SOME REMARKS ON PARAMEDIAL SEMIGROUPS

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Abstract. Semigroups satisfying some type of generalized commutativity were considered in quite a number of papers. S. Lajos, A. Nagy and M. Yamada dealt with externally commutative semigroups. N. Stevanović and P. V. Protić in [Structure of weakly externally commutative semigroups, Algebra Colloq. 13:3 (2006) 441-446], introduced the notion of weakly externally commutative semigroup and gave a structural description for some subclasses of this class of semigroups. In this paper we consider a class which is a generalization of the class of externally commutative semigroups.

1. Introduction

A semigroup \( S \) in which the following holds
\[
(\forall x, y, z \in S) \quad xyz = zyx
\]
is an externally commutative semigroup [4]. The class of externally commutative semigroups appears as a natural generalization of the class of commutative semigroups.

Now we are going to introduce the concept of paramedial semigroups as a generalization of externally commutative semigroups.

**Definition 1.** A semigroup \( S \) is a paramedial semigroup if the paramedial law
\[
(\forall a, b, c, d \in S) \quad abcd = dbca
\]
holds in \( S \).

If \( S \) is an externally commutative semigroup, then \( abcd = dbca \) for all \( a, b, c, d \in S \), thus \( S \) is a paramedial semigroup. Hence, the class of externally commutative semigroups is a subclass of the class of paramedial semigroups.
Let $S$ be a paramedial semigroup, $a, c \in S$ and $b \in S^2$; then $abc = cba$. It follows by above that if $S$ is a paramedial semigroup, then $S^2$ is an externally commutative subsemigroup of $S$.

A semigroup $S$ is a universal (global idempotent) semigroup if it satisfies $S^2 = S$. Therefore a universal paramedial semigroup $S$ is an externally commutative semigroup. Also, by [8, Proposition 1.1], an universal externally commutative semigroup is commutative. Hence, each universal paramedial semigroup is commutative.

The concept of weakly externally commutative semigroup was introduce in [7].

Definition 2. [7] A semigroup $S$ in which the following holds

$$\exists a \in S \forall x, y \in S \ x ay = yax,$$

is a weakly externally commutative semigroup.

It is clear from the definition that the class of paramedial semigroups is included in the class of weakly externally commutative semigroups.

2. Some general properties of paramedial semigroups

Lemma 1. Let $S$ be a simple paramedial semigroup. Then $E(S) \neq \emptyset$.

Proof. Since $S$ is a simple semigroup, it follows that $S = SaS$ for all $a \in S$. Now, for $x \in S$, from $x^2 \in Sx^4S$ it follows that $x^2 = ux^4v$ for some $u, v \in S$. Consequently,

$$(ux^2v)^2 = ux^2vux^2v = uxxvux^2v = uuxvxxxv
= uuxxxxxv = u(ux^4v)v = ux^2v.$$ 

Hence, $ux^2v \in E(S)$.

If $S$ is a semigroup, then $C(S) = \{a \in S \mid \forall x \in S \ xa = ax\}$ is the well known center of $S$.

Lemma 2. Let $S$ be a paramedial semigroup. If $E(S) \neq \emptyset$ then $E(S)$ is a semilattice and $E(S) \subseteq C(S)$.

Proof. Let $e, f \in E(S)$ be arbitrary elements. Then $ef = eef = fefe = fe$ and so

$$(ef)^2 = efef = eeff = ef.$$ 

Consequently, $E(S)$ is a commutative subsemigroup of $S$.

Let $e \in E(S)$ and $x \in S$ be arbitrary elements, then $ex = eex = xeex = xe$, hence $E(S) \subseteq C(S)$.

