LIGHTLIKE SUBMANIFOLDS OF INDEFINITE PARA-SASAKIAN MANIFOLDS

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Abstract. In this paper, we study invariant, slant and screen slant lightlike submanifolds of indefinite para-Sasakian manifolds. We obtain necessary and sufficient conditions for existence of slant and screen slant lightlike submanifolds of indefinite para-Sasakian manifolds and also provide non-trivial examples of such submanifolds. We obtain integrability conditions of distributions $D$ and $RadTM$ on screen slant lightlike submanifolds of indefinite para-Sasakian manifold. Further we obtain sufficient condition for induced connection on screen slant lightlike submanifolds of indefinite para-Sasakian manifold to be metric connection.

1. Introduction

A submanifold of a semi-Riemannian manifold is called a lightlike submanifold if the induced metric on it is degenerate. In [3], Duggal and Bejancu introduced the geometry of arbitrary lightlike submanifolds of semi-Riemannian manifolds. Lightlike geometry has its applications in general relativity, particularly in black hole theory, which gave impetus to study lightlike submanifolds of semi-Riemannian manifolds equipped with certain structures. Lightlike submanifolds of an indefinite Sasakian manifold have been studied by Duggal and Sahin in [5]. In 2009, Sahin [9] study screen slant lightlike submanifolds of indefinite Kaehler manifold. In [11], authors introduced the notion of an $\epsilon$-para-Sasakian structure and gave some examples.

In this article, we study lightlike submanifolds of an $\epsilon$-para-Sasakian manifold, which is called an indefinite para-Sasakian manifold. The paper is arranged as follows. Section 2 contains some basic results and definitions. In Section 3, we study invariant lightlike submanifolds of an indefinite para-Sasakian manifold giving some examples. Section 4 deals with slant lightlike submanifolds of an indefinite para-Sasakian manifold. In Section 5, we study screen slant lightlike submanifolds of an indefinite para-Sasakian manifold and obtain integrability conditions of distributions $D$ and $RadTM$.

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2. Preliminaries

A semi-Riemannian manifold \((\overline{M}, \overline{g})\) is called an \(\epsilon\)-almost paracontact metric manifold [11] if there exists a \((1, 1)\) tensor field \(\phi\), a vector field \(V\) called the characteristic vector field and a 1-form \(\eta\), satisfying

\[
\phi^2 X = X - \eta(X)V, \quad \eta(V) = \epsilon, \quad \eta \circ \phi = 0, \quad \phi V = 0,
\]

\[
\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(T\overline{M}),
\]

where \(\epsilon = 1\) or \(-1\). It follows that

\[
\overline{g}(V, V) = \epsilon, \quad \overline{g}(V, X) = \eta(X), \quad \overline{g}(X, \phi Y) = \overline{g}(\phi X, Y), \quad \forall X, Y \in \Gamma(T\overline{M}).
\]

Then \((\phi, V, \eta, \overline{g})\) is called an \(\epsilon\)-almost paracontact metric structure on \(\overline{M}\).

An \(\epsilon\)-almost paracontact metric structure \((\phi, V, \eta, \overline{g})\) is called an indefinite para-Sasakian structure [11] if

\[
(\nabla_X \phi)Y = -\overline{g}(\phi X, \phi Y)V - \epsilon \eta(Y)\phi^2 X, \quad \forall X, Y \in \Gamma(T\overline{M}),
\]

where \(\nabla\) is Levi-Civita connection with respect to \(\overline{g}\).

A semi-Riemannian manifold endowed with an indefinite para-Sasakian structure is called an indefinite para-Sasakian manifold. From (2.4), we get

\[
(\nabla_X V) = \phi X, \quad \forall X \in \Gamma(T\overline{M}).
\]

Let \((\overline{M}, \overline{g}, \phi, V, \eta)\) be an \(\epsilon\)-almost paracontact metric manifold. If \(\epsilon = 1\), then \(\overline{M}\) is said to be a spacelike \(\epsilon\)-almost paracontact metric manifold and if \(\epsilon = -1\), then \(\overline{M}\) is called a timelike \(\epsilon\)-almost paracontact metric manifold. In this paper we consider indefinite para-Sasakian manifold with spacelike characteristic vector field \(V\).

A submanifold \((M^m, g)\) immersed in a semi-Riemannian manifold \((\overline{M}^{m+n}, \overline{g})\) is called a lightlike submanifold [3] if the metric \(g\) induced from \(\overline{g}\) is degenerate and the radical distribution \(RadTM\) is of rank \(r\), where \(1 \leq r \leq m\). Let \(S(TM)\) be a screen distribution which is a semi-Riemannian complementary distribution of \(RadTM\) in \(TM\), that is

\[
TM = RadTM \oplus_{\text{orth}} S(TM).
\]

Now consider a screen transversal vector bundle \(S(TM^\perp)\), which is a semi-Riemannian complementary vector bundle of \(RadTM\) in \(TM^\perp\). Since for any local basis \(\{\xi_i\}\) of \(RadTM\), there exists a local null frame \(\{N_j\}\) of sections with values in the orthogonal complement of \(S(TM^\perp)\) in \([S(TM)]^\perp\) such that \(\overline{g}(\xi_i, N_j) = \delta_{ij}\) and \(\overline{g}(N_i, N_j) = 0\), it follows that there exists a lightlike transversal vector bundle \(ltr(TM)\) locally spanned by \(\{N_i\}\). Let \(tr(TM)\) be complementary (but not orthogonal) vector bundle to \(TM\) in \(T\overline{M}|_M\). Then

\[
tr(TM) = ltr(TM) \oplus_{\text{orth}} S(TM^\perp),
\]

\[
T\overline{M}|_M = TM \oplus tr(TM),
\]

\[
T\overline{M}|_M = S(TM) \oplus_{\text{orth}} [RadTM \oplus ltr(TM)] \oplus_{\text{orth}} S(TM^\perp).
\]
The following are four cases of a lightlike submanifold \((M, g, S(TM), S(TM^\perp))\):

- **Case 1.** \(r\)-lightlike if \(r < \min(m, n)\),
- **Case 2.** co-isotropic if \(r = n < m\), \(S(TM^\perp) = \{0\}\),
- **Case 3.** isotropic if \(r = m < n\), \(S(TM) = \{0\}\),
- **Case 4.** totally lightlike if \(r = m = n\), \(S(TM) = S(TM^\perp) = \{0\}\).

