GLOBAL EXISTENCE AND BOUNDEDNESS OF SOLUTIONS
FOR A GENERAL ACTIVATOR-INHIBITOR MODEL

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Abstract. The purpose of this paper is to prove global existence in time of solutions
for a class of multi-component reaction diffusion systems called multiple Gierer-Meinhardt type.
The system describes, following Gierer-Meinhardt’s scheme, “m” substances in interaction. The
nonlinearities present a difficulty since they are fractions. We prove the global existence by using
a series of Lyapunov functionals.

1. Introduction

We consider the reaction-diffusion system
\[
\frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i = \prod_{k=1}^{m} u_k^{p_{ik}}, \quad \text{on} \ \mathbb{R}^+ \times \Omega, \quad i = 1, \ldots, m,
\]
where we assume that there is no flux through the boundary, i.e., we impose Neu-
mann boundary conditions
\[
\frac{\partial u_i}{\partial \eta} = 0 \quad \text{on} \ \mathbb{R}^+ \times \partial \Omega, \quad i = 1, \ldots, m,
\]
the initial data
\[
u_i(0, x) = u_i^0(x), \quad \text{in} \ \Omega, \quad i = 1, \ldots, m,
\]
are assumed to be nonnegative and in \(L^\infty(\Omega)\). The open domain \(\Omega\) is bounded and
of class \(C^1\), with boundary \(\partial \Omega\) and \(\partial / \partial \eta\) denotes the outward normal derivative
on \(\partial \Omega\). The positive constants \(d_i, i = 1, \ldots, m\), are the diffusion coefficients of the
system. We suppose that the reactions are fractions: for some \(l = 1, \ldots, m\), the
exponents \(p_{ik}\) satisfy
\[
p_{ik} > 0 \quad \text{with} \quad p_{kk} > 1, \quad k = 1, \ldots, l \quad \text{and} \quad p_{ik} < 0, \quad k = l + 1, \ldots, m, \quad \forall i = 1, \ldots, m,
\]

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and that for all $i = 1, \ldots, l$, there exists $j = l + 1, \ldots, m$, such that

$$0 < \frac{p_{ii} - 1}{p_{ji}} < 1,$$

(5)

$$\left(\frac{p_{ii} - 1}{p_{ji}}\right)\left(\frac{p_{jj} - 1}{p_{ij}}\right) < 1.$$

(6)

If we have more than two equations ($m > 2$), we suppose

$$\frac{p_{ik}}{p_{ii} - 1} < \frac{p_{jk}}{p_{ji}}, \text{ for all } k = 1, \ldots, m, \ k \neq i, j.$$

(7)

In the case of Coupled Reaction-Diffusion equations ($m = 2$), problem (1)–(3) describes the pattern formation of spatial tissue structures of hydra in morphogenesis. This mathematical model was proposed by A. Gierer and H. Meinhardt [3] following an idea of A. Turing [14]. The system in the form considered in [5] is the following

$$\begin{cases}
\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = \sigma_1 (x) - a_1 u_1 + \rho_1 (u_1, u_2, x) u_1^{p_{11}} u_2^{p_{12}}, & \text{on } \mathbb{R}_+ \times \Omega, \\
\frac{\partial u_2}{\partial t} - d_2 \Delta u_2 = \sigma_2 (x) - a_2 u_2 + \rho_2 (u_1, u_2, x) u_1^{p_{21}} u_2^{p_{22}}, & \text{on } \mathbb{R}_+ \times \Omega,
\end{cases}$$

(8)

where $u_1$ and $u_2$ are the unknowns representing the concentrations of two substances called the activator and the inhibitor and $a_1$ and $a_2$ are positive constants. The functions $\sigma_1$ and $\sigma_2$ are positive and continuously differentiable on $\Omega$ and $\rho_1$ and $\rho_2 \in C^1 (\mathbb{R}_+^2 \times \Omega) \cap L^\infty (\mathbb{R}_+^2 \times \Omega)$. Under the conditions (4)–(6) on the exponents $p_{ij} (i, j = 1, 2)$ the authors in [2] and [5] later, proved global existence of solutions for system (8) with homogenous Neumann boundary conditions and positive bounded initial conditions.