Lemma 3. Let $S$ be a paramedial semigroup. Then $S^3$ is a commutative semigroup.
Some remarks on paramedial semigroups

Proof. Let \( a, b \in S^3 \). Then there exist elements \( x, y, z, u, v, w \in S \) such that 
\[
ab = xyzuvw = uwyxxw = wxyvw = uvxyz = ba.
\]

Lemma 4. Let \( S \) be a paramedial semigroup and \( x, y \in S \) be its arbitrary elements. Then \((xy)^2 = y^2x^2\) and for \( n \in \mathbb{N} \) and \( n \geq 3 \) it follows that 
\[
(xy)^n = x^n y^n = y^n x^n = (yx)^n.
\] (1)

Proof. Let \( x, y \in S \). Then 
\[
(xy)^2 = xyxy = yxx = y^2 x^2.
\]
We are going to prove the second part of lemma by induction. For \( n = 3 \) it follows 
\[
(xy)^3 = xyxyxy = xy(xyxy) = xyyyxx = y^3 x^3.
\]
By Lemma 3, it follows that \( x^3 y^3 = y^3 x^3 \). Hence, \((xy)^3 = x^3 y^3 = y^3 x^3 = (yx)^3\).

Let \((xy)^n = x^n y^n = y^n x^n = (yx)^n\). Now we get 
\[
(xy)^{n+1} = (xy)^n xy = x^n y^n xy = y^n x^nx^n = y^{n+1} x^{n+1}.
\]
Since \( S^n, n \geq 3, \) is a commutative semigroup, it follows that \( x^{n+1} y^{n+1} = y^{n+1} x^{n+1}, \)
which gives \((xy)^{n+1} = x^{n+1} y^{n+1} = y^{n+1} x^{n+1} = (yx)^{n+1}\) and the lemma is proved.■

By above, if \( S \) is a paramedial semigroup, \( m, n \in \mathbb{N}, x_1, x_2, \ldots, x_m \in S\), then 
\[
(x_1 x_2 \cdots x_m)^n = (x_{p(1)} x_{p(2)} \cdots x_{p(m)})^n = x_{p(1)}^n x_{p(2)}^n \cdots x_{p(m)}^n,
\]
where \( \{p(1), p(2), \ldots, p(n)\} \) is a permutation of \( \{1, 2, \ldots, n\} \).

A semigroup \( S \) is a \((\text{well known})\) \( E\)-semigroup if \((xy)^{m} = x^{m} y^{m}, m \geq 2 \)
holds for some \( m \in \mathbb{N} \) and for all \( x, y \in S \).

By the above lemma, every paramedial semigroup is an \( E\)-semigroup for all \( m \geq 3 \).

3. Semilattice decomposition of paramedial semigroups

Theorem 1. Let \( S \) be a paramedial semigroup. Then the relation \( \rho \) defined on \( S \) by 
\[
\rho \iff (\forall x, y \in S)(\exists m, n \in \mathbb{N}) \ xa^my \in xbyS, \ xb^ny \in xayS
\]
is a semilattice congruence.

Proof. Let \( a, x, y \in S \). Since \( S \) is a paramedial semigroup, then 
\[
xy = xaaaay = xayaa \in xayS
\]
and so $apa$. Clearly, $\rho$ is a symmetric relation. Let $a, b, c \in S$ and

$$ a \rho b \iff (\forall x, y \in S)(\exists m, n \in N) x a^m y \in xbyS, \ x b^n y \in xayS, $$

$$ b \rho c \iff (\forall x, y \in S)(\exists p, q \in N) x b^p y \in xcyS, \ x c^q y \in xbyS. $$

Now

$$ x a^m a^{p+1} y = x a^m \ldots a^m y \subseteq xbyS a^m \ldots a^m \subseteq xbyS \ b \subseteq xbySS a^m \ldots a^m b^2 \subseteq xbySSS a^m \ldots a^m b^3 \subseteq \ldots = x a^m S S \ldots S b^p \subseteq xbySS \ldots S b^p = xb^p y S S \ldots S b \subseteq xb^p y S \subseteq xcyS. $$

Similarly, $x a^{n+1} \subseteq xayS$. Hence, $a \rho c$ and so the relation $\rho$ is transitive. It follows that $\rho$ is an equivalence relation.

