DING PROJECTIVE MODULES WITH RESPECT TO A SEMIDUALIZING MODULE

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Abstract. In this paper, for a fixed semidualizing module $C$, we introduce the notion of $D_C$-projective modules which are the special setting of $G_C$-projective modules introduced by White [D. White, Gorenstein projective dimension with respect to a semidualizing module, J. Commut. Algebra 2(1) (2010) 111–137]. Then we investigate the properties of $D_C$-projective modules and dimensions, in particular, we give descriptions of the finite $D_C$-projective dimensions.

1. Introduction

Auslander and Bridger in [1], introduced the notion of so-called G-dimension for finitely generated modules over commutative Noetherian rings. Enochs and Jenda defined in [4] a homological dimension, namely the Gorenstein projective dimension, $Gpd_R(-)$, for any $R$-module as an extension of G-dimension. Let $R$ be any associative ring. Recall that an $R$-module $M$ is said to be Gorenstein projective (for short $G$-projective; see [4]) if there is an exact sequence

$$0 \rightarrow \mathbf{P} \rightarrow \mathbf{P}_1 \rightarrow \mathbf{P}_0 \rightarrow \mathbf{P}^0 \rightarrow \mathbf{P}^1 \rightarrow \cdots$$

of projective modules with $M = \text{Ker}(\mathbf{P}^0 \rightarrow \mathbf{P}^1)$ such that $\text{Hom}(\mathbf{P}, \mathbf{Q})$ is exact for each projective $R$-module $\mathbf{Q}$. Such exact sequence is called a complete projective resolution. We use $\mathcal{GP}(R)$ to denote the class of all $G$-projective $R$-modules. We say that $M$ has Gorenstein projective dimension at most $n$, denoted $Gpd_R(M) \leq n$, if there is a Gorenstein projective resolution, i.e., there is an exact sequence

$$0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0,$$

where all $G_i$ are $G$-projective $R$-modules, and say $Gpd_R(M) = n$ if there is not a shorter Gorenstein projective resolution.

In [3], an $R$-module $M$ is called strongly Gorenstein flat if there is an exact sequence

$$0 \rightarrow \mathbf{P} \rightarrow \mathbf{P}_1 \rightarrow \mathbf{P}_0 \rightarrow \mathbf{P}^0 \rightarrow \mathbf{P}^1 \rightarrow \cdots$$

of projective modules with $M = \text{Ker}(\mathbf{P}^0 \rightarrow \mathbf{P}^1)$ such that $\text{Hom}(\mathbf{P}, \mathbf{Q})$ is exact for each flat $R$-module $\mathbf{Q}$. It is clear that strongly Gorenstein flat $R$-modules are

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Gorenstein projective. But no one knows whether there is a Gorenstein projective $R$-module which is not strongly Gorenstein flat. Following [8, 19], the strongly Gorenstein flat $R$-modules are called Ding projective, since strongly Gorenstein flat $R$-modules are not necessarily Gorenstein flat [3, Example 2.19] and strongly Gorenstein flat $R$-modules were first introduced by Ding and his coauthors. In [3], the authors gave a lot of wonderful results about Ding projective $R$-modules over coherent rings. Mahdou and Tamekkante in [14], generalized some of these results over arbitrary associative rings. In this paper, we use $DP(R)$ to denote the class of all Ding projective $R$-modules.

In [7], the author initiated the study of semidualizing modules; see Definition 2.1. Over a noetherian ring $R$, Vasconcelos [17] studied them too. Golod [9] used these to define $G_C$-dimension for finitely generated modules, which is a refinement of projective dimension. Holm and Jørgensen [11] have extended this notion to arbitrary modules over a noetherian ring. Moreover, for semi-dualizing $R$-module $C$ and the trivial extension of $R$ by $C$: $R \ltimes C$, that is, the ring $R \oplus C$ equipped with the product: $(r, c)(r', c') = (rr', rc' + r'c)$, they considered the ring changed Gorenstein dimensions, $Gpd_{R \ltimes C} M$ and proved that $M$ is $G_C$-projective $R$-module if and only if $M$ is $G$-projective $R \ltimes C$-module [11, Theorem 2.16]. In [18], White unified and generalized treatment of this concept over any commutative rings and showed many excellent $G_C$-projective properties shared by $G$-projectives. Recall that an $R$-module $M$ is called $G_C$-projective if there exists a complete PC-resolution of $M$, which means that

$$P = \cdots \to P_1 \to P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

is an exact complex such that $M \cong \text{Coker}(P_1 \to P_0)$ and each $P_i$ and $P^i$ is projective and such that the complex $\text{Hom}_R(P, C \otimes_R Q)$ is exact for every projective $R$-module $Q$. We use $G_C P(R)$ to denote the class of all $G_C$-projective $R$-modules.

