FIXED POINT FOR FUZZY CONTRACTION MAPPINGS
SATISFYING AN IMPLICIT RELATION

Ismat Beg and M. A. Ahmed

Abstract. We prove a common fixed point theorem for generalized fuzzy contraction mappings satisfying an implicit relation.

1. Introduction and preliminaries

Heilpern [8] introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler [11]. Afterwards several fixed point theorems for fuzzy contractive mappings have appeared in the literature (see, [1–5, 12, 13, 15]). In this paper, we prove a common fixed point theorem for fuzzy mappings satisfying an implicit relations. Our results generalize and extend results in Rashwan and Ahmed [14], Arora and Sharma [1, Lemma 3.1] and Lee and Cho [10, Proposition 3.2].

Let \((X, d)\) be a metric linear space [8]. A fuzzy set in \(X\) is a function with domain \(X\) and values in \([0, 1]\). If \(A\) is a fuzzy set and \(x \in X\), then the function-value \(A(x)\) is called the grade of membership of \(x\) in \(A\). The collection of all fuzzy sets in \(X\) is denoted by \(\mathcal{F}(X)\). A fuzzy mapping on a set \(X\) is a usual mapping from \(X\) into \(\mathcal{F}(X)\).

Let \(A \in \mathcal{F}(X)\) and \(\alpha \in [0, 1]\). The \(\alpha\)-level set of \(A\), denoted by \(A_\alpha\), is defined by

\[ A_\alpha = \{ x : A(x) \geq \alpha \} \quad \text{if} \quad \alpha \in (0, 1], \quad A_0 = \{ x : A(x) > 0 \}, \]

whenever \(B\) is the closure of set (nonfuzzy) \(B\).

Definition 1.1. [8] A fuzzy set \(A\) in \(X\) is an approximate quantity if and only if its \(\alpha\)-level set is a nonempty compact convex subset (nonfuzzy) of \(X\) for each \(\alpha \in [0, 1]\).

2010 Mathematics Subject Classification: 47H10; 54H25

Keywords and phrases: Fuzzy sets; fuzzy map; fuzzy contractive mappings; common fixed points.
The set of all approximate quantities, denoted by $W(X)$, is a subcollection of $\mathcal{Y}(X)$.

**Definition 1.2.** [11] Let $A, B \in W(X)$, $\alpha \in [0, 1]$ and $CP(X)$ be a set of all nonempty compact subsets of $X$. Then

$$p_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y), \quad \delta_\alpha(A, B) = \sup_{x \in A_\alpha, y \in B_\alpha} d(x, y)$$

and $D_\alpha(A, B) = H(A_\alpha, B_\alpha)$, where $H$ is the Hausdorff metric between two sets in the collection $CP(X)$.

We also define the following functions

$$p(A, B) = \sup_\alpha p_\alpha(A, B), \quad \delta(A, B) = \sup_\alpha \delta_\alpha(A, B)$$

and $D(A, B) = \sup_\alpha D_\alpha(A, B)$. It is noted that $p_\alpha$ is a nondecreasing function of $\alpha$.

**Definition 1.3.** [11] Let $A, B \in W(X)$. Then $A$ is said to be more accurate than $B$ (or that $B$ includes $A$), denoted by $A \subset B$, if and only if $A(x) \leq B(x)$ for each $x \in X$.

The relation $\subset$ induces a partial order on $W(X)$.

**Definition 1.4.** [4] Let $X$ be an arbitrary set and $Y$ be a metric linear space. The mapping $T$ is said to be a fuzzy mapping if and only if $T$ is a mapping from the set $X$ into $W(Y)$, i.e., $T(x) \in W(Y)$ for each $x \in X$.

The following proposition is used in the sequel.

**Proposition 1.5.** [11] If $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.

Let $\Psi$ be the family of real valued lower semi-continuous functions $F : [0, \infty)^6 \to \mathbb{R}$, satisfying the following conditions:

$(\psi_1)$ $F$ is non-decreasing in $1^{st}$ coordinate and $F$ is non-increasing in $3^{rd}$, $4^{th}$, $5^{th}$, $6^{th}$ coordinate variable,

$(\psi_2)$ there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with

$(\psi_{21}) \; F(u, v, u, u + v, 0) \leq 0$ or $(\psi_{22}) \; F(u, v, u, v, 0, u + v) \leq 0$,

we have $u \leq hv$, and

$(\psi_3)$ $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

Conditions $\psi_i \; (i = 1, 2, 3)$ are called implicit conditions and we refer for examples and their applications in fixed point theory to Beg and Butt [6, 7].
2. Main results

Let \((X,d)\) be a metric space. We consider a subcollection of \(\mathcal{G}(X)\) denoted by \(W^*(X)\); for any \(A \in W^*(x)\), its \(\alpha\)-level set is a nonempty compact subset (nonfuzzy) of \(X\) for each \(\alpha \in [0,1]\). It is obvious that each element \(A \in W(X)\) leads to \(A \in W^*(X)\) but the converse is not true.

