KOROVKIN TYPE APPROXIMATION THEOREM 
IN $A^2_I$-STATISTICAL SENSE

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Abstract. In this paper we consider the notion of $A^2_I$-statistical convergence for real double sequences which is an extension of the notion of $A^2$-statistical convergence for real single sequences introduced by Savas, Das and Dutta. We primarily apply this new notion to prove a Korovkin type approximation theorem. In the last section, we study the rate of $A^2_I$-statistical convergence.

1. Introduction and background

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. Approximation theory has important applications in the theory of polynomial approximation in various areas of functional analysis. For a sequence $\{T_n\}_{n \in \mathbb{N}}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset $X$ of real numbers, Korovkin [20] first established the necessary and sufficient conditions for the uniform convergence of $\{T_n f\}_{n \in \mathbb{N}}$ to a function $f$ by using the test functions $e_0 = 1$, $e_1 = x$, $e_2 = x^2$ [1]. The study of the Korovkin type approximation theory has a long history and is a well-established area of research. As is mentioned in [12] in particular, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence of various sequences of positive linear operators such as the interpolation operators of Hermite-Fejér [6]. In recent years, using the concept of uniform statistical convergence various statistical approximation results have been proved [9,10]. Erkuş and Duman [15] studied a Korovkin type approximation theorem via $A$-statistical convergence in the space $H_w(I^2)$ where $I^2 = [0, \infty) \times [0, \infty)$ which was extended for double sequences of positive linear operators of two variables in $A$-statistical sense by Demirci and Dirik in [12]. Our primary interest in this paper is to obtain a general Korovkin type approximation theorem for double sequences of positive linear operators of two variables from $H_w(K)$ to $C(K)$ where $K = [0, A] \times [0, B]$, $A, B \in (0, 1)$, in the sense of $A^2_I$-statistical convergence.

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The concept of convergence of a sequence of real numbers was extended to statistical convergence by Fast [17]. Further investigations started in this area after the pioneering works of Šalát [31] and Fridy [18]. The notion of $I$-convergence of real sequences was introduced by Kostyrko et al. [23] as a generalization of statistical convergence using the notion of ideals (see [3,4,5] for further references). Later the idea of $I$-convergence was also studied in topological spaces in [24]. On the other hand statistical convergence was generalized to $A$-statistical convergence by Kolk [21,22]. Later a lot of works have been done on matrix summability and $A$-statistical convergence (see [2,7,8,11,16,19,21,22,25,29]). In particular, very recently in [33] and [34] the two above mentioned approaches were unified and the very general notion of $A^2$-statistical convergence was introduced and studied. In this paper we consider an extension of this notion to double sequences, namely $A^2$-statistical convergence.

A real double sequence $\{x_{mn}\}_{m,n \in \mathbb{N}}$ is said to be convergent to $L$ in Pringsheim’s sense if for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ for all $m,n > N(\varepsilon)$ and denoted by $\lim_{m,n} x_{mn} = L$. A double sequence is called bounded if there exists a positive number $M$ such that $|x_{mn}| \leq M$ for all $(m,n) \in \mathbb{N} \times \mathbb{N}$. A real double sequence $\{x_{mn}\}_{m,n \in \mathbb{N}}$ is statistically convergent to $L$ if for every $\varepsilon > 0$,

$$\lim_{j,k} \frac{|\{m \leq j, n \leq k : |x_{mn} - L| \geq \varepsilon \}|}{jk} = 0$$

[27,28].

Recall that a family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set $Y$ is said to be an ideal in $Y$ if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; (ii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of $Y$ further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is a non-trivial proper ideal in $Y$ (i.e. $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$) then the family of sets $F(\mathcal{I}) = \{M \subset Y : \text{there exists } A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in $Y$. It is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\set{\{i\} \times \mathbb{N}}$ and $\mathbb{N} \times \set{i}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also. Let $\mathcal{I}_0 = \{A \subset \mathbb{N} \times \mathbb{N} : \exists m(A) \in \mathbb{N} \text{ such that } i,j \geq m(A) \Rightarrow (i,j) \notin A\}$. Then $\mathcal{I}_0$ is a non-trivial strongly admissible ideal [14]. Let $A = (a_{nk})$ be a non-negative regular matrix. For an ideal $\mathcal{I}$ of $\mathbb{N}$ a sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be $A^2$-statistically convergent to $L$ if for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \sum_{k \in K(\varepsilon)} a_{nk} \geq \delta \right\} \in \mathcal{I},$$

where $K(\varepsilon) = \set{k \in \mathbb{N} : |x_k - L| \geq \varepsilon}$ [33,34].