Motivated by the above, in this paper, we define the concept of Ding projective $R$-modules with respect to a fixed semidualizing module $C$, for short, $D_C$-projective and show properties of $D_C$-projective modules and dimensions. It is organized as follows:

Section 2 is devoted to the study of the $D_C$-projective modules and dimensions. White proved that every module that is either projective or $C$-projective is $G_C$-projective [18, Proposition 2.6]. Moreover, we show that they are also $D_C$-projective, see Proposition 2.7. Further, we give homological descriptions of the $D_C$-projective dimension, see Proposition 2.11. And then characterize modules with the finite $D_C$-projective dimension as follows,

**Theorem 1.1.** Let $M$ be an $R$-module and $n$ be a non-negative integer. Then the following are equivalent,

1. $D_C$-$pd_R(M) \leq n$;
2. For some integer $k$ with $1 \leq k \leq n$, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that $P_i$ is $D_C$-projective if $0 \leq i < k$ and $P_j$ is $P_C$-projective if $j \geq k$. 

(3) For any integer \( k \) with \( 1 \leq k \leq n \), there is an exact sequence \( 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0 \) such that \( P_i \) is \( D_C \)-projective if \( 0 \leq i < k \) and \( P_j \) is \( P_C \)-projective if \( j \geq k \).

**Theorem 1.2.** Let \( M \) be an \( R \)-module and \( n \) be a non-negative integer. Then the following are equivalent,

1. \( D_C\text{-pd}_R(M) \leq n \);
2. For some integer \( k \) with \( 0 \leq k \leq n \), there is an exact sequence \( 0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0 \) such that \( A_k \) is \( D_C \)-projective and other \( A_i \) projective or \( P_C \)-projective.

Although we do not know whether there is a \( G_C \)-projective \( R \)-module which is not \( D_C \)-projective, we think that this article gives new things. Proposition 2.7, Proposition 2.11, Proposition 2.20 and the above two theorems add a new message to \( G_C \)-projective \( R \)-modules if \( G_C \)-projective \( R \)-modules and \( D_C \)-projective \( R \)-modules happen to coincide.

**Setup and notation.** Throughout this paper, \( R \) denotes a commutative ring. \( C \) is a fixed semidualizing \( R \)-module. \( _RM \) denotes the category of \( R \)-modules, and \( P(R) \) and \( I(R) \) denote the class of projective and injective modules, respectively.

## 2. Properties of \( D_C \)-projective modules

Now we begin with recall of the definition on semidualizing \( R \)-modules.

**Definition 2.1.** An \( R \)-module \( C \) is semidualizing if

(a) \( C \) admits a degreewise finite projective resolution, that is, there is an exact complex \( \cdots \to P_1 \to P_0 \to C \to 0 \) with all \( P_i \) finitely generated projective \( R \)-modules,

(b) the natural homothety map \( \chi_R^C : R \to \text{Hom}_R(C, C) \) is an isomorphism, where \( \chi_R^C \) satisfies that \( \chi_R^C(r)(c) = rc \) for each \( r \in R \) and \( c \in C \), and

(c) \( \text{Ext}^{n>1}_R(C, C) = 0 \).

For any noetherian ring \( R \), a finitely generated \( R \)-module \( C \) is semidualizing if and only if \( \mathbb{R}_{\text{Hom}} R \) is in \( D(R) \), the derived category of the category of \( R \)-modules. Clearly, \( R \) is a semidualizing \( R \)-module.

**Definition 2.2.** The \( C \)-projective dimension of an \( R \)-module \( M \) is \( P_C\text{-pd}(M) = \inf \{ n \mid 0 \to X_n \to \cdots \to X_0 \to M \to 0 \} \) is exact with all \( X_i \) \( C \)-projective\}. The class of \( C \)-flat
modules, denoted by $\mathcal{F}_C$ and $\mathcal{F}_C$-flat dimension of $M$, denoted by $\mathcal{F}_C$-$fd(M)$ are defined similarly.

**Definition 2.3.** An $R$-module $M$ is called $D_C$-projective if there exists a complete PC-resolution of $M$, which means that

$$P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow C \otimes_R P^0 \rightarrow C \otimes_R P^1 \rightarrow \cdots$$

is an exact complex such that $M \cong \text{Coker}(P_1 \rightarrow P_0)$ and each $P_i$ and $P^i$ is projective and such that the complex $\text{Hom}_R(P, C \otimes_R Q)$ is exact for every flat $R$-module $Q$. We use $D_C\mathcal{P}(R)$ to denote the class of all $D_C$-projective $R$-modules.