Let

$$ a \rho b \iff (\forall x, y \in S)(\exists m, n \in N) x a^m y \in xbyS, \ x b^n y \in xayS $$

and $c \in S$ be an arbitrary element. Then, by Lemma 4,

$$ x(ac)^{m+3} y = x a^{m+3} c^{m+3} = x a^{m+3} c^{m+1} c y = x a^{m+3} y c^{m+3} = x a^{m+3} c a y c^{m+3} $$

$$ = x a^{m+3} y a^{m+3} c^{m+2} = x b y S a^{m+3} c^{m+2} c = x b y S a^{m+3} c^{m+2} y = x b y a^{m+3} c^{m+2} S $$

$$ \subseteq x b y S a^{m+3} c^{m+2} S \subseteq x b y S. $$

Similarly, $x(be)^{n} y \in xacyS$. Hence, $acpb$ and so $\rho$ is a left congruence on $S$. Also,

$$ x(ca)^{n+3} y = x c^{n+3} a^{n+3} = x c^{n+1} c a^{n+3} = x c^{n+1} a^{n+2} a^{3} y $$

$$ = x c^{n+1} a^{n+2} a^{3} c \subseteq x b y S c^{n+2} a^{3} c = x c y S c^{n+2} a^{3} c = x c y S c^{n+2} a^{3} y $$

$$ = x c b y c^{n+2} a^{3} S \subseteq x c b y S. $$

Similarly, $x(cb)^{n} y \in xacyS$. Hence, $capb$ and so $\rho$ is a right congruence on $S$.

Hence, $\rho$ is a congruence relation on $S$.

Let $a, x, y \in S$ be arbitrary elements. Then

$$ x a^5 y = x a^2 a a y = x a^2 y a a a \in x a^2 y S, $$

$$ x(a^2)^2 y = x a a a y = x a y a^3 \in x a y S. $$

Now, $apa^2$ and so $\rho$ is a band congruence on $S$.

Let $a, b, x, y \in S$ be arbitrary elements. Then

$$ x(ab)^3 y = x ab (ab) by = x b a b a b y = x b a y a b b \in x b a y S. $$

Similarly, $x(ba)^3 y \in x a b y S$. Hence, $abpb$ and it follows that $\rho$ is a semilattice congruence on $S$. ■
Corollary 1. If $S$ is a paramedial semigroup, then $S$ is a semilattice of Archimedean paramedial subsemigroups on $S$.

Proof. Let $S$ be a paramedial semigroup. Then the relation $\rho$ defined as in the above theorem, is a semilattice congruence. We prove that $\rho$-classes are Archimedean semigroups. Hence, $S = \bigcup_{\alpha \in Y} S_\alpha$, $Y$ is a semilattice and $S_\alpha$ are $\rho$-classes. Let $a, b \in S_\alpha$. Then

\[ ab \Leftrightarrow (\forall x, y \in S)(\exists m, n \in \mathbb{N}) \ x a^m y \in x y S, \ x b^n y \in x a y S. \]

For $x = a = y$ it follows that $a^{m+2} \in abaS \subseteq SbS$. Now $a^{m+2} = ubv$ for some $u, v \in S$. Let $u \in S_\beta$, $v \in S_\gamma$. Since $a^{m+2} \in S_\alpha$ and $a^{m+2} = ubv \in S_\beta S_\alpha S_\gamma \subseteq S_{\beta \alpha \gamma}$, we have $\alpha = \beta \alpha \gamma$. Also, since $Y$ is a semilattice, we have $\beta \alpha \gamma \beta = \alpha$, $\gamma \beta \alpha \gamma = \alpha$ and so $ubvu, vubv \in S_\alpha$. Now

\[ a^{3(m+2)} = ubv \cdot ubv \cdot ubv = (ubvu)b(vubv) \in S_\alpha bS_\alpha. \]

Hence, $S_\alpha$ is an Archimedean semigroup. ■

References


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