Earlier partial results of global existence were proved when $N = 3$, $p_{11} = 2$, $p_{21} = 2$, $p_{12} = -1$, $p_{22} = 0$ by F. Rothe [13] in 1984. In 1987, K. Masuda and K. Takahashi [9] extended the result to $\frac{p_{11} - 1}{p_{21}} < \frac{2}{N+2}$. On the other hand, when

$$\frac{p_{ii} - 1}{p_{ji}} > 1,$$

it is shown in [10] that there exist initial values such that the solution blows up in finite time. The equality case in the above inequality and (6) seems to be open.

To get a more general form of system (1) and for reasons of biological applications (especially when $m = 2$ and 3), the reaction terms in (1) are perturbed (following K. Masuda and K. Takahashi [9]) by linear terms to become

$$f_i = \sigma_i - a_i u_i + \rho_i \prod_{k=1}^{m} u_k^{p_{ik}}, \ i = 1, \ldots, m,$$

where $a_i$, $\sigma_i$, $\rho_i$ are positive constants. As an example of biological applications of system (1) with more than three equations, we cite the extended secretion model
in the case of multiple Gierer-Meinhardt systems described by R. Bauer and al. [1] as follows:

\[
\begin{align*}
\frac{\partial a_i}{\partial t} - D_a \frac{\partial^2 a_i}{\partial x^2} &= \rho_a - \mu_a a_i + \rho \frac{a_i^2}{b_i} - \sum_{j \neq i}^m r_{ai}, \\
\frac{\partial b_i}{\partial t} - D_b \frac{\partial^2 b_i}{\partial x^2} &= \rho_b - \mu_b b_i + \rho a_i^2,
\end{align*}
\]

(9)

where \(a_i\) and \(b_i\) indicate the activator and inhibitor substance concentrations respectively, \(i = 1, \ldots, m\).

In this paper, we generalize these results, concerning global existence of solutions to reaction diffusion systems with \(m\) equations by using a series of Lyapunov functionals for coupled components of \((u_1, \ldots, u_m)\) analogous to that considered by the authors in [2] and [5] for \((u_1, u_2)\) (see R. D. Parshad, S. Kouachi and J. B. Gutierrez [11] and S. Kouachi [6,7,8]).

2. Notations and preliminaries

The usual norms in the spaces \(L^p(\Omega), p \geq 1, \ L^\infty(\Omega)\) and \(C(\Omega)\) are respectively denoted by:

\[
\|u\|_p = \left( \frac{1}{|\Omega|} \int_\Omega |u(x)|^p \, dx \right)^{1/p},
\]

\[
\|u\|_\infty = \text{ess. sup}_{x \in \Omega} |u(x)|.
\]

In order to show global existence in time of a solution, we start with the following standard local existence and uniqueness result which follows from the basic existence theory for abstract semilinear differential equations (see D. Henry [4]).

**Proposition 1.** If the initial conditions are positive and uniformly bounded in \(\Omega\), then the problem (1)–(3) admits a unique classical and positive solution \(u = (u_1, \ldots, u_m)\) on \([0, T_{\text{max}}] \times \Omega\). If \(T_{\text{max}} < \infty\) then

\[
\lim_{t \nearrow T_{\text{max}}} \|u(t, \cdot)\|_\infty = \infty.
\]