For any $R$-module $M$, we say that $M$ has $D_C$-projective dimension at most $n$, denoted $D_C$-$pd_R(M) \leq n$, if $M$ has a $D_C$-projective resolution of length $n$, that is, there is an exact complex of the form $0 \rightarrow D_n \rightarrow \cdots \rightarrow D_0 \rightarrow M \rightarrow 0$, where all $D_i$ are $D_C$-projective $R$-modules, and say $D_C$-$pd_R(M) = n$ if there is not a shorter $D_C$-projective resolution.

**Remark 2.4.** It is clear that $D_C\mathcal{P}(R) \subseteq G_C\mathcal{P}(R)$. When $C = R$, $D_C\mathcal{P}(R) = \mathcal{D}\mathcal{P}(R)$.

From Definition 2.3 one can obtain the following characterization of $D_C$-projective $R$-modules.

**Proposition 2.5.** $M$ is $D_C$-projective if and only if $\text{Ext}_R^{n \geq 1}(M, C \otimes_R Q) = 0$ and there exists an exact sequence of the form:

$$X = 0 \rightarrow M \rightarrow C \otimes_R P^0 \rightarrow C \otimes_R P^1 \rightarrow \cdots$$

such that $\text{Hom}_R(X, C \otimes_R Q)$ is exact for any flat $R$-module $Q$.

Recall that White in [18] proved that for any projective $P$, $P$ and $C \otimes_R P$ are $G_C$-projective. Moreover, we can show that $P$ and $C \otimes_R P$ are $D_C$-projective. First we give the following lemma,

**Lemma 2.6.** Let $P$ be a projective $R$-module and $X$ be a complex. For an $R$-module $A$, if the complex $\text{Hom}_R(X, A)$ is exact, then the complex $\text{Hom}_R(P \otimes_R X, A)$ is exact. Thus, if $X$ is a complete PC-resolution of an $R$-module $M$, then $P \otimes_R X$ is a complete PC-resolution of an $R$-module $P \otimes_R M$. The converses hold in case $P$ is faithfully projective.

**Proof.** Since $\text{Hom}_R(P, -)$ is an exact functor, by the isomorphism of complexes given by Hom-tensor adjointness

$$\text{Hom}_R(P \otimes_R X, A) \cong \text{Hom}_R(P, \text{Hom}_R(X, A)),$$

exactness of the complex $\text{Hom}_R(X, A)$ implies that the complex $\text{Hom}_R(P \otimes_R X, A)$ is exact. The remains are trivial. ■

**Proposition 2.7.** (1) $C$ and $R$ are $D_C$-projective;

(2) For any projective $P$, $P$ and $C \otimes_R P$ are $D_C$-projective.
Proof. (1) Since $C$ is semidualizing, there is an exact sequence of the form:
$$X = \cdots \to R^{n_1} \to R^{n_0} \to C \to 0$$
with all $n_i$ being positive integer numbers. By [18, Lemma 1.11 (b)], $\text{Hom}_R(X, C \otimes_R Q)$ is exact for any flat $R$-module $Q$. On the other hand, there is an exact sequence of the form:
$$Y = 0 \to C \to C \to C \to \cdots .$$
By tensor evaluation homomorphism; see [2, p. 11],
$$\text{Hom}_R(Y, C \otimes_R Q) \cong \text{Hom}_R(Y, C) \otimes_R Q \cong Q$$
is exact, where $Q$ is the following exact sequence
$$\cdots \to Q \to Q \to Q \to 0 .$$
Therefore, $C$ is $D_C$-projective.

It is clear that the complex $\text{Hom}_R(X, C) = 0 \to R \to C^{n_0} \to C^{n_1} \to \cdots$ is exact. Since $R$ and all $C^{n_i}$ are finitely generated, for any flat $R$-module $F$,
$$\text{Hom}_R(\text{Hom}_R(X, C), C \otimes_R F) \cong \text{Hom}_R(\text{Hom}_R(X, C), C) \otimes_R F \cong X \otimes_R F$$
is exact. Thus $R$ is $D_C$-projective.

(2) By Lemma 2.6 and (1), for any projective $P$, $P$ and $C \otimes_R P$ are $D_C$-projective.

Using a standard argument, we can get the following proposition.

**Proposition 2.8.** If $X$ is a complete $PC$-resolution, and $L$ is an $R$-module with $\mathcal{F}_{C\text{-}fd}(L) < \infty$, then the complex $\text{Hom}_R(X, L)$ is exact. Thus if $M$ is $D_C$-projective, then $\text{Ext}^1_R(M, L) = 0$.