The Gauss and Weingarten formulae are given as

\[
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \quad (2.6)
\]
\[
\nabla_X V = -A_V X + \nabla_X^t V, \quad \forall V \in \Gamma(tr(TM)), \quad (2.7)
\]

where \(\{\nabla_X Y, A_V X\}\) and \(\{h(X, Y), \nabla_X^t V\}\) belong to \(\Gamma(TM)\) and \(\Gamma(tr(TM))\) respectively. \(\nabla\) and \(\nabla^t\) are linear connections on \(M\) and on the vector bundle \(tr(TM)\) respectively. The second fundamental form \(h\) is a symmetric \(F(M)\)-bilinear form on \(\Gamma(TM)\) with values in \(\Gamma(tr(TM))\) and the shape operator \(A_V\) is a linear endomorphism of \(\Gamma(TM)\). From (2.6) and (2.7), we have

\[
\nabla_X Y = \nabla_X Y + h^t(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM), \quad (2.8)
\]
\[
\nabla_X N = -A_N X + \nabla_X^t (N) + D^s(X, N), \quad \forall N \in \Gamma(ltr(TM)), \quad (2.9)
\]
\[
\nabla_X W = -A_W X + \nabla_X^t (W) + D^t(X, W), \quad \forall W \in \Gamma(S(TM^\perp)), \quad (2.10)
\]

where \(h^t(X, Y) = L(h(X, Y))\), \(h^s(X, Y) = S(h(X, Y))\), \(D^t(X, W) = L(\nabla_X W)\), \(D^s(X, N) = S(\nabla_X N)\). \(L\) and \(S\) are the projection morphisms of \(tr(TM)\) on \(ltr(TM)\) and \(S(TM^\perp)\) respectively. \(\nabla^t\) and \(\nabla^s\) are linear connections on \(ltr(TM)\) and \(S(TM^\perp)\) called the lightlike connection and screen transversal connection on \(M\) respectively. For any vector field \(X\) tangent to \(M\), we put

\[
\phi X = PX + FX, \quad (2.11)
\]

where \(PX\) and \(FX\) are tangential and transversal parts of \(\phi X\) respectively.

Now by using (2.6), (2.8)–(2.10) and metric connection \(\nabla\), we obtain

\[
\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^t(X, W)) = g(A_W X, Y),
\]
\[
\bar{g}(D^s(X, N), W) = g(N, A_W X).
\]

Denote the projection of \(TM\) on \(S(TM)\) by \(\mathcal{P}\). Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

\[
\nabla_X \mathcal{P}Y = \nabla_X^s \mathcal{P}Y + h^s(X, \mathcal{P}Y), \quad \forall X, Y \in \Gamma(TM),
\]
\[
\nabla_X \xi = -A^\xi X + \nabla_X^t \xi, \quad \xi \in \Gamma(RadTM).
\]

By using above equations, we obtain

\[
\bar{g}(h^t(X, \mathcal{P}Y), \xi) = g(A^\xi X, \mathcal{P}Y),
\]
\[
\bar{g}(h^s(X, \mathcal{P}Y), N) = g(A_N X, \mathcal{P}Y),
\]
\[
\bar{g}(h^t(X, \xi), \xi) = 0, \quad A^\xi \xi = 0. \quad (2.12)
\]
It is important to note that in general \( \nabla \) is not a metric connection. Since \( \nabla \) is metric connection, by using (2.8), we get

\[
(\nabla_X g)(Y, Z) = \bar{g}(h^i(X, Y), Z) + \bar{g}(h^i(X, Z), Y).
\]

**Definition 2.1.** [3] A submanifold \( M \) of semi-Riemannian manifold \( (\bar{M}, \bar{g}) \) is said to be totally geodesic lightlike submanifold of \( \bar{M} \) if any geodesic of \( M \), with respect to Levi-Civita connection \( \nabla \), is a geodesic of \( \bar{M} \), i.e., \( h^i = h^s = 0 \) on \( M \).

**Definition 2.2.** [1] A lightlike submanifold \( (M, g, S(TM), S(TM^\perp)) \) of a semi-Riemannian manifold \( (\bar{M}, \bar{g}) \) is minimal if \( h^s = 0 \) on \( \text{Rad}(TM) \) and \( tr(h) = 0 \), where \( \text{trace} \) is written with respect to \( g \) restricted to \( S(TM) \).

**Definition 2.3.** [4] A lightlike submanifold \( (M, g, S(TM), S(TM^\perp)) \) of a semi-Riemannian manifold \( (\bar{M}, \bar{g}) \) is said to be totally umbilical in \( \bar{M} \) if there is a smooth transversal vector field \( H \in \Gamma(tr(TM)) \) on \( M \), called the transversal curvature vector field of \( M \), such that

\[
h(X, Y) = H\bar{g}(X, Y), \quad \forall X, Y \in \Gamma(TM).
\]

From (2.8) and (2.13), it is easy to see that \( M \) is totally umbilical if and only if on each coordinate neighbourhood \( U \), there exist smooth vector fields \( H^i \in \Gamma(ltr(TM)) \) and \( H^s \in \Gamma(S(TM^\perp)) \), such that

\[
h^i(X, Y) = H^i \bar{g}(X, Y) \quad \text{and} \quad h^s(X, Y) = H^s \bar{g}(X, Y), \quad \forall X, Y \in \Gamma(TM).
\]

### 3. Invariant lightlike submanifolds

**Definition 3.1.** A lightlike submanifold \( M \), tangent to the structure vector field \( V \), of an indefinite para-Sasakian manifold \( \bar{M} \) is said to be invariant lightlike submanifold if the following condition is satisfied:

\[
\phi(\text{Rad}TM) = \text{Rad}TM \quad \text{and} \quad \phi(D) = D,
\]

where \( S(TM) = D \perp \{V\} \) and \( D \) is complementary nondegenerate distribution to \( \{V\} \) in \( S(TM) \).

From (2.4), (2.5), (2.8) and (3.1), we get

\[
h^i(X, V) = 0, \quad h^s(X, V) = 0, \quad \nabla_X V = PX,
\]

\[
h(X, \phi Y) = \phi h(X, Y) = h(\phi X, Y), \quad \forall X, Y \in \Gamma(TM).
\]

Let \( (\mathbb{R}^{2m+1}_q, \bar{g}, \phi, \eta, V) \) denote the manifold \( \mathbb{R}^{2m+1}_q \) with its usual para-Sasakian structure given by

\[
\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), \quad V = 2\partial z,
\]

\[
\bar{g} = \eta \otimes \eta + \frac{1}{4}(- \sum_{i=1}^{2} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=2}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),
\]

\[
\phi(\sum_{i=1}^{m} (X_i \partial x_i + Y_i \partial y_i) + Z \partial z) = \sum_{i=1}^{m} (Y_i \partial x_i + X_i \partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z,
\]

where \((x^i; y^i; z)\) are the cartesian coordinates on \( \mathbb{R}^{2m+1}_q \). Now we construct some examples of invariant lightlike submanifolds of an indefinite para-Sasakian manifold.
Example 1. Let $(\mathbb{R}^7_2, \mathcal{F}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\mathcal{F}$ is of signature $(-, +, +, -, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial y_1, \partial y_2, \partial y_3, \partial z\}$. Suppose $M$ is a submanifold of $\mathbb{R}^7_2$ given by

$$x^1 = y^2 = u_1, \quad x^2 = y^1 = u_2, \quad x^3 = u_4, \quad y^3 = u_4, \quad z = u_5.$$ 