The proof of the local existence of a solution to a multi-component reaction-diffusion system such as (1) with certain regularity requirement is not a trivial issue. The arguments of the proof are classical applied in \(L^2(\Omega)\) (see also A. Pazy [12, Th.6.1.4]), except that we need additional \(L^\infty\) growth estimates and positivity control. Thus we combine the local Lipschitz continuity of \(f_i\) which is in \(C^1(\mathbb{R}_+^m, \mathbb{R}_+)\), \(i = 1, \ldots, m\), the Gronwall inequality, and the following non-expansiveness property of the semigroup \(S_i(t)\) associated to the operator \(d_i \Delta\) in \(L^\infty(\Omega)\)

\[
\forall p \in [1, +\infty], \quad \|S_i(t) \varphi\|_p \leq \|\varphi\|_p \text{ for all } t \in [0, T] \text{ and } \varphi \in L^\infty(\Omega), \quad i = 1, \ldots, m.
\]

The Banach fixed-point theorem gives the following unique local solution (called mild solution) with values in \(L^\infty(\Omega)\):

\[
u_i(t, \cdot) = S_i(t)u_{i0} + \int_0^t S_i(t-s)f_i(u_1(s, \cdot), \ldots, u_m(s, \cdot)) \, ds, \quad t \in [0, T], \quad i = 1, \ldots, m,
\]
for some positive $T$. We verify by standard arguments that this solution belongs to the space $H^1_{loc}((0,T);L^2(\Omega)) \cap C((0,T);L^2(\Omega))$ and satisfies, in the classical sense, the problem (1)–(3). Finally, the continuation principle is used to get the solution on a maximal interval $[0,T_{\text{max}}]$.

Also, we can show by comparison arguments for parabolic equations (D. Henry [4] and F. Rothe [13]) that if the initial data are nonnegative, then the solutions are nonnegative on $[0,T_{\text{max}}] \times \Omega$. Moreover, from similar arguments using the maximum principle, we have

$$u_i(t,x) > u_0^0, \quad \text{on } [0,T_{\text{max}}] \times \Omega, \quad i = 1, \ldots, m, \quad (10)$$

where $u_0^0 = \min_{\Omega} u_0 > 0$. To see this, since the solutions are nonnegative, we have

$$\frac{\partial u_i}{\partial t} - d_i \Delta u_i > \frac{dy_i}{dt}, \quad \text{on } [0,T_{\text{max}}] \times \Omega, \quad i = 1, \ldots, m,$$

where $y_i$ are the solutions of the following ordinary differential system

$$\frac{dy_i}{dt} = 0, \quad \text{on } [0,T_{\text{max}}], \quad i = 1, \ldots, m,$$

with initial data

$$y_i(0) = u_0^0, \quad i = 1, \ldots, m,$$

then by comparison arguments using the maximum principle, we have

$$u_i(t,x) > y_i > 0, \quad \text{on } [0,T_{\text{max}}] \times \Omega, \quad i = 1, \ldots, m,$$

and since the solutions of the corresponding ordinary differential system are $y_i \equiv u_0^0$, $i = 1, \ldots, m$, this gives (10).

3. Global existence

It is well known that to prove global existence of solutions to (1)–(3) (see Henry [4]), it suffices, thanks to the $L^p$-regularity theory for the heat operator, to derive a uniform estimate of $\|f_i\|_p$ on $[0,T_{\text{max}}]$ for some $p > N/2$.

The main result of this paper is the following theorem.

**Theorem.** Let $u(t,\cdot) = (u_1(t,\cdot), \ldots, u_m(t,\cdot))$ be any solution of the problem (1)–(3); then under conditions (4)–(7) the functional

$$t \mapsto L(t) = \int_{\Omega} \prod_{k=1}^m u_k^{\alpha_k} \, dx, \quad (11)$$

is bounded on $[0,T_{\text{max}}]$ for all $\alpha_j > 1$, $j = 1, \ldots, l$ and all $\alpha_j < 0$, $j = l+1, \ldots, m$.