In [3, Lemma 2.4], the authors proved that for a $D$-projective $R$-module $M$, either $M$ is projective or $\text{fd}_R(M) = \infty$. Now we generalize it as follows:

**Proposition 2.9.** If $R$-module $M$ is $D_C$-projective, then either $M$ is $C$-flat or $\mathcal{F}_{C\text{-}fd_R}(M) = \infty$.

**Proof.** Suppose that $\mathcal{F}_{C\text{-}fd_R}(M) = n$ with $1 \leq n < \infty$. We show by induction on $n$ that $M$ is $C$-flat. First assume that $n = 1$, then there is an exact sequence $0 \to X_1 \to X_0 \to M \to 0$ with $X_0$ and $X_1$ $C$-flat. Thus by Proposition 2.8, $\text{Ext}^1_R(M, X_1) = 0$. So the above short exact sequence is split, and $M$ is a direct summand of $X_0$. By [13, Proposition 5.5], $M$ is $C$-flat. Then assume that $n \geq 2$. There is a short exact sequence $0 \to K \to X \to M \to 0$ with $X$ $C$-flat and $\mathcal{F}_{C\text{-}fd_R}(K) \leq n - 1$. By induction, we conclude that $K$ is $C$-flat. Thus $\mathcal{F}_{C\text{-}fd_R}(M) \leq 1$. By the above discussion, $M$ is $C$-flat. $

It is easy to prove the following two results using standard arguments. We leave the proofs to readers.

**Proposition 2.10.** The class of $D_C$-projective $R$-modules is projectively resolving and closed under direct summands.
Proposition 2.11. Let $M$ be an $R$-module with $D_C$-pd$_R(M) < \infty$ and $n$ be a positive integer. The following are equivalent.

(1) $D_C$-pd$_R(M) \leq n$.

(2) $\text{Ext}^i_R(M, L) = 0$ for all $i > n$ and all $R$-modules $L$ with $F_C$-fd$(L) < \infty$.

(3) $\text{Ext}^i_R(M, C \otimes_R F) = 0$ for all $i > n$ and all flat $R$-modules $F$.

(4) For any exact sequence $0 \to K \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ with all $G_i$ $D_C$-projective, $K_n$ is $D_C$-projective.

We give the following lemma which plays a crucial role in this paper.

Lemma 2.12. Let $0 \to A \to G_1 \to G_0 \to M \to 0$ be an exact sequence with $G_0$ and $G_1$ $D_C$-projective. Then there are two exact sequences $0 \to A \to C \otimes_R P \to G \to M \to 0$ with $P$ projective and $G$ $D_C$-projective and $0 \to A \to H \to Q \to M \to 0$ with $Q$ projective and $H$ $D_C$-projective.

Proof. Set $K = \text{Im}(G_1 \to G_0)$. Since $G_1$ is $D_C$-projective, there is a short exact sequence $0 \to G_1 \to C \otimes_R P \to G'_1 \to 0$ with $P$ projective and $G'_1$ $D_C$-projective. Consider the following pushout diagram:

Then consider the following pushout diagram:
By Proposition 2.10, \( G \) is \( DC \)-projective, since \( G_0 \) and \( G_1' \) are \( DC \)-projective. Therefore, we can obtain exact sequence \( 0 \to A \to C \otimes_R P \to G \to M \to 0 \). Similarly, we use pullbacks and can obtain the other exact sequence. 

**Theorem 2.13.** Let \( M \) be an \( R \)-module and \( n \) be a non-negative integer. Then the following are equivalent,

1. \( DC \)-pd\(_R\)(\( M \)) \( \leq n \);
2. For some integer \( k \) with \( 1 \leq k \leq n \), there is an exact sequence \( 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0 \) such that \( P_i \) is \( DC \)-projective if \( 0 \leq i < k \) and \( P_j \) is \( C \)-projective if \( j \geq k \).
3. For any integer \( k \) with \( 1 \leq k \leq n \), there is an exact sequence \( 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0 \) such that \( P_i \) is \( DC \)-projective if \( 0 \leq i < k \) and \( P_j \) is \( C \)-projective if \( j \geq k \).