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$$Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z), \quad Z_2 = 2(\partial x_2 + \partial y_1 + y^2 \partial z),$$

$$Z_3 = 2(\partial x_3 + y^3 \partial z), \quad Z_4 = 2\partial y_3 \quad \text{and} \quad Z_5 = V = 2\partial z.$$ 

Hence $RadTM = span\{Z_1, Z_2\}$, $S(TM) = span\{Z_3, Z_4, V\}$ and $ltr(TM)$ is spanned by $N_1 = \partial x_1 - \partial y_2 + y^1 \partial z$, $N_2 = -\partial x_2 + \partial y_1 - y^2 \partial z$.

It follows that $\phi Z_1 = Z_2$, $\phi Z_2 = Z_1$, $\phi Z_3 = Z_4$, $\phi Z_4 = Z_3$, $\phi N_1 = N_2$ and $\phi N_2 = N_1$. Thus $\phi RadTM = RadTM$, $\phi D = D$ and $\phi ltr(TM) = ltr(TM)$. Hence $M$ is an invariant 2-lightlike submanifold of $\mathbb{R}^7_2$.

Example 2. Let $(\mathbb{R}^8_2, \mathcal{F}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\mathcal{F}$ is of signature $(-, +, +, -, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Suppose $M$ is a submanifold of $\mathbb{R}^8_2$ given by $x^1 = y^2 = u_1$, $x^2 = y^1 = u_2$, $-x^3 = y^4 = u_3$, $-x^4 = y^3 = u_4$, $z = u_5$.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$$Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z), \quad Z_2 = 2(\partial x_2 + \partial y_1 + y^2 \partial z),$$

$$Z_3 = 2(-\partial x_3 + \partial y_4 - y^3 \partial z), \quad Z_4 = 2(-\partial x_4 + \partial y_3 - y^4 \partial z), \quad Z_5 = V = 2\partial z.$$ 

Hence $RadTM = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2 - y^1 \partial z$, $N_2 = \partial x_2 - \partial y_1 + y^2 \partial z$ and $S(TM^\perp)$ is spanned by $W_1 = 2(\partial x_3 + \partial y_4 + y^3 \partial z)$, $W_2 = 2(\partial x_4 + \partial y_3 + y^4 \partial z)$.

It follows that $\phi Z_1 = Z_2$, $\phi Z_2 = Z_1$, $\phi Z_3 = -Z_4$, $\phi Z_4 = -Z_3$, $\phi N_1 = N_2$, $\phi N_2 = N_1$, $\phi W_1 = W_2$ and $\phi W_2 = W_1$. Thus $\phi RadTM = RadTM$, $\phi D = D$, $\phi ltr(TM) = ltr(TM)$ and $\phi S(TM^\perp) = S(TM^\perp)$. Hence $M$ is an invariant 2-lightlike submanifold of $\mathbb{R}^8_2$.

Theorem 3.1. Let $(M, g, S(TM), S(TM^\perp))$ be an invariant lightlike submanifold, tangent to the structure vector field $V$ of an indefinite para-Sasakian manifold $\overline{M}$. If the second fundamental forms $h^l$ and $h^s$ of $M$ are parallel then $M$ is totally geodesic.

Proof. Suppose $h^l$ is parallel. Then $(\nabla_X h^l)(Y, V) = 0, \forall X, Y \in \Gamma(TM)$, which implies

$$\nabla_X h^l(Y, V) - h^l(\nabla_X Y, V) - h^l(Y, \nabla_X V) = 0, \quad \forall X, Y \in \Gamma(TM). \quad (3.4)$$

From (3.2) and (3.4), we get $h^l(Y, \nabla_X V) = 0, \forall X, Y \in \Gamma(TM)$. Thus from above, we have $h^l(Y, PX) = 0, \forall X, Y \in \Gamma(TM)$. Hence $h^l = 0$. Similarly $h^s = 0$. Thus $M$ is totally geodesic.

Theorem 3.2. Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold, tangent to the structure vector field $V$ of an indefinite para-Sasakian manifold $\overline{M}$. If $M$ is totally umbilical then it is totally geodesic.
Proof. Let $M$ be a totally umbilical lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then, from (2.8), we have
\[ \nabla_X V = \nabla_X V + h^!(X, V) + h^s(X, V), \quad \forall X \in \Gamma(TM). \] (3.5)
From (2.5), (2.11) and (3.5), we get
\[ PX + FX = \nabla_X V + h^!(X, V) + h^s(X, V), \quad \forall X \in \Gamma(TM). \] (3.6)
Equating transversal parts in (3.6), we get
\[ h^!(X, V) + h^s(X, V) = FX. \] (3.7)
Replacing $X$ by $V$ in (3.7), we get
\[ h^!(V, V) + h^s(V, V) = FV. \] (3.8)
Now from (2.1), (2.11) and (3.8), we get
\[ h^!(V, V) = 0 \quad \text{and} \quad h^s(V, V) = 0. \] (3.9)
From (2.14) and (3.9), we have $H^!(\overline{V}, V) = 0$ and $H^s(\overline{V}, V) = 0$.

Since $V$ is non-null vector, we have $H^! = H^s = 0$. Thus from (2.14), we obtain $h^!(X, Y) = 0$ and $h^s(X, Y) = 0$. Hence, $M$ is totally geodesic. $\blacksquare$

Theorem 3.3. Let $(M, g, S(TM), S(TM^⊥))$ be a lightlike submanifold of nullity degree two of an indefinite para-Sasakian manifold $\overline{M}$. Then, $\text{RadTM}$ defines a totally geodesic foliation on $M$.

Proof. Let $M$ be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. By definition of lightlike submanifold, $\text{RadTM}$ defines a totally geodesic foliation if and only if $g(\nabla_X Y, Z) = 0$, $\forall X, Y \in \Gamma(\text{RadTM})$ and $Z \in \Gamma(S(TM))$.