Next, we introduce the improvements of the lemmas in Heilpern [8] as follows.

**Lemma 2.1.** If \(\{x_0\} \subset A\) for each \(A \in W^*(X)\) and \(x_0 \in X\), then \(p_\alpha(x_0,B) \leq D_\alpha(A,B)\) for each \(B \in W^*(X)\).

**Lemma 2.2.** \(p_\alpha(x,A) \leq d(x,y) + p_\alpha(y,A)\) for all \(x,y \in X\) and \(A \in W^*(X)\).

**Lemma 2.3.** Let \(x \in X\), \(A \in W^*(X)\) and \(\{x\}\) be a fuzzy set with membership function equal to a characteristic function of the set \(\{x\}\). Then \(\{x\} \subset A\) if and only if \(p_\alpha(x,A) = 0\) for each \(\alpha \in [0,1]\).

**Proof.** If \(\{x\} \subset A\), then \(x \in \overline{A}_\alpha\) for each \(\alpha \in [0,1]\). It implies that \(p_\alpha(x,A) = \inf_{y \in A_\alpha} d(x,y) = 0\) for each \(\alpha \in [0,1]\).

Conversely, if \(p_\alpha(x,A) = 0\), then \(\inf_{y \in A_\alpha} d(x,y) = 0\). It follows that \(x \in \overline{A}_\alpha = A_\alpha\) for each \(\alpha \in [0,1]\). Thus \(\{x\} \subset A\). \(\blacksquare\)

Next, we state and prove a new lemma.

**Lemma 2.4.** Let \((X,d)\) be a complete metric space, \(T : X \rightarrow W^*(X)\) be a fuzzy map and \(x_0 \in X\). Then there exists \(x_1 \in X\) such that \(\{x_1\} \subset T(x_0)\).

**Proof.** For \(n \in \mathbb{N}\), \(((T(x_0))_{n/(n+1)})\) is a decreasing sequence of nonempty compact subsets of \(X\). Thus we have from [16, Prop. 11.4 and Remark 11.5 on page 495-496] that \(\bigcap_{n=1}^{\infty} (T(x_0))_{n/(n+1)}\) is nonempty and compact. Let \(x_1 \in \bigcap_{n=1}^{\infty} (T(x_0))_{n/(n+1)}\). Then \(\frac{n}{n+1} \leq \frac{(T(x_0))(x_1)}{1}\). As \(n \to \infty\), we get that \((T(x_0))(x_1) = 1\). It implies that \(\{x_1\} \subset T(x_0)\). \(\blacksquare\)

**Remark 2.5.** Lemma 2.4 is a generalization of Arora and Sharma [1, Lemma 3.1] and Lee and Cho [10, Prop.3.2].

Now, we prove our main theorem.

**Theorem 2.6.** Let \((X,d)\) be a complete metric space and \(T_1, T_2\) be fuzzy mappings from \(X\) into \(W^*(X)\). If there is an \(F \in \Psi\) such that for all \(x,y \in X\),

\[
F(D(T_1(x), T_2(y)), d(x,y), p(x,T_1(x)), p(y,T_2(y)), p(x,T_2(y)), p(y,T_1(x))) \leq 0,
\]

then there exists \(z \in X\) such that \(\{z\} \subset T_1(z)\) and \(\{z\} \subset T_2(z)\).

**Proof.** Let \(x_0 \in X\). Then by Lemma 2.4, there exists an element \(x_1 \in X\) such that \(\{x_1\} \subset T_1(x_0)\). For \(x_1 \in X\), \((T_2(x_1))_1\) is a nonempty compact subset of \(X\). Since \((T_1(x_0))_1, (T_2(x_1))_1 \in CP(X)\) and \(x_1 \in (T_1(x_0))_1\), then Proposition 1.5
asserts that there exists \( x_2 \in (T_2(x_1))_1 \) such that \( d(x_1, x_2) \leq D_1(T_1(x_0), T_2(x_1)) \). So, we have from Lemma 2.3 and the property \((\psi_1)\) of \( F \) that

\[
F(d(x_1, x_2), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0) \\
\leq F(D_1(T_1(x_0), T_2(x_1)), d(x_0, x_1), p(x_0, T_1(x_0)), p(x_1, T_2(x_1)), \\
p(x_0, T_2(x_1)), p(x_1, T_1(x_0))) \\
\leq F(D(T_1(x_0), T_2(x_1)), (d(x_0, x_1), p(x_0, T_1(x_0)), p(x_1, T_2(x_1)), \\
p(x_0, T_2(x_1)), p(x_1, T_1(x_0))) \leq 0.
\]