**Proof.** (3) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (1): It is clear.

(1) \( \Rightarrow \) (3): Let \( 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \) be an exact sequence with all \( G_i \) \( DC \)-projective. We prove (3) by induction on \( n \). Let \( n = 1 \). Since \( G_1 \) is \( DC \)-projective, there is a short exact sequence \( 0 \to G_1 \to P_1 \to N \to 0 \) with \( P_1 \) \( C \)-projective and \( N \) \( DC \)-projective. Consider the following pushout diagram:

\[
\begin{array}{ccc}
0 & \to & G_1 \\
\downarrow & & \downarrow \\
0 & \to & P_1
\end{array}
\quad
\begin{array}{ccc}
& & G_0 \\
& & \downarrow \\
& & M \to 0
\end{array}
\quad
\begin{array}{ccc}
& & \downarrow \\
& & \downarrow \\
& & N
\end{array}
\quad
\begin{array}{ccc}
& & \downarrow \\
& & \downarrow \\
& & 0
\end{array}
\]

By Proposition 2.10, \( D_0 \) is \( DC \)-projective, since \( G_0 \) and \( N \) are \( DC \)-projective. Now assume that \( n > 1 \). Set \( A = Ker(G_0 \to M) \), then \( DC \)-pd\(_R\)(\( A \)) \( \leq n - 1 \). By the induction hypothesis, for any integer \( k \) with \( 2 \leq k \leq n \), there is an exact sequence \( 0 \to P_n \to \cdots \to P_1 \to A \to 0 \) such that \( P_i \) is \( DC \)-projective if \( 1 \leq i < k \) and \( P_j \) is \( C \)-projective if \( j \geq k \). Therefore, there is an exact sequence \( 0 \to P_n \to \cdots \to P_1 \to G_0 \to M \to 0 \). Set \( B = Ker(P_1 \to G_0) \). For the exact sequence \( 0 \to B \to P_1 \to G_0 \to M \to 0 \), by Lemma 2.16, there is an exact sequence \( 0 \to B \to P'_1 \to G'_0 \to M \to 0 \) with \( P'_1 \) \( C \)-projective and \( G'_0 \) \( DC \)-projective. Therefore, we get the wanted exact sequence \( 0 \to P_n \to \cdots \to P_2 \to P'_1 \to G'_0 \to M \to 0 \).

Let \( \mathcal{F} \) be a class of \( R \)-modules. A morphism \( \varphi: F \to M \) of \( \mathcal{A} \) is called an \( \mathcal{F} \)-precover of \( M \) if \( F \in \mathcal{F} \) and \( Hom(F', F) \to Hom(F', M) \to 0 \) is exact for all \( F' \in \mathcal{F} \). \( \varphi \) is called an epic \( \mathcal{F} \)-precover of \( M \) if it is an \( \mathcal{F} \)-precover and is an epimorphism. If every \( R \)-module admits an (epic) \( \mathcal{F} \)-precover, then we say \( \mathcal{F} \) is an
(epic) precovering class. $M$ is said to have a special $\mathcal{F}$-precover if there is an exact sequence

$$0 \rightarrow C \rightarrow F \rightarrow M \rightarrow 0$$

with $F \in \mathcal{F}$ and $\text{Ext}^1(\mathcal{F}, C) = 0$. It is clear that $M$ has an epic $\mathcal{F}$-precover if it has a special $\mathcal{F}$-precover. For more details about precovers, readers can refer to [5, 6, 16].

The authors in [14, Theorem 2.2] proved the following result: If $M$ is an $R$-module with $D$-pd$_R(M) < \infty$, then $M$ admits a special $D$-projective precover $\varphi : G \rightarrow M$ where pd$_R(\text{Ker}\varphi) = n - 1$ if $n > 0$ and $\text{Ker}\varphi = 0$ if $n = 0$. We can use the above theorem to generalize it to the below form,

**Corollary 2.14.** If $M$ is an $R$-module with $D_C$-pd$_R(M) = n < \infty$, then $M$ admits a special $D_C$-projective precover $\varphi : G \rightarrow M$ where $P_C$-pd$_R(\text{Ker}\varphi) \leq n - 1$ if $n > 0$ and $\text{Ker}\varphi = 0$ if $n = 0$.

**Proof.** If $n = 0$, it is trivial. Now assume that $n > 0$. By Theorem 2.13, there is an exact sequence $0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow G \rightarrow M \rightarrow 0$ such that $G$ is $D_C$-projective and any $P_j$ is $P_C$-projective. Then the remainder is trivial. ■

**Remark 2.15.** In [18, Definition 3.1], the author called a bounded $G_C$-projective resolution of $R$-module $M$ a strict $G_C$-projective resolution if there is an exact sequence

$$0 \rightarrow G_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$$

with all $G_i$ $C$-projective for $i \geq 1$ and $G_0$ $G_C$-projective. And it is proved that every $R$-module $M$ of finite $G_C$-projective dimension always admits a strict $G_C$-projective resolution [18, Theorem 3.6]. Using the different method (Theorem 2.13), we can prove that the $R$-module $M$ of finite $D_C$-projective dimension has the similar property.