Since $\text{rank}(\text{RadTM}) = 2$, we can write $X, Y \in \Gamma(\text{RadTM})$ as a linear combination of $\xi$ and $\phi_\xi$, that is $X = A_1 \xi + B_1 \phi_\xi$ and $Y = A_2 \xi + B_2 \phi_\xi$. Now since $\overline{\nabla}$ is a metric connection, using (2.8), we get
\[ g(\nabla_X Y, Z) = X g(Y, Z) - g(Y, \nabla_X Z) \]
\[ = -g(Y, \nabla_X Z) = -g(Y, h^!(X, Z)) \]
\[ = -g(A_2 \xi + B_2 \phi_\xi, h^!(A_1 \xi + B_1 \phi_\xi, Z)) \]
\[ = -A_1 A_2 g(\xi, h^!(\xi, Z)) - B_1 A_2 g(\phi_\xi, h^!(\phi_\xi, Z)) - B_2 A_1 g(\phi_\xi, h^!(\xi, Z)) \]
\[ - B_2 A_2 g(\phi_\xi, h^!(\phi_\xi, Z)), \quad \text{for all} \ X, Y \in \text{RadTM} \ and \ Z \in \Gamma(S(TM)). \] (3.10)
From (2.12), (3.3) and (3.10), we get $g(\nabla_X Y, Z) = 0$, which completes the proof. $\blacksquare$

4. Slant lightlike submanifolds

At first, we state the following lemmas for later use:

Lemma 4.1. Let $M$ be an $r$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to $M$. Suppose that $\phi \text{RadTM}$ is a distribution on $M$ such that $\text{RadTM} \cap \phi \text{RadTM} = \{0\}$. 

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Then $\phi \text{ltr}(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi \text{Rad}TM \cap \phi \text{ltr}(TM) = \{0\}$.

**Lemma 4.2.** Let $M$ be a $q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$, of index $2q$ with structure vector field tangent to $M$. Suppose $\text{Rad}TM$ is a distribution on $M$ such that $\text{Rad}TM \cap \phi \text{Rad}TM = \{0\}$. Then any complementary distribution to $\phi \text{ltr}(TM) \oplus \phi \text{Rad}TM$ in $S(TM)$ is Riemannian.

The proofs of Lemma 4.1 and Lemma 4.2 follow as in Lemma 3.1 and Lemma 3.2 respectively of [10], so we omit them.

**Definition 4.1.** Let $M$ be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then we say that $M$ is slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) $\text{Rad}TM$ is a distribution on $M$ such that $\phi \text{Rad}TM \cap \text{Rad}TM = \{0\}$,

(ii) For each non-zero vector field $X$ tangent to $D$ at $x \in U \subset M$, the angle $\theta(X)$ between $\phi X$ and the vector space $D_x$ is constant, i.e. it is independent of the choice of $x \in U \subset M$ and $X \in D_x$, where $D$ is complementary distribution to $(\phi \text{Rad}TM \oplus \phi \text{ltr}(TM)) \perp \{V\}$ in the screen distribution $S(TM)$.

This constant angle $\theta(X)$ is called slant angle of distribution $D$. A slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TM = \text{Rad}TM \perp (\phi \text{Rad}TM \oplus \phi \text{ltr}(TM)) \perp D \perp \{V\}.$$ (4.1)

From Definition 4.1, we conclude that the class of slant lightlike submanifolds does not include invariant lightlike submanifolds of an indefinite para-Sasakian manifold.

**Example 1.** Let $(\mathbb{R}^6, \mathcal{g}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\mathcal{g}$ is of signature $(-, +, +, +, - ,+ , +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial z\}$. Suppose $M$ is a submanifold of $\mathbb{R}^6$ given by $-x^1 = y^2 = u_1, x^2 = u_2, x^3 = 0, x^4 = u_3, y^1 = u_4, y^2 = u_5 \sin \theta, y^3 = u_5 \cos \theta, z = u_6$, where $\theta \in (0, \frac{\pi}{2})$.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$Z_1 = 2(-\partial x_1 + \partial y_2 - y^1 \partial z), Z_2 = 2(\partial x_2 + y^2 \partial z), Z_3 = 2(\partial x_4 + y^4 \partial z), Z_4 = 2 \partial y_1, Z_5 = 2(\sin \theta \partial y_3 + \cos \theta \partial y_4), Z_6 = V = 2 \partial z$.

Hence $\text{Rad}TM = \text{span} \{Z_1\}$ and $S(TM) = \text{span} \{Z_2, Z_3, Z_4, Z_5, V\}$.

Now $\text{ltr}(TM)$ is spanned by $N = \partial x_1 + \partial y_2 + y^1 \partial z$ and $S(TM^\perp)$ is spanned by $W_1 = 2(\partial x_3 + y^3 \partial z), W_2 = 2(\cos \theta \partial y_3 - \sin \theta \partial y_4)$. It follows that $\phi Z_1 = 2(\partial x_2 - \partial y_1 + y^2 \partial z) = Z_2 - Z_4, \phi N = \partial x_2 + \partial y_1 + y^2 \partial z = \frac{1}{2}(Z_2 + Z_4)$ and $g(\phi Z_1, \phi N) = 1$.

Thus $\phi \text{Rad}TM$ and $\phi \text{ltr}(TM)$ are distributions on $M$ and $D = \text{span} \{Z_3, Z_5\}$ is a slant distribution with slant angle $\theta$. Then $TM = \text{Rad}TM \perp (\phi \text{Rad}TM \oplus \phi \text{ltr}(TM)) \perp D \perp \{V\}$. Hence $M$ is a slant lightlike submanifold of $\mathbb{R}^6$.

**Theorem 4.3.** Let $M$ be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$ such that $\phi \text{Rad}TM \cap
RadTM = \{0\}. Then M is slant lightlike submanifold if and only if there exists a constant \( \lambda \in [0, 1] \) such that \( P^2X = \lambda(X - \eta(X)V) \), \( \forall X \in \Gamma(D) \).

**Proof.** Let M be a lightlike submanifold of an indefinite para-Sasakian manifold \( \mathcal{M} \). Suppose there exists a constant \( \lambda \), such that \( P^2X = \lambda(X - \eta(X)V) = \lambda \phi^2X \), \( \forall X \in \Gamma(D) \). Now

\[
\cos(\theta(X)) = \frac{g(\phi X, PX)}{|\phi X||PX|} = \frac{g(X, \phi PX)}{|\phi X||PX|} = \frac{g(X, P^2X)}{|\phi X||PX|} = \lambda \frac{g(X, \phi^2X)}{|\phi X||PX|} = \lambda \frac{g(\phi X, \phi X)}{|\phi X||PX|}.
\]

From above equation, we get

\[
\cos(\theta(X)) = \lambda \frac{|\phi X|}{|PX|}. \tag{4.2}
\]

Also \( |PX| = |\phi X| \cos(\theta(X)) \), which implies

\[
\cos(\theta(X)) = \frac{|PX|}{|\phi X|}. \tag{4.3}
\]

From (4.2) and (4.3), we get \( \cos^2(\theta(X)) = \lambda \) (constant). Hence, M is a slant lightlike submanifold.