For the proof of the Theorem, we need the following Lemma

**Lemma 1.** Let $\alpha > 0$ and $\beta < 0$ be two numbers; then for all positive numbers $u_1, \ldots, u_m$ ($u_0 = 0$) and all $p_k$ and $p_j$, $1 \leq k \leq m$, satisfying (4)–(7), there exist $\theta(\alpha) \in (0,1)$ and a positive constant $C_1(\alpha)$ such that

$$\frac{\alpha}{u_i} \left(u_i^\alpha u_j^\beta\right)^{\prod_{k=1}^m u_k^{p_k}} \leq -\frac{\beta}{u_j} \left(u_i^\alpha u_j^\beta\right)^{\prod_{k=1}^m u_k^{p_k}} + C_1 \left(u_i^\alpha u_j^\beta\right)^\theta, \quad (12)$$

for all $u_k \geq u_k$, $k = 1, \ldots, m$. 
Proof. Relation (12) is equivalent to
\[
\frac{\alpha}{u_i} \prod_{k=1}^{m} u_k^{p_{ik}} \leq -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{ik}} + C_1 \left( u_i^{\alpha} u_j^{\beta} \right)^{\theta - 1}, \quad \text{for all } u_k \geq u_k, \ k = 1, \ldots, m. \tag{13}
\]
We have
\[
\frac{\alpha}{u_i} \prod_{k=1}^{m} u_k^{p_{ik}} = C_2 \left( -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{ik}} \right) = C_2 \left( -\frac{\beta}{u_j} \prod_{k=1, k \neq i,j}^{m} u_k^{p_{ik}} \right) = C_2 \left( -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{ik}} \right)
\]
\[
= C_2 \left( -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{ik}} \right) = C_2 \left( -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{ik}} \right)
\]
\[
= C_2 \left( -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{ik}} \right) = C_2 \left( -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{ik}} \right)
\]
\[
\text{where } C_2 \text{ is a positive constant. As } p_{ij} < 0 \text{ and } u_j \geq u_j > 0, \text{ then under condition (6) and for } \epsilon \text{ sufficiently small, we have}
\]
\[
p_{ij} \left[ 1 - \left( \frac{p_{ij} - 1}{u_j} \right) \right] + \epsilon \left( \frac{p_{ij} - 1}{u_j} \right) \leq C_3,
\]
\[
\text{where } C_3 \text{ is a positive constant. Since } p_{ii} > 1 \text{ and } u_k \geq u_k > 0, \ k = 1, \ldots, m, \text{ and } k \neq i, j, \text{ then under condition (7) and for } \epsilon \text{ sufficiently small, we have}
\]
\[
\prod_{k=1, k \neq i,j}^{m} u_k^{p_{ik} - p_{ij} \left( \frac{p_{ij} - 1}{u_j} \right) - \epsilon p_{ij}} \leq C_4,
\]
\[
\text{where } C_4 \text{ is a positive constant. This implies that}
\]
\[
\frac{\alpha}{u_i} \prod_{k=1}^{m} u_k^{p_{ik}} \leq C_5 \left( -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{ik}} \right) = C_5 \left( -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{ik}} \right)
\]
\[
\text{for all } u_k > u_k, \ k = 1, \ldots, m, \text{ where } C_5 \text{ is a positive constant. Now, under condition (5), Young’s inequality in the form}
\]
\[
abla e_a^{p} + C(\epsilon) b^{q}, \quad \frac{1}{p} + \frac{1}{q} = 1,
\]
\[
\text{for}
\]
\[
\frac{1}{p} = \frac{p_{ii} - 1}{p_{ij}} + \epsilon,
\]
\[
\text{where } \epsilon \text{ is sufficiently small, is applicable to the right-hand side of inequality (14) to deduce (13) with}
\]
\[
\theta = 1 - \frac{(p_{ij})}{1 - \frac{p_{ii} - 1}{p_{ij}} - \epsilon}.
\]
This completes the proof of the Lemma. \(\blacksquare\)
The proof of the above Lemma can be found in [2].