**Corollary 2.16.** (1) Let $0 \rightarrow G_1 \rightarrow G \rightarrow M \rightarrow 0$ be a short exact sequence with $G_1$ and $G$ $D_C$-projective and $\text{Ext}^1_R(M, F) = 0$ for any $C$-flat $R$-module $F$. Then $M$ is $D_C$-projective.

(2) If $M$ is an $R$-module with $D_C$-pd$_R(M) = n$, then there exists an exact sequence $0 \rightarrow M \rightarrow H \rightarrow N \rightarrow 0$ with $P_C$-pd$_R(H) \leq n$ and $N$ $D_C$-projective.

**Proof.** (1) Since $D_C$-pd$_R(M) \leq 1$, by Corollary 2.14, there is an exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ where $G$ is $D_C$-projective and $K$ is $C$-projective. By the hypothesis $\text{Ext}^1_R(M, K) = 0$, the exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ is split and by Proposition 2.10, $M$ is $D_C$-projective.

(2) If $n = 0$, by the definition of $D_C$-projective $R$-modules, there is an exact sequence $0 \rightarrow M \rightarrow C \otimes_R P \rightarrow K \rightarrow 0$ where $P$ is projective and $K$ is $D_C$-projective. If $n \geq 1$, by Corollary 2.14, there is an exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ with $P_C$-pd$_R(K) \leq n - 1$. Since $G$ is $D_C$-projective, there is $0 \rightarrow G \rightarrow C \otimes_R Q \rightarrow N \rightarrow 0$ where $Q$ is projective and $N$ is $D_C$-projective. Consider the following pushout diagram:
Then $\mathcal{P}_{C,pd_R}(H) \leq n$. □

**Theorem 2.17.** Let $M$ be an $R$-module and $n$ be a non-negative integer. Then the following are equivalent,

1. $D_{C,pd_R}(M) \leq n$;

2. For some integer $k$ with $0 \leq k \leq n$, there is an exact sequence $0 \rightarrow A_n \rightarrow \cdots \rightarrow A_1 \rightarrow A_0 \rightarrow M \rightarrow 0$ such that $A_k$ is $D_{C}$-projective and other $A_i$ projective or $C$-projective.

3. For any integer $k$ with $0 \leq k \leq n$, there is an exact sequence $0 \rightarrow A_n \rightarrow \cdots \rightarrow A_1 \rightarrow A_0 \rightarrow M \rightarrow 0$ such that $A_k$ is $D_{C}$-projective and other $A_i$ projective or $C$-projective.

**Proof.** (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1): It is clear.

(1) $\Rightarrow$ (3): Let $0 \rightarrow G_n \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an exact sequence with all $G_i$ $D_{C}$-projective. We prove (3) by induction on $n$. If $n = 1$, by Lemma 2.12, the assertion is true. Now we assume that $n \geq 2$. Set $K = \text{Ker}(G_1 \rightarrow G_0)$. For the exact sequence $0 \rightarrow K \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$, by Lemma 2.12, we get two exact sequences $0 \rightarrow K \rightarrow G'_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ with $G'_1$ $D_{C}$-projective and $P_0$ projective and $0 \rightarrow G_n \rightarrow \cdots \rightarrow G_2 \rightarrow G'_1 \rightarrow P_0 \rightarrow M \rightarrow 0$. Set $N = \text{Ker}(P_0 \rightarrow M)$, then $D_{C,pd_R}(N) \leq n - 1$. By the induction hypothesis, for any integer $k$ with $1 \leq k \leq n$, there is an exact sequence $0 \rightarrow A_n \rightarrow \cdots \rightarrow A_1 \rightarrow N \rightarrow 0$ such that $A_k$ is $D_{C}$-projective and other $A_i$ are projective or $C$-projective. Therefore, we get the wanted exact sequence $0 \rightarrow A_n \rightarrow \cdots \rightarrow A_1 \rightarrow P_0 \rightarrow M \rightarrow 0$. Now we prove the case $k = 0$. Set $A = \text{Ker}(G_0 \rightarrow M)$, then $D_{C,pd_R}(A) \leq n - 1$. By the induction hypothesis, there is an exact sequence $0 \rightarrow B_n \rightarrow \cdots \rightarrow B_1 \rightarrow A \rightarrow 0$ such that $B_1$ is $D_{C}$-projective and other $B_i$ projective or $C$-projective. So we have an exact sequence $0 \rightarrow B_n \rightarrow \cdots \rightarrow B_1 \rightarrow G_0 \rightarrow M \rightarrow 0$. Set $B = \text{Ker}(B_1 \rightarrow G_0)$. For the exact sequence $0 \rightarrow B \rightarrow B_1 \rightarrow G_0 \rightarrow M \rightarrow 0$, by Lemma 2.12, we get an exact sequence $0 \rightarrow B \rightarrow P'' \rightarrow G \rightarrow M \rightarrow 0$ with $G$ $D_{C}$-projective and $P''$ $C$-projective. Hence the exact sequence $0 \rightarrow B_n \rightarrow \cdots \rightarrow B_2 \rightarrow P'' \rightarrow G \rightarrow M \rightarrow 0$ is wanted.