Conversely, suppose that M is a slant lightlike submanifold. Then \( \cos^2(\theta(X)) = \lambda \), where \( \lambda \) is a constant. From (4.3), we have \( \frac{|PX|^2}{|\phi X|^2} = \lambda \). Now \( g(PX, PX) = \lambda g(\phi X, \phi X) \), which gives \( g(X, P^2X) = \lambda g(X, \phi^2X) \). Thus \( g(X, (P^2 - \lambda \phi^2)X) = 0 \). Since \( X \) is non-null vector, we have \( (P^2 - \lambda \phi^2)X = 0 \). Hence, \( P^2X = \lambda \phi^2X = \lambda(X - \eta(X)V) \), \( \forall X \in \Gamma(D) \). \( \blacksquare \)

**Corollary 4.4.** Let M be a slant lightlike submanifold of an indefinite para-Sasakian manifold \( \mathcal{M} \) with slant angle \( \theta \). Then

\[
g(PX, PY) = \cos^2(\theta)g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(D),
\]

\[
g(FX, FY) = \sin^2(\theta)g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(D).
\]

**Proof.** Since \( g(PX, PY) = g(X, P^2Y) = g(X, \lambda \phi^2Y) = \lambda g(X, \phi^2Y) = \lambda g(\phi X, \phi Y) \), \( \forall X, Y \in \Gamma(D) \), we have

\[
g(PX, PY) = \cos^2(\theta)g(\phi X, \phi Y), \quad \forall X, Y \in \Gamma(D). \tag{4.4}
\]

Thus \( g(PX, PY) = \cos^2(\theta)(g(X, Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D) \).

From (4.4), we obtain \( g(FX, FY) = (1 - \sin^2(\theta))g(\phi X, \phi Y), \quad \forall X, Y \in \Gamma(D) \), which implies \( g(\phi X, \phi Y) - g(FX, FY) = \sin^2(\theta)g(\phi X, \phi Y), \quad \forall X, Y \in \Gamma(D) \), which gives \( g(FX, FY) = \sin^2(\theta)(g(X, Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D) \). This completes the proof. \( \blacksquare \)

Now, we denote the projections on \( RadTM, \phi RadTM, \phi ltr(TM) \) and \( D \) in \( TM \) by \( P_1, P_2, P_3 \) and \( P_4 \), respectively. Similarly, we denote the projections on \( ltr(TM) \) and \( S(TM^+) \) by \( Q_1 \) and \( Q_2 \), respectively. Then, we get

\[
X = P_1X + P_2X + P_3X + P_4X + \eta(X)V, \quad \forall X \in \Gamma(TM). \tag{4.5}
\]

\[
W = Q_1W + Q_2W, \quad \forall W \in \Gamma(tr(TM)). \tag{4.6}
\]
Now applying $\phi$ to (4.5), we have
$$\phi X = \phi P_1 X + \phi P_2 X + \phi P_3 X + fP_4 X + FP_4 X, \quad \forall X \in \Gamma(TM),$$
where $fP_4 X$ (resp. $FP_4 X$) denotes the tangential (resp. screen transversal) component of $\phi P_4 X$. Thus we get
$$\phi P_1 X \in \phi \text{Rad}TM, \quad \phi P_2 X \in \Gamma(\text{Rad}TM), \quad \phi P_3 X \in \Gamma(\text{ltr}(TM)), \quad fP_4 X \in \Gamma(D), \quad FP_4 X \in \Gamma(S(TM^\perp)).$$

Applying $\phi$ to (4.6), we obtain $\phi W = \phi Q_1 W + BQ_2 W + CQ_2 W$, where $BQ_2 W$ (resp. $CQ_2 W$) denote the tangential (resp. transversal) component of $\phi Q_2 W$.

Now, by using (2.4), (4.5) and (2.8)–(2.10) and equating tangential, lightlike transversal and screen transversal components, we obtain

$$-\bar{g}(\phi X, \phi Y) V - \eta(Y) \phi^2 X = \nabla_X \phi P_1 X + \nabla_X \phi P_2 X - A_{\phi P_2 X} X + \nabla_X fP_4 Y$$
$$- A_{FP_4 Y} X - \phi P_1 \nabla_X Y - \phi P_2 \nabla_X Y$$
$$- fP_4 \nabla_X Y - \phi h^i(X, Y) - Bh^s(X, Y), \quad (4.7)$$

$$h^i(X, \phi P_1 Y) + h^i(X, \phi P_2 Y) + h^i(X, fP_4 Y) = -\nabla^i_X \phi P_3 Y - D^i(X, FP_4 Y)$$
$$+ \phi P_3 \nabla_X Y,$$

$$h^s(X, \phi P_1 Y) + h^s(X, \phi P_2 Y) + h^s(X, fP_4 Y) = -D^s(X, \phi P_3 Y) - \nabla^s_X FP_4 Y$$
$$+ FP_4 \nabla_X Y - Ch^s(X, Y).$$

**Theorem 4.5.** Let $M$ be a proper slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field $V$ tangent to $M$. Then induced connection $\nabla$ is never a metric connection.

**Proof.** Suppose that the induced connection is a metric connection. Then $\nabla_X \phi P_2 Y \in \Gamma(\text{Rad}TM)$ and $h^i(X, Y) = 0$. Thus for $Y \in \phi \text{Rad}TM$ and $X \in \phi \text{ltr}(TM)$, (4.7) becomes

$$-\bar{g}(X, Y) V = \nabla_X \phi P_2 X - \phi P_1 \nabla_X Y - \phi P_2 \nabla_X Y - fP_4 \nabla_X Y - Bh^s(X, Y).$$

Since $TM = \text{Rad}TM \oplus \phi \text{Rad}TM \oplus \phi \text{ltr}(TM) \oplus D \oplus V$, from (4.8), we get

$$\phi P_1 \nabla_X Y = 0, \quad \nabla_X \phi P_2 X + \phi P_2 \nabla_X Y = 0,$$

$$\bar{g}(X, Y) V = 0, \quad fP_4 \nabla_X Y + Bh^s(X, Y) = 0. \quad (4.9)$$

Now, taking $X = \phi N$ and $Y = \phi \xi$ in (4.9), we get $\bar{g}(N, \xi) V = 0$. Thus $V = 0$, which is a contradiction. Hence $M$ does not have a metric connection. ■

5. Screen slant lightlike submanifolds

At first, we state the following lemma for later use:

**Lemma 5.1.** Let $M$ be a $2q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$, of index $2q$ such that $2q < \dim(M)$ with structure vector field tangent...
to $M$. Then the screen distribution $S(TM)$ of lightlike submanifold $M$ is Riemannian.

The proof of above lemma follows as in Lemma 4.1 of [10], so we omit it.