Let $A = (a_{jk})$ be a four dimensional summability matrix. For a given double sequence $\{x_{mn}\}_{m,n \in \mathbb{N}}$, the $A$-transform of $x$, denoted by $Ax := ((Ax)_{jk})$, is given by

$$(Ax)_{jk} = \sum_{(m,n) \in \mathbb{N}^2} a_{jkmn} x_{mn}$$

provided the double series converges in Pringsheim sense for every $(j,k) \in \mathbb{N}^2$. In 1926, Robison [30] presented a four dimensional analog of the regularity by
considering an additional assumption of boundedness. This assumption was made because a convergent double sequence is not necessarily bounded.

Recall that a four dimensional matrix \( A = (a_{jkmn}) \) is said to be RH-regular if it maps every bounded convergent double sequence into a convergent double sequence with the same limit. The Robison-Hamilton conditions state that a four dimensional matrix \( A = (a_{jkmn}) \) is RH-regular if and only if

(i) \( \lim_{j,k} a_{jkmn} = 0 \) for each \( (m, n) \in \mathbb{N}^2 \),

(ii) \( \lim_{j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{jkmn} = 1 \),

(iii) \( \lim_{j,k} \sum_{m \in \mathbb{N}} |a_{jkmn}| = 0 \) for each \( n \in \mathbb{N} \),

(iv) \( \lim_{j,k} \sum_{n \in \mathbb{N}} |a_{jkmn}| = 0 \) for each \( m \in \mathbb{N} \),

(v) \( \sum_{(m,n) \in \mathbb{N}^2} |a_{jkmn}| \) is convergent,

(vi) there exist finite positive integers \( M_0 \) and \( N_0 \) such that \( \sum_{m,n > N_0} |a_{jkmn}| < M_0 \) holds for every \( (j, k) \in \mathbb{N}^2 \).

Let \( A = (a_{jkmn}) \) be a non-negative RH-regular summability matrix and let \( K \subset \mathbb{N}^2 \). Then the \( A \)-density of \( K \) is given by

\[
\delta^{(2)}_A \{ K \} = \lim_{j,k} \sum_{(m,n) \in K} a_{jkmn}
\]

provided the limit exists. A real double sequence \( x = \{x_{mn}\}_{m,n \in \mathbb{N}} \) is said to be \( A \)-statistically convergent to a number \( L \) if for every \( \varepsilon > 0 \)

\[
\delta^{(2)}_A \{ (m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon \} = 0.
\]

We denote \( I^{(2)}_A = \left\{ C \subset \mathbb{N}^2 : \delta^{(2)}_A \{ C \} = 0 \right\} \) which is an admissible ideal in \( \mathbb{N} \times \mathbb{N} \). Throughout we use \( I \) as a non-trivial strongly admissible ideal on \( \mathbb{N} \times \mathbb{N} \).

2. A Korovkin type approximation theorem

Recently the concept of \( I \)-statistical convergence for real single sequences has been introduced by Das and Savas as a notion of convergence which is strictly weaker than the notion of statistical convergence (see [32] for details). Consequently this notion has been further investigated in [13]. Very recently it has been further generalized by using a summability matrix \( A \) into \( A^T \)-statistical convergence for real single sequences by Savas, Das and Dutta [33,34]. In this paper we consider the following natural extension of these convergence for real double sequences.

The following definition is due to E. Savas (who has informed about it in a personal communication).

**Definition 2.1.** A real double sequence \( \{x_{m,n}\}_{m,n \in \mathbb{N}} \) is said to be \( I^2 \)-statistically convergent to \( L \) if for each \( \varepsilon > 0 \) and \( \delta > 0 \),

\[
\left\{ (j,k) \in \mathbb{N}^2 : \frac{1}{jk} \sum_{m \leq j, n \leq k} |x_{mn} - L| \geq \varepsilon \right\} \in I.
\]

We now introduce the main definition of this paper.
**Definition 2.2.** Let \( A = (a_{jkmn}) \) be a non-negative RH-regular summability matrix. Then a real double sequence \( \{x_{mn}\}_{m,n \in \mathbb{N}} \) is said to be \( A^2_{2}\)-statistically convergent to a number \( L \) if for every \( \varepsilon > 0 \) and \( \delta > 0 \),

\[
\left\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n) \in K_2(\varepsilon)} a_{jkmn} \geq \delta \right\} \in \mathcal{I},
\]

where \( K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon \} \). In this case, we write \( A^2_{2}\)-st-lim\(_{m,n} x_{mn} = L \).