Let $\mathcal{F}$ be a class of $R$-modules. $\mathcal{F}^\perp$ will denote the right orthogonal class of $\mathcal{F}$, that is, $\mathcal{F}^\perp = \{M \mid \text{Ext}_R^1(F, M) = 0, \forall F \in \mathcal{F}\}$. Analogously, $^\perp \mathcal{F} = \{M \mid \text{Ext}_R^1(M, F) = 0, \forall F \in \mathcal{F}\}$. A cotorsion theory is a pair of classes $(\mathcal{F}, \mathcal{C})$ of
$R$-modules such that $\mathcal{F}^\perp = \mathcal{C}$ and $^\perp \mathcal{C} = \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called complete if every $R$-module has a special $\mathcal{F}$-precover and a special $\mathcal{C}$-preenvelope. It is called hereditary if for any exact sequence $0 \to F' \to F \to F'' \to 0$ with $F, F'' \in \mathcal{F}$ implies that $F' \in \mathcal{F}$. For more details about cotorsion theory, readers can refer to [5, 6, 16]. Let $glGpd(R) = \sup\{G_C-pd_R(M) \mid \forall M \in R \mathcal{M}\}$. We in [12, Theorem 5.1] proved that $(G_C \mathcal{P}(R), G_C \mathcal{P}(R)^\perp)$ is a complete hereditary cotorsion theory if $glGpd(R) < \infty$ and [12, Corollary 5.2] $(\mathcal{P}(R), \mathcal{P}(R)^\perp)$ is a complete hereditary cotorsion theory if $glGpd(R) < \infty$. Similarly, we prove that $(\mathcal{D}_C \mathcal{P}(R), \mathcal{D}_C \mathcal{P}(R)^\perp)$ is a complete hereditary cotorsion theory if $glDpd(R) < \infty$, where $glDpd(R) = \sup\{D_C-pd_R(M) \mid \forall M \in R \mathcal{M}\}$.

**Theorem 2.18.** Assume that $glDpd(R) < \infty$. Then $(\mathcal{D}_C \mathcal{P}(R), \mathcal{D}_C \mathcal{P}(R)^\perp)$ is a complete hereditary cotorsion theory.

**Proof.** We begin with proving that $^\perp (\mathcal{D}_C \mathcal{P}(R)^\perp) = \mathcal{D}_C \mathcal{P}(R)$. It is clear that $^\perp (\mathcal{D}_C \mathcal{P}(R)^\perp) \supseteq \mathcal{D}_C \mathcal{P}(R)$ because $Ext^1_R(A, B) = 0$ for any $A \in \mathcal{D}_C \mathcal{P}(R)$ and $B \in \mathcal{D}_C \mathcal{P}(R)^\perp$ by definition. By Corollary 2.14, there is an exact sequence $0 \to K \to G \to M \to 0$ such that $G$ is $\mathcal{D}_C$-projective and $\mathcal{P}_Cpd(K) < \infty$. By Proposition 2.8, $K \in \mathcal{D}_C \mathcal{P}(R)^\perp$. So $Ext^1_R(M, K) = 0$, and then $0 \to K \to G \to M \to 0$ is split, i.e., $M$ is a direct summand of $G$. By Proposition 2.10, $M$ is $\mathcal{D}_C$-projective.

By Proposition 2.10, $(\mathcal{D}_C \mathcal{P}(R))$ is projectively resolving, $(\mathcal{D}_C \mathcal{P}(R)^\perp)$ is injectively resolving, so $(\mathcal{D}_C \mathcal{P}(R), \mathcal{D}_C \mathcal{P}(R)^\perp)$ is hereditary. By Corollary 2.14, $(\mathcal{D}_C \mathcal{P}(R), \mathcal{D}_C \mathcal{P}(R)^\perp)$ is complete. $\blacksquare$

**Corollary 2.19.** If $glDpd(R) = \sup\{Dpd_R(M) \mid \forall M \in R \mathcal{M}\} < \infty$, $(\mathcal{D}_C \mathcal{P}(R), \mathcal{D}_C \mathcal{P}(R)^\perp)$ is a complete hereditary cotorsion theory.