**Definition 5.1.** Let $M$ be a $2q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ of index $2q$ such that $2q < \dim(M)$ with structure vector field tangent to $M$. Then we say that $M$ is screen slant lightlike submanifold of $\overline{M}$ if following conditions are satisfied:

(i) $\text{Rad}T M$ is invariant with respect to $\phi$, i.e. $\phi(\text{Rad}T M) = \text{Rad}T M$,

(ii) For each non-zero vector field $X$ tangent to $D$ at $x \in U \subset M$, the angle $\theta(X)$ between $\phi X$ and the vector space $D_x$ is constant, i.e. it is independent of the choice of $x \in U \subset M$ and $X \in D_x$, where $D$ is complementary nondegenerate distribution to $\{V\}$ in $S(TM)$ such that $S(TM) = D \perp \{V\}$.

This constant angle $\theta(X)$ is called the slant angle of distribution $D$. A screen slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TM = \text{Rad}T M \perp D \perp \{V\}. \tag{5.1}$$

From Definitions 4.1 and 5.1, we conclude that the class of screen slant lightlike submanifolds does not include slant lightlike submanifolds of an indefinite para-Sasakian manifold and vice-versa.

**Theorem 5.2.** Let $M$ be a screen slant lightlike submanifold of $\overline{M}$. Then $M$ is invariant (resp. screen real) if and only if $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$).

Proof of the above theorem follows from Proposition 4.1 of [10].

**Example 1.** Let $(\mathbb{R}^9_2, \overline{\mathcal{G}}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\overline{\mathcal{G}}$ is of signature $(-, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Suppose $M$ is a submanifold of $\mathbb{R}^9_2$ given by $x^1 = y^2 = u_1, x^2 = y^1 = u_2, x^3 = u_3 \cos \theta, x^4 = u_3 \sin \theta, y^3 = u_4 \sin \theta, y^4 = u_4 \cos \theta, z = u_5$.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z), \quad Z_2 = 2(\partial x_2 + \partial y_1 + y^2 \partial z),$

$Z_3 = 2(\cos \theta \partial x_3 + \sin \theta \partial x_4 + y^1 \cos \theta \partial z + y^4 \sin \theta \partial z),$

$Z_4 = 2(\sin \theta \partial y_3 + \cos \theta \partial y_4), \quad Z_5 = V = 2\partial z.$

Hence $\text{Rad}T M = \text{span} \{Z_1, Z_2\}$ and $S(TM) = \text{span} \{Z_3, Z_4, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2 - y^1 \partial z, \quad N_2 = \partial x_2 - \partial y_1 + y^2 \partial z$ and $S(TM)^\perp$ is spanned by

$W_1 = 2(- \sin \theta \partial x_3 + \cos \theta \partial x_4 - y^3 \sin \theta \partial z + y^4 \cos \theta \partial z),$

$W_2 = 2(\cos \theta \partial y_3 - \sin \theta \partial y_4).$

It follows that $\phi Z_1 = Z_2, \phi Z_2 = Z_1$, which implies that $\text{Rad}TM$ is invariant, i.e., $\phi\text{Rad}TM = \text{Rad}TM$. On other hand, we can see that $D = \text{span} \{Z_3, Z_4\}$ is a slant
distribution with slant angle $2\theta$. Hence $M$ is screen slant 2-lightlike submanifold of $\mathbb{R}^3$.

Now, we denote the projections on $\text{Rad}TM$ and $D$ in $TM$ by $P_1$ and $P_2$ respectively. Similarly, we denote the projections on $\text{ltr}(TM)$ and $S(TM^\perp)$ by $Q_1$ and $Q_2$ respectively. Then, we get

$$X = P_1 X + P_2 X + \eta(X)V, \quad \forall X \in \Gamma(TM).$$

(5.2)

Now applying $\phi$ to (5.2), we have $\phi X = \phi P_1 X + \phi P_2 X$, which gives

$$\phi X = \phi P_1 X + fP_2 X + FP_2 X, \quad \forall X \in \Gamma(TM),$$

(5.3)

where $fP_2 X$ (resp. $FP_2 X$) denotes the tangential (resp. transversal) component of $\phi P_2 X$. Thus we get $\phi P_1 X \in \text{Rad}TM, fP_2 X \in \Gamma(D), FP_2 X \in \Gamma(S(TM^\perp))$. Also, we have

$$W = Q_1 W + Q_2 W, \quad \forall W \in \Gamma(\text{ltr}(TM)).$$

(5.4)

Applying $\phi$ to (5.4), we obtain

$$\phi W = \phi Q_1 W + \phi Q_2 W,$$

(5.5)

which gives

$$\phi W = \phi Q_1 W + BQ_2 W + CQ_2 W,$$

(5.6)

where $BQ_2 W$ (resp. $CQ_2 W$) denotes the tangential (resp. transversal) component of $\phi Q_2 W$.

Now, by using (2.4), (5.3), (5.6) and (2.8)–(2.10) and equating tangential, lightlike transversal and screen transversal components, we obtain

$$-\overline{g}(\phi X, \phi Y)V - \eta(Y)\phi^2 X = \nabla_X \phi P_1 Y + \nabla_X fP_2 Y - AF_{P_2} Y X$$

$$- \phi P_1 \nabla_X Y - \phi P_2 \nabla_X Y + Bh^s(X, Y),$$

(5.7)

$$h^l(X, \phi P_1 Y) + h^l(X, fP_2 Y) = \phi h^l(X, Y) - D^l(X, FP_2 Y),$$

$$h^s(X, \phi P_1 Y) + h^s(X, fP_2 Y) = Ch^s(X, Y) - \nabla_X^s FP_2 Y - FP_2 \nabla_X Y.$$  

(5.8)

**Theorem 5.3.** Let $M$ be a 2$q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then $M$ is screen slant lightlike submanifold if and only if

(i) the lightlike transversal vector bundle $\text{ltr}(TM)$ is invariant with respect to $\phi$,

(ii) there exists a constant $\lambda \in [0, 1]$ such that $P^2 X = \lambda(X - \eta(X)V), \forall X \in \Gamma(D)$.

**Proof.** Let $M$ be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then its radical distribution $\text{Rad}TM$ is invariant with respect to $\phi$, i.e., $\phi X = X \forall X \in \Gamma\text{Rad}TM$.

Now, for $N \in \Gamma\text{ltr}(TM)$ and $X \in \Gamma D$, using (2.3) and (5.3), we obtain

$$\overline{g}(\phi N, X) = \overline{g}(N, \phi X) = \overline{g}(N, fX + FX) = \overline{g}(N, fX) + \overline{g}(N, FX) = 0.$$  

Thus $\phi N$ does not belong to $\Gamma(D)$. 