It should be noted that, if we take \( A = C(1,1) \), the double Cesàro matrix [26] defined as follows

\[
a_{jkmn} = \begin{cases} \frac{1}{jk} & \text{for } m \leq j, n \leq k; \\ 0 & \text{otherwise}, \end{cases}
\]

then \( A^2_{2}\)-statistical convergence coincides with the notion of \( I_2\)-statistical convergence. Again if we replace the matrix \( A \) by the identity matrix for four dimensional matrices and \( \mathcal{I} = \mathcal{I}_0 \) then \( A^2_{2}\)-statistical convergence reduces to the Pringsheim convergence for double sequences. For the ideal \( \mathcal{I} = \mathcal{I}_0 \), \( A^2_{2}\)-statistical convergence implies \( A\)-statistical convergence for double sequences. The basic properties of \( A^2_{2}\)-statistically convergent double sequences are similar to \( A^2\)-statistical convergent single sequences and can be obtained analogously as in [32,33]. So our main aim here is to present an application of this notion in approximation theory.

Throughout this section, let \( \mathcal{K} = [0,A] \times [0,B] \), \( A, B \in (0,1) \) and denote the space of all real valued continuous functions on \( \mathcal{K} \) by \( C(\mathcal{K}) \). This space is endowed with the supremum norm \( \|f\| = \sup_{(x,y)\in \mathcal{K}} |f(x,y)| \), \( f \in C(\mathcal{K}) \). Consider the space \( H_w(\mathcal{K}) \) of real valued functions \( f \) on \( \mathcal{K} \) satisfying

\[
|f(u,v) - f(x,y)| \leq w(f; \delta) \sqrt{\left(\frac{u}{1-u} - \frac{x}{1-x}\right)^2 + \left(\frac{v}{1-v} - \frac{y}{1-y}\right)^2}
\]

where \( w \) is the modulus of continuity for \( \delta > 0 \) given by

\[
w(f; \delta) = \sup \{|f(u,v) - f(x,y)| : (u,v), (x,y) \in \mathcal{K}, \sqrt{(u-x)^2 + (v-y)^2} \leq \delta \}.
\]

Then it is clear that any function in \( H_w(\mathcal{K}) \) is continuous and bounded on \( \mathcal{K} \).

We will use the following test functions \( f_0(x,y) = 1 \), \( f_1(x,y) = \frac{x}{1-x} \), \( f_2 = \frac{y}{1-y} \), \( f_3(x,y) = \left(\frac{x}{1-x}\right)^2 + \left(\frac{y}{1-y}\right)^2 \) and we denote the value of \( T\) at a point \((u,v) \in \mathcal{K}\) by \( T(f;u,v) \).

Now we establish the Korovkin type approximation theorem in \( A^2_{2}\)-statistical sense.

**Theorem 2.1.** Let \( \{T_{mn}\}_{m,n \in \mathbb{N}} \) be a sequence of positive linear operators from \( H_w(\mathcal{K}) \) into \( C(\mathcal{K}) \) and let \( A = (a_{jkmn}) \) be a non-negative RH-regular summability matrix. Then for any \( f \in H_w(\mathcal{K}) \),

\[
A^2_{2}\text{-st-lim}_{m,n} \|T_{mn}f - f\| = 0
\]

is satisfied if the following holds

\[
A^2_{2}\text{-st-lim}_{m,n} \|T_{mn}f_i - f_i\| = 0, \; i = 0, 1, 2, 3.
\]
Proof. Assume that (2) holds. Let $f \in H_w(K)$. Our objective is to show that for given $\varepsilon > 0$ there exist constants $C_0$, $C_1$, $C_2$, $C_3$ (depending on $\varepsilon > 0$) such that

$$\|T_{mn}f - f\| \leq \varepsilon + C_3\|T_{mn}f_3 - f_3\| + C_2\|T_{mn}f_2 - f_2\| + C_1\|T_{mn}f_1 - f_1\| + C_0\|T_{mn}f_0 - f_0\|.$$ 

If this is done then our hypothesis implies that for any $\varepsilon > 0$, $\delta > 0$,

$$\left\{(j, k) \in \mathbb{N}^2 : \sum_{(m, n) \in K_2(\varepsilon)} a_{jkn} \geq \delta \right\} \in I,$$

where $K_2(\varepsilon) = \{(m, n) \in \mathbb{N}^2 : \|T_{mn}f - f\| \geq \varepsilon\}$.