**Proposition 2.20.** (1) $Ext^n_R(G, M) = 0$ for all $n \geq 1$, $G \in \mathcal{D}_C \mathcal{P}(R)$ and $M \in \mathcal{D}_C \mathcal{P}(R)^\perp$.

(2) $\mathcal{P}_C = \mathcal{D}_C \mathcal{P}(R) \cap \mathcal{D}_C \mathcal{P}(R)^\perp$.

(3) If $M$ be an $R$-module with $\mathcal{P}_Cpd_R(M) < \infty$, then $\mathcal{P}_Cpd_R(M) = \mathcal{D}_Cpd_R(M)$.

(4) If $M$ be an $R$-module with $\mathcal{D}_Cpd_R(M) < \infty$, then $\mathcal{D}_Cpd_R(M) = \mathcal{D}_Cpd_R(M)$.

(5) If $M$ be an $R$-module with $pd_R(M) < \infty$, then $pd_R(M) = \mathcal{D}_Cpd_R(M)$.

**Proof.** (1) For any $\mathcal{D}_C$-projective $R$-module $G$, there is an exact sequence

$$0 \to G' \to P_{n-1} \to \cdots \to P_1 \to P_0 \to G \to 0$$

where all $P_i$ are projective and $G'$ is $\mathcal{D}_C$-projective. So for any $M \in \mathcal{D}_C \mathcal{P}(R)^\perp$, $Ext^n_R(G, M) = Ext^n_R(G', M) = 0$.

(2) By Propositions 2.7 and 2.8, $\mathcal{P}_C \subseteq \mathcal{D}_C \mathcal{P}(R) \cap \mathcal{D}_C \mathcal{P}(R)^\perp$. Let $M \in \mathcal{D}_C \mathcal{P}(R) \cap \mathcal{D}_C \mathcal{P}^\perp$. There is a short exact sequence $0 \to M \to C \otimes_R P \to M' \to 0$ where $P$ is projective and $M'$ is $\mathcal{D}_C$-projective. So $Ext^n_R(M', M) = 0$ and
0 → M → C ⊗_R P → M′ → 0 is split. Therefore M ∈ PC and PC ⊇ DC(P(R)) ∩ DC(P(R)⁺).

(3) It is clear that PC-pdR(M) ≥ DC-pdR(M), since every C-projective R-module is DC-projective. Now we prove that PC-pdR(M) ≤ DC-pdR(M). For doing this we assume that DC-pdR(M) = n < ∞. Since PC is precovering [13, Proposition 5.10] and projectively resolving [13, Corollary 6.8], there is an exact sequence

0 → K → C ⊗_R P_{n−1} → · · · → C ⊗_R P₁ → C ⊗_R P₀ → M → 0

with K DC-projective. Since M be an R-module with PC-pdR(M) < ∞, PC-pdR(K) < ∞. By (2), K is C-projective.

(4) It is clear that GC-pdR(M) ≤ DC-pdR(M), since every DC-projective R-module is GC-projective. Now we assume that DC-pdR(M) = n < ∞. By [18, Proposition 2.1,2], it is sufficient to find a projective R-module P such that Ext^n_R(M, C ⊗_R P) ≠ 0. By Proposition 2.11, there is a flat R-module F such that Ext^n_R(M, C ⊗_R F) ≠ 0. Since PC is precovering [13, Proposition 5.10] and FC is projectively resolving [13, Corollary 6.8], there is a short exact sequence

0 → K → C ⊗_R P → C ⊗_R F → 0

where K is C-flat. By [15, Theorem 7.3], there is a long exact sequence ...

Ext^n_R(M, C ⊗_R P) → Ext^n_R(M, C ⊗_R F) → Ext^n_R(M, K) → · · · , where Ext^n+1_R(M, K) = 0. So Ext^n_R(M, C ⊗_R P) ≠ 0.

(5) It is clear that GC-pdR(M) ≤ DC-pdR(M) ≤ pdR(M). It is well-known that pdR(M) = GC-pdR(M) if pdR(M) < ∞. So pdR(M) = DC-pdR(M). □

We round off this paper with the following questions:

(1) Recall that the author in [14, Theorem 3.1] proved that for any ring R, r.glGdim(R) = r.glDdim(R). So we conjecture that glGCpd(R) = glDCpd(R), is it true?

(2) Whether is there a GC-projective R-module which is not DC-projective?

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References


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