For $N \in \Gamma \ltr(TM)$ and $W \in \Gamma S(TM^\perp)$, from (2.3) and (5.6), we have
\[ \bar{g}(\phi N, W) = \bar{g}(N, \phi W) = \bar{g}(N, BW + CW) = \bar{g}(N, BW) + \bar{g}(N, CW) = 0. \]
Hence we conclude that $\phi N$ does not belong to $\Gamma S(TM^\perp)$.

Now, suppose that $\phi N \in \Gamma (\Rad TM)$. Then $\phi(\phi N) = \phi^2 N = -N + \eta(N)V \in \Gamma (ltr TM) \oplus \span \{V\}$, which contradicts that $\Rad TM$ is invariant. Hence $ltr TM$ is invariant with respect to $\phi$.

Since $|PX| = |\phi X| \cos \theta(X)$, $\forall X \in \Gamma(D)$, we have
\[ \cos \theta(X) = \frac{|PX|}{|\phi X|}. \quad (5.9) \]
In view of (5.9), we get
\[ \cos^2 \theta(X) = \frac{|P X|^2}{|\phi X|^2} = \frac{\bar{g}(P X, P X)}{\bar{g}(\phi X, \phi X)} = \frac{\bar{g}(X, \phi^2 X)}{\bar{g}(X, \phi^2 X)} = \frac{\bar{g}(X, P^2 X)}{\bar{g}(X, P^2 X)}, \]
which gives
\[ g(X, P^2 X) = \cos^2 \theta g(X, \phi^2 X). \quad (5.10) \]
Since $M$ is screen slant lightlike submanifold, $\cos^2 \theta(X) = \lambda \text{(constant)} \in [0, 1]$.

Therefore from (5.10), we get
\[ g(X, P^2 X) = \lambda g(X, \phi^2 X) = g(X, \lambda \phi^2 X), \]
which implies $g(X, (P^2 - \lambda^2)X) = 0$. Since $X$ is non-null vector, we have $(P^2 - \lambda^2)X = 0$, which implies
\[ P^2 X = \lambda \phi^2 X = \lambda (X - \eta(X)V), \quad \forall X \in \Gamma(D). \]
This proves (ii).

Conversely suppose that conditions (i) and (ii) are satisfied. We can show that $\Rad TM$ is invariant in similar way that $ltr TM$ is invariant. From (ii) we have
\[ P^2 X = \lambda \phi^2 X, \quad \forall X \in \Gamma(D), \]
where $\lambda \text{(constant)} \in [0, 1]$.

Now, $\cos \theta(X) = \frac{\bar{g}(\phi X, P X)}{|\phi X||PX|} = \frac{\bar{g}(X, \phi^2 X)}{|\phi X||PX|} = \frac{\bar{g}(X, P^2 X)}{|\phi X||PX|} = \lambda \frac{\bar{g}(\phi X, \phi X)}{|\phi X||PX|}.$
From the above equation, we get
\[ \cos \theta(X) = \lambda \frac{|\phi X|}{|PX|}. \quad (5.11) \]
Therefore (5.9) and (5.11) give $\cos^2 \theta(X) = \lambda \text{(constant)}$. Hence $M$ is a screen slant lightlike submanifold. \blacksquare

**Corollary 5.4** Let $M$ be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with slant angle $\theta$, then
\[ g(P X, P Y) = \cos^2 \theta(g(X, Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D), \]
\[ g(F X, F Y) = \sin^2 \theta(g(X, Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D). \quad (5.12) \]

The proof of above corollary follows using the steps as in proof of Corollary 3.2 of [9].

**Lemma 5.5.** Let $M$ be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then we have
From (2.3), (5.15) and (5.16), we have

\[ g(\nabla_X Y, V) = -\overline{g}(Y, \phi X), \quad \forall X, Y \in \Gamma(TM) - \{V\}, \]

(5.17)

From (2.5) and (5.14), we obtain

\[ g([X, Y], V) = 0, \quad \forall X, Y \in \Gamma(TM) - \{V\}. \]

(5.18)

On interchanging \( M \) then from (2.8), we have

\[ g(\nabla_X Y, V) = -\overline{g}(Y, \nabla_X V), \quad \forall X, Y \in \Gamma(TM) - \{V\}. \]

(5.19)

From (5.7), we have

\[ g(\nabla_Y X, V) = -\overline{g}(X, \phi Y), \quad \forall X, Y \in \Gamma(TM) - \{V\}. \]

(5.20)

From (2.3), (5.18) and (5.20), we have

\[ g([X, Y], V) = 0, \quad \forall X, Y \in \Gamma(TM) - \{V\}. \]

(5.21)

Proof of (i) follows from (5.17), (5.20) and (5.23).

Theorem 5.6. Let \( M \) be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \) with structure vector field tangent to \( M \). Then

(i) the radical distribution \( \text{Rad}TM \) is integrable if and only if

\[ h^*(Y, \phi X) = h^*(X, \phi Y) \quad \text{and} \quad (\nabla_X \phi F) Y = (\nabla_Y \phi F) X, \quad \forall X, Y \in \Gamma(\text{Rad}TM), \]

(ii) the distribution \( D \) is integrable if and only if

\[ P_1(\nabla_X f - \nabla_Y f) = P_1(A_F Y - A_F X), \quad \forall X, Y \in \Gamma(D). \]

Proof. Let \( M \) be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). From (5.8), we get

\[ h^*(X, \phi Y) = Ch^*(X, Y) - FP_2 \nabla_X Y, \quad \forall X, Y \in \Gamma(\text{Rad}TM). \]

(5.22)

Interchanging \( X \) and \( Y \) in (5.18), we get

\[ h^*(Y, \phi X) = Ch^*(Y, X) - FP_2 \nabla_Y X, \quad \forall X, Y \in \Gamma(\text{Rad}TM). \]

(5.23)

From (5.18) and (5.19), we get

\[ h^*(Y, \phi X) - h^*(X, \phi Y) = FP_2(\nabla_X Y - \nabla_Y X) = FP_2[X, Y]. \]

(5.24)

From (5.7), we have

\[ \nabla_X \phi F Y - \phi F_1 \nabla_X Y - f P_2 \nabla_X Y + Bh^*(X, Y) = 0, \quad \forall X, Y \in \Gamma(\text{Rad}TM). \]

(5.25)

On interchanging \( X \) and \( Y \) in (5.21), we get

\[ \nabla_Y \phi F_1 X - \phi F_1 \nabla_Y X - f P_2 \nabla_Y X + Bh^*(Y, X) = 0, \quad \forall X, Y \in \Gamma(\text{Rad}TM). \]

(5.26)

From (5.21) and (5.22), we have

\[ (\nabla_X \phi F_1 Y - (\nabla_Y \phi F_1) X = f P_2([X, Y]), \quad \forall X, Y \in \Gamma(\text{Rad}TM). \]

(5.27)