To this end, start by observing that for each $(u, v) \in K$ the function $0 \leq g_{uv} \in H_w(K)$ defined by $g_{uv}(s, t) = \left(\frac{s}{1-u} - \frac{u}{1-v}\right)^2 + \left(\frac{t}{1-t} - \frac{v}{1-v}\right)^2$ satisfies $g_{uv} = \left(\frac{s}{1-u} - \frac{u}{1-v}\right)^2 + \left(\frac{t}{1-t} - \frac{v}{1-v}\right)^2$. Since each $T_{mn}$ is a positive operator, $T_{mn}g_{uv}$ is a positive function. In particular, we have for each $(u, v) \in K$,

$$0 \leq T_{mn}g_{uv}(u, v)$$

$$= [T_{mn}\left(\left(\frac{s}{1-u}\right)^2 + \left(\frac{u}{1-v}\right)^2 - \frac{2u}{1-u} \frac{x}{1-z} - \frac{2v}{1-v} \frac{y}{1-w} + \left(\frac{u}{1-u}\right)^2 + \left(\frac{v}{1-v}\right)^2; u, v\right)]$$

$$= [T_{mn}\left(\left(\frac{s}{1-u}\right)^2 + \left(\frac{u}{1-v}\right)^2; u, v\right) - \left(\frac{u}{1-u}\right)^2 - \left(\frac{v}{1-v}\right)^2]$$

$$- \frac{2u}{1-u}[T_{mn}\left(\frac{s}{1-u}; u, v\right) - \frac{u}{1-u}] - \frac{2v}{1-v}[T_{mn}\left(\frac{u}{1-v}; u, v\right) - \frac{v}{1-v}]$$

$$+ \{(\frac{u}{1-u}\right)^2 + \left(\frac{v}{1-v}\right)^2\}T_{mn}f_0 - f_0\right\}$$

$$\leq \|T_{mn}f_3 - f_3\| + \frac{2u}{1-u}\|T_{mn}f_1 - f_1\|$$

$$+ \frac{2v}{1-v}\|T_{mn}f_2 - f_2\| + \{(\frac{u}{1-u}\right)^2 + \left(\frac{v}{1-v}\right)^2\}\|T_{mn}f_0 - f_0\|.$$ 

Let $M = \|f\|$ and $\varepsilon > 0$. By the uniform continuity of $f$ on $K$ there exists a $\delta > 0$ such that $-\varepsilon < f(s, t) - f(u, v) < \varepsilon$ holds whenever

$$\sqrt{\left(\frac{s}{1-s} - \frac{u}{1-u}\right)^2 + \left(\frac{t}{1-t} - \frac{v}{1-v}\right)^2} < \delta,$$

$(s, t), (u, v) \in K$. Next observe that

$$-\varepsilon - \frac{2M}{\delta^2} \left\{ \left(\frac{s}{1-s} - \frac{u}{1-u}\right)^2 + \left(\frac{t}{1-t} - \frac{v}{1-v}\right)^2 \right\}$$

$$\leq f(s, t) - f(u, v)$$

$$\leq \varepsilon + \frac{2M}{\delta^2} \left\{ \left(\frac{s}{1-s} - \frac{u}{1-u}\right)^2 + \left(\frac{t}{1-t} - \frac{v}{1-v}\right)^2 \right\}$$

(3)

Indeed, if $\sqrt{\left(\frac{s}{1-s} - \frac{u}{1-u}\right)^2 + \left(\frac{t}{1-t} - \frac{v}{1-v}\right)^2} < \delta$ then (3) follows from

$$-\varepsilon < f(s, t) - f(u, v) < \varepsilon.$$
On the other hand, if 

\[ \sqrt{(\frac{s}{1-s} - \frac{u}{1-u})^2 + (\frac{t}{1-t} - \frac{v}{1-v})^2} \geq \delta \]

then (3) follows from

\[ -\varepsilon - \frac{2M}{\delta^2} \left\{ \left( \frac{s}{1-s} - \frac{u}{1-u} \right)^2 + \left( \frac{t}{1-t} - \frac{v}{1-v} \right)^2 \right\} \leq -2M \leq f(s, t) - f(u, v) \leq 2M \]

\[ \leq \varepsilon + \frac{2M}{\delta^2} \left\{ \left( \frac{s}{1-s} - \frac{u}{1-u} \right)^2 + \left( \frac{t}{1-t} - \frac{v}{1-v} \right)^2 \right\} . \]

Since each \( T_{mn} \) is positive and linear it follows from (3) that

\[ -\varepsilon T_{mn}f_0 - \frac{2M}{\delta^2} T_{mn}g_{uv} \leq T_{mn}f - f(u, v)T_{mn}f_0 \leq \varepsilon T_{mn}f_0 