Proof of the Theorem. Put

\[ L_{ij}(t) = \int_{\Omega} u_i^{\alpha_i} u_j^{\alpha_j} \, dx. \] (15)

Differentiating \( L_{ij} \) with respect to \( t \) yields

\[
L'_{ij}(t) = \int_{\Omega} \left( \frac{\alpha_i}{u_i} \frac{\partial u_i}{\partial t} + \frac{\alpha_j}{u_j} \frac{\partial u_j}{\partial t} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx
= \int_{\Omega} \left( \frac{\alpha_i}{u_i} \Delta u_i + \frac{\alpha_j}{u_j} \Delta u_j \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx
+ \int_{\Omega} \left( \frac{\alpha_i}{u_i} \prod_{k=1}^m u_k^{\nu_{ik}} + \frac{\alpha_j}{u_j} \prod_{k=1}^m u_k^{\nu_{jk}} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx
= I + J. \] (16)

By a simple use of Green’s formula, we get

\[
I = - \int_{\Omega} u_i^{\alpha_i} u_j^{\alpha_j} \left( d_i \left( -\alpha_i + \alpha_i^2 \right) \left| \nabla u_i \right| \right)^2
+ (d_i + d_j) \alpha_i \alpha_j \frac{\nabla u_i}{u_i} \frac{\nabla u_j}{u_j} + d_j \left( -\alpha_j + \alpha_j^2 \right) \left| \nabla u_j \right|^2 \, dx.
\]

Therefore, \( I \leq 0 \), if

\[
\left( (d_i + d_j) \alpha_i \alpha_j \right)^2 - 4d_i d_j \left( -\alpha_i + \alpha_i^2 \right) \left( -\alpha_j + \alpha_j^2 \right) < 0,
\]

that is

\[
\left( \frac{\alpha_i}{\alpha_i} - 1 \right) \left( \frac{\alpha_j}{\alpha_j} - 1 \right) > \frac{(d_i + d_j)^2}{4d_i d_j}.
\]

If for all \( \alpha_i > 1 \), we choose \( \alpha_j < 0 \) sufficiently close to zero, we get \( I \leq 0 \).

For the integral \( J \) given by (16), we use (12) to get

\[
J \leq C_1 \int_{\Omega} \left( u_i^{\alpha_i} u_j^{\alpha_j} \right)^\theta \, dx. \] (17)

Since \( 0 < \theta < 1 \), by application of Hölder’s inequality, we get

\[
J \leq C_6 \left( \int_{\Omega} u_i^{\alpha_i} u_j^{\alpha_j} \, dx \right)^\theta, \] (18)

where \( C_6 \) is a positive constant. Since \( I \leq 0 \), we get

\[
L'_{ij} \leq C_6 L_{ij}^\theta, \quad \text{on } [0, T_{\text{max}}]. \] (19)

If \( T_{\text{max}} < +\infty \), a simple integration gives the boundedness of the functional \( L_{ij} \) on the interval \([0, T_{\text{max}}]\). As \( u_j \) is bounded below, we get the boundedness of the functional \( L_{ij} \) on the interval \([0, T_{\text{max}}]\).
Since for each $\alpha_i > 1$, $i = 1, \ldots, l$, there exists an $\alpha_j < 0$, $j = l+1, \ldots, m$, such that the functional $L_{ij}$ is bounded on the interval $[0, T_{\max}]$, we have $l$ functionals $L_{1j_1}, L_{2j_2}, \ldots, L_{lj_l}$ of the form (15) which are bounded on $[0, T_{\max}]$.

By application of Hölder’s inequality to the functional $L_i$ defined by

$$L_i(t) = \int_{\Omega} \prod_{i=1}^{l} u_i^{\alpha_i} \, dx,$$  \hspace{1cm} (20)

we get

$$L_i(t) \leq \prod_{i=1}^{l} \left( \int_{\Omega} (u_i^{\alpha_i} \, dx)^{p_i} \right)^{\frac{1}{p_i}},$$

where the exponents $p_i > 1$, $i = 1, \ldots, l$, satisfy $\sum_{i=1}^{l} \frac{1}{p_i} = 1$. As the functionals $L_{ij_i}, i = 1, \ldots, l$ are bounded on $[0, T_{\max}]$, for all $\alpha_i > 1$ and all $\alpha_j < 0$, $i = 1, \ldots, l$, the functional $L_i$ given by (20) is bounded on $[0, T_{\max}]$. For the remaining non-positive exponents $\alpha_j$, intervening in the expression of the functional $L$ defined by (11) and which are not present in the expression of the functional $L_i$ (if they exist), we use the fact that the corresponding components $u_j$ of the solution $u$ are uniformly bounded below on $[0, T_{\max}]$ by a positive constant. This gives the boundedness of the functional $L$ and ends the proof of the theorem. \[\blacksquare\]

