y}{256} - \frac{u}{128} - \frac{v}{256} + \frac{w}{128} - \frac{z}{256} \right)^2 \]
\[ \leq \left( \frac{1}{128} |x - u| + \frac{1}{256} |y - v| + \frac{1}{128} |u - z| + \frac{1}{256} |v - w| + \frac{1}{128} |z - x| \right. \]
\[ + \frac{1}{256} |w - y| \]
\[ = \left( \frac{1}{32} \left( \frac{x}{4} - \frac{u}{4} + \frac{u}{4} - \frac{z}{4} + \frac{z}{4} - \frac{x}{4} \right) + \frac{1}{64} \left( \frac{y}{4} - \frac{v}{4} + \frac{v}{4} - \frac{w}{4} + \frac{w}{4} - \frac{y}{4} \right) \right)^2 \]
\[ \leq \frac{2}{32} G(gx, gu, gz) + \frac{2}{64^2} G(gy, gv, gw) \]
\[ \leq \frac{1}{4} G(gx, gu, gz) + G(gy, gv, gw) \]
\[ \leq \frac{1}{4} \phi(G(gx, gu, gz) + G(gy, gv, gw)) \]

holds for all \( x, y, u, v, z, w \in X \). It is easy to see that \( F \) and \( g \) satisfy all the hypothesis of Theorem 2.1. Thus \( F \) and \( g \) have a unique common fixed point. Here \( F(0, 0) = g(0) = 0 \).

**Example 2.2.** Let \( X \) and \( G \) be as in Example 2.1. Define a map

\[ F : X \times X \to X \]
by
\[ F(x, y) = \frac{1}{16} x^2 + \frac{1}{16} y^2 + \frac{1}{8} \]

for \( x, y \in X \). Then \( F(X \times X) = [\frac{1}{5}, \frac{3}{4}] \). Also,

\[ G(F(x, y), F(u, v), F(z, w)) \]
\[ = (2|F(x, y) - F(u, v)|^2) \]
\[ = \frac{1}{64} (|x^2 - u^2 + y^2 - v^2|)^2 \]
\[ \leq \frac{1}{64} (|x^2 - u^2| + |y^2 - v^2|)^2 \]
\[ \leq \frac{1}{32} (4|x - u|^2 + 4|y - v|^2) = \frac{1}{32} (G(x, u, u) + G(y, v, v)) \]
\[ \leq \frac{1}{2} \phi(G(x, u, u) + G(y, v, v)) \]

Then by Corollary 2.2, \( F \) has a unique fixed point. Here \( x = 4 - \sqrt{15} \) is the unique fixed point of \( F \), that is, \( F(x, x) = x \).
Now we present an example for the main result in an asymmetric $G_b$-metric space.

**Example 2.3.** Let $X = \{0, 1, 2\}$ and let

$$A = \{(2, 0, 0), (0, 2, 0), (0, 0, 2)\}, \quad B = \{(2, 2, 0), (2, 0, 2), (0, 2, 2)\}$$

and $C = \{(x, x, x) : x \in X\}$.

Define $G : X^3 \to \mathbb{R}^+$ by

$$G(x, y, z) = \begin{cases} 1, & \text{if } (x, y, z) \in A \\ 3, & \text{if } (x, y, z) \in B \\ 4, & \text{if } (x, y, z) \in X^3 - (A \cup B \cup C) \\ 0, & \text{if } x = y = z. \end{cases}$$

It is easy to see that $(X, G)$ is an asymmetric $G_b$-metric space with coefficient $s = \frac{3}{2}$. Also, $(X, G)$ is complete. Indeed, for each $(x_n)$ in $X$ such that $G(x_n, x_m, x_m) \to 0$, then there is a $k \in \mathbb{N}$ such that for each $n \geq k$, $x_n = x_m = x$ for an $x \in X$, so $G(x_n, x_n, x) \to 0$.

Define mappings $F$ and $g$ by

$$F = \begin{pmatrix} (0, 0) & (0, 1) & (1, 0) & (1, 1) & (1, 2) & (2, 1) & (2, 2) & (2, 0) & (0, 2) \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

We see that $F(X \times X) \subseteq gX$, $g$ is continuous and commutes with $F$, and $g(X)$ is complete.

Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{27}{4} \ln(\frac{27}{4} + 1)$. Since

$$(F(x, y), F(u, v), F(z, w)), (gx, gu, gz), (gy, gv, gw) \in A \cup B,$$

we have

$$G(F(x, y), F(u, v), F(z, w)), G(gx, gu, gz), G(gy, gv, gw) \in \{0, 1, 3\}.$$ 

Hence, one can easily check that the contractive condition (2) is satisfied for every $x, y, z, u, v, w \in X$.

Thus, all the conditions of Theorem 2.1 are fulfilled and $F$ and $g$ have a unique common fixed point. Here $F(0, 0) = g(0) = 0.$

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**References**

Coupled fixed point theorems in Gb-metric spaces


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