Proof of (i) follows from (5.17), (5.20) and (5.23).
Now from (5.7) and (2.2), we obtain
\[ \varphi(\phi X, \phi Y)V + \nabla_X fY - A_{FY}X = \phi P_1 \nabla_X Y + f P_2 \nabla_X Y - Bh^s(X, Y), \quad \forall X, Y \in \Gamma(D). \] (5.24)

Interchanging \( X \) and \( Y \) in (5.24), we have
\[ \varphi(\phi Y, \phi X) V + \nabla_Y fX - A_{FX}X = \phi P_1 \nabla_Y X + f P_2 \nabla_Y X - Bh^s(X, Y), \quad \forall X, Y \in \Gamma(D). \] (5.25)

From (5.24) and (5.25), we get
\begin{align*}
\nabla_X fY - \nabla_Y fX + A_{FX}Y - A_{FY}X &= \phi P_1 \nabla_Y X - \phi P_1 \nabla_Y X + f P_2 \nabla_X Y - f P_2 \nabla_Y X \\
&= \phi P_1 [X, Y] + f P_2 [X, Y], \quad \forall X, Y \in \Gamma(D).
\end{align*}
(5.26)

The equation (5.26) implies
\[ P_1(\nabla_X fY - \nabla_Y fX) + P_1(A_{FX}Y - A_{FY}X) = \phi P_1 [X, Y], \quad \forall X, Y \in \Gamma(D). \] (5.27)

Proof of (ii) follows from (5.17) and (5.27). \( \blacksquare \)

**Theorem 5.7.** Let \( M \) be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold \( \mathcal{M} \) with structure vector field tangent to \( M \). Then \( S(TM) \) defines a totally geodesic foliation if and only if \( \nabla_X fY - A_{FY}X \) has no component in \( RadTM \), \( \forall X, Y \in \Gamma(D) \).

**Proof.** Let \( M \) be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold \( \mathcal{M} \). From (2.2) and (2.8), we get
\[ \varphi(\nabla_X Y, N) = \varphi(-\nabla_X \phi Y + \nabla_X \phi Y, \phi N), \quad \forall X, Y \in \Gamma(D) \quad \text{and} \quad N \in \text{ltr}(TM). \]

Using (2.4) in above equation, we get
\[ \varphi(\nabla_X Y, N) = \varphi(\varphi(X, \phi Y)V + \eta(Y)\phi^2 X + \nabla_X \phi Y, \phi N). \] (5.28)

From (2.1) and (5.28), we obtain
\[ \varphi(\nabla_X Y, N) = \varphi(\nabla_X \phi Y, \phi N), \quad \forall X, Y \in \Gamma(D) \quad \text{and} \quad N \in \text{ltr}(TM). \] (5.29)

From (2.8), (2.10), (5.3) and (5.29), we get
\[ \varphi(\nabla_X Y, N) = \varphi(\nabla_X fY + h^l(X, fY) + h^s(X, fy) - A_{FY}X + \nabla_X fY + D^l(X, FY), \phi N). \]

From the above equation, we get
\[ \varphi(\nabla_X Y, N) = \varphi(\nabla_X fY - A_{FY}X, \phi N), \quad \forall X, Y \in \Gamma(D) \quad \text{and} \quad N \in \text{ltr}(TM). \]

which completes the proof. \( \blacksquare \)

**Theorem 5.8.** Let \( M \) be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold \( \mathcal{M} \) with structure vector field tangent to \( M \). If \( Bh^s(X, Y) = 0, \quad \forall X \in \Gamma(TM) \) and \( Y \in \Gamma(RadTM) \) then the induced connection \( \nabla \) is a metric connection.

**Proof.** Let \( M \) be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold \( \mathcal{M} \). Then the induced connection \( \nabla \) on \( M \) is a metric connection if and only if \( RadTM \) is parallel distribution with respect to \( \nabla \) (3). Since
$Bh^s(X, Y) = 0$, $\forall X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$, we have $g(Bh^s(X, Y), Z) = 0$, $\forall X, Z \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$. Thus from (5.5) and (5.6), we obtain
\[ g(\phi h^s(X, Y), Z) = 0, \quad \forall X, Z \in \Gamma(TM) \quad \text{and} \quad Y \in \Gamma(RadTM). \quad (5.30) \]
Using (2.3) and (5.3) in (5.30), we get
\[ g(h^s(X, Y), F\nabla^2 Z) = 0, \quad \forall X, Z \in \Gamma(TM) \quad \text{and} \quad Y \in \Gamma(RadTM). \quad (5.31) \]
Now from (2.8), we get
\[ g(F\nabla^2 \nabla X Y, \phi h^s(X, Y)) = g(F\nabla^2 \nabla X Y, \nabla X \phi Y) - g(F\nabla^2 \nabla X Y, F\nabla^2 \nabla X Y), \quad \forall X \in \Gamma(TM) \quad \text{and} \quad Y \in \Gamma(RadTM). \quad (5.32) \]
Since $ltr(TM)$ is invariant, from (2.4), (5.3) and (5.32), we get
\[ g(F\nabla^2 \nabla X Y, \phi h^s(X, Y)) = g(F\nabla^2 \nabla X Y, F\nabla^2 \nabla X Y). \quad (5.33) \]
From (2.8) and (5.33), we obtain
\[ g(F\nabla^2 \nabla X Y, \phi h^s(X, Y)) = \sin^2 \theta g(P\nabla^2 \nabla X Y, P\nabla^2 \nabla X Y), \quad \forall X \in \Gamma(TM) \quad \text{and} \quad Y \in \Gamma(RadTM). \quad (5.34) \]
Now from (2.2) and (5.3), we have
\[ g(F\nabla^2 \nabla X Y, \phi h^s(X, Y)) = g(F\nabla^2 \nabla X Y, \phi h^s(X, Y)), \quad \forall X \in \Gamma(TM) \quad \text{and} \quad Y \in \Gamma(RadTM). \quad (5.35) \]
The equations (5.30) and (5.36) imply
\[ g(F\nabla^2 \nabla X Y, \phi h^s(X, Y)) = 0, \quad \forall X \in \Gamma(TM) \quad \text{and} \quad Y \in \Gamma(RadTM). \quad (5.37) \]
From (5.35) and (5.37), we get
\[ \sin^2 \theta g(P\nabla^2 \nabla X Y, P\nabla^2 \nabla X Y) = 0, \quad \forall X \in \Gamma(TM) \quad \text{and} \quad Y \in \Gamma(RadTM). \]
Since $M$ is proper screen slant lightlike submanifold and $D$ is Riemannian, we get $P\nabla^2 \nabla X Y = 0$. Hence $\nabla^2 \nabla Y \in \Gamma(RadTM)$, i.e., radical distribution $RadTM$ is parallel, which completes the proof.

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