**Corollary 1.** Under conditions (5)–(7), all solutions of problem (1)–(3) with positive initial data in $L^\infty(\Omega)$ are global.

**Proof.** Using the Theorem, all reaction terms $f_i = \prod_{k=1}^{m} u_k^{p_{ik}}$, $i = 1, \ldots, m$, are in $L^\infty(0, T_{\max}; L^p(\Omega))$ for all $p \geq 1$. We take $p > N/2$ to derive a uniform estimate of $\|f_i\|_p$ on $[0, T_{\max}]$. This gives, from the preliminary observations, that the solution will never blow up in $L^\infty(\Omega)$ at any finite time $T_{\max}$, hence it exists globally ($T_{\max} = +\infty$). \[\blacksquare\]

**Corollary 2.** The solution of problem (1)–(3) remains global when the reaction terms are perturbed by linear terms to get the following form

$$f_i = \sigma_i(x) - a_i u_i + \rho_i(x, u) \prod_{k=1}^{m} u_k^{p_{ik}}, \quad i = 1, \ldots, m,$$  \hspace{1cm} (21)

where $a_i$ are positive constants, $\sigma_i \in C^1(\Omega)$, $\sigma_i \geq 0$, $\rho_i \in C^1(\overline{\Omega} \times \mathbb{R}^m) \cap L^\infty(\overline{\Omega} \times \mathbb{R}^m)$, with $\rho_i > 0$, $i = 1, \ldots, m$.

**it Proof.** In this case the integral $I$ remains non-positive and

$$J = \int_{\Omega} \left( \frac{a_i \sigma_i}{u_i} + \frac{a_j \sigma_j}{u_j} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx + (a_i + a_j) \int_{\Omega} u_i^{\alpha_i} u_j^{\alpha_j} \, dx$$

$$+ \int_{\Omega} \left( \rho_i \frac{a_i}{u_i} \prod_{k=1}^{m} u_k^{p_{ik}} + \rho_j \frac{a_j}{u_j} \prod_{k=1}^{m} u_k^{p_{jk}} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx. \hspace{1cm} (22)$$
Using the fact that the \( u_i \) respectively the \( \sigma_i \) are uniformly bounded below on \([0, T_{\text{max}}] \) by a positive constant, respectively uniformly bounded on \( \Omega \), we get for the first integral in (22)

\[
\int_{\Omega} \left( \frac{\alpha_i \sigma_i}{u_i} + \frac{\alpha_j \sigma_j}{u_j} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx \leq C_T L_{ij} \text{ on } [0, T_{\text{max}}],
\]

where \( C_T \) is a positive constant. Finally, since \( \rho_i / \rho_j \) are uniformly bounded on \( \Omega \times \mathbb{R}^m_+ \) using the Lemma, we obtain an analogous differential inequality to (19). Following the same reasoning as in the Theorem, we deduce the global existence of solutions to problem (1)–(3) with the reaction terms (21).

**Remark 1.** In the case when \( p_{ij} = 0 \) for some \( i = 1, \ldots, l \) and \( j = l+1, \ldots, m \), condition (6) is replaced by

\[
(p_{jj} - 1) \left( \frac{p_{ii} - 1}{p_{ji}} \right) > 0.
\]

**Remark 2.** Condition (6) can be replaced by the following simpler but stronger condition

\[
\frac{p_{jj} - 1}{p_{ij}} < 1,
\]

which, together with (5), implies the condition (6).

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