From the property \((\psi_2)\) of \( F \in \Psi \), there exists \( h \in (0, 1) \) such that \( d(x_1, x_2) \leq h d(x_0, x_1) \). Similarly, one can deduce from the property \((\psi_2')\) of \( F \in \Psi \) that there exists \( h \in (0, 1) \) such that \( d(x_2, x_3) \leq h d(x_1, x_2) \). By induction, we have a sequence \( (x_n) \) of points in \( X \) such that, for all \( n \in N \cup \{0\} \),

\[
\{x_{2n+1}\} \subset T_1(x_{2n}), \quad \{x_{2n+2}\} \subset T_2(x_{2n+1}).
\]

It follows by induction that \( d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \). Since

\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
\leq h^n d(x_0, x_1) + h^{n+1} d(x_0, x_1) + \cdots + h^{m-1} d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1),
\]

then \( \lim_{n,m \to \infty} d(x_n, x_m) = 0 \). Therefore, \( (x_n) \) is a Cauchy sequence. Since \( X \) is a complete metric space, then there exists \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \).

Next, we show that \( \{z\} \subset T_i(z), \ i = 1, 2 \). We get from Lemma 2.1 and Lemma 2.2 that

\[
p_\alpha(z, T_2(z)) \leq d(z, x_{2n+1}) + p_\alpha(x_{2n+1}, T_2(z)) \leq d(z, x_{2n+1}) + D_\alpha(T_1(x_{2n}), T_2(z)),
\]

for each \( \alpha \in [0, 1] \). Taking supremum on \( \alpha \) in the last inequality, we obtain from the property \((\psi_1)\) of \( F \) that

\[
F(p(x_{2n+1}, T_2(z)), d(x_n, z), d(x_n, x_{2n+1}), p(z, T_2(z)), p(x_2n, T_2(z)), d(z, x_{2n+1})) \\
\leq F(D_1(T_1(x_{2n}), T_2(z)), d(x_n, z), p(x_2n, T_1(x_{2n})), p(z, T_2(z)), p(x_{2n}, T_2(z)), \\
p(z, T_1(x_{2n}))) \\
\leq F(D(T_1(x_{2n}), T_2(z)), d(x_n, z), p(x_2n, T_1(x_{2n})), p(z, T_2(z)), p(x_{2n}, T_2(z)), \\
p(z, T_1(x_{2n}))) \leq 0.
\]

As \( n \to \infty \), we have

\[
F(p(z, T_2(z)), 0, 0, p(z, T_2(z)), p(z, T_2(z)), 0) \leq 0.
\]

From the property \((\psi_3)\) of \( F \in \Psi \), it yields that \( p(z, T_2(z)) = 0 \). So, we get from Lemma 2.3 that \( \{z\} \subset T_2(z) \). Similarly, it can be shown that \( \{z\} \subset T_1(z) \). ■
**Example 2.7.** Let $X = [0, 1]$ be endowed with the metric $d$ defined by $d(x, y) = |x - y|$. It is clear that $(X, d)$ is a complete metric space. Assume that $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{3}{4}t_2$ for every $t_1, t_2, t_3, t_4, t_5, t_6 \in [0, \infty)$. It is obvious that $F \in \Psi$. Let $T_1 = T_2 = T$. Define a fuzzy mapping $T$ on $X$ such that for all $x \in X$, $T(x)$ is the characteristic function for $\left\{ \frac{1}{2}x \right\}$. For each $x, y \in X$,

$$F(D(F(x), F(y)), d(x, y), p(x, F(x)), p(y, F(y)), p(x, F(y)), p(y, F(x))) = D(F(x), F(y)) - \frac{3}{4}d(x, y) = \frac{3}{4}d(x, y) - \frac{3}{4}d(x, y) = 0.$$ 

The characteristic function for $\{0\}$ is the fixed point of $T$.

**Remark 2.8.** (I) If there is an $F \in \Psi$ such that, for each $x, y \in X$,

$$F(\delta(T_1(x), T_2(y)), d(x, y), p(x, T_1(x)), p(y, T_2(y)), p(x, T_2(y)), p(y, T_1(x))) \leq 0,$$

then the conclusion of Theorem 2.6 remains valid. This result is considered as a special case of Theorem 2.6 because $D(T_1(x), T_2(y)) \leq \delta(T_1(x), T_2(y))$ [9, page 414].

(II) Park and Jeong [12, Theorems 3.1 and 3.4] and Rashwan and Ahmed [14, Theorem 2.1] are special cases of Theorem 2.6.

**Acknowledgement.** The present version of the paper owes much to the precise and kind remarks of the learned referees.

**References**


(received 07.12.2012; in revised form 20.03.2013; available online 20.04.2013)

I. Beg, Centre for Mathematics and Statistical Sciences, Lahore School of Economics, 53200-Lahore, Pakistan
E-mail: begismat@yahoo.com

M.A. Ahmed, Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt
E-mail: mahmed68@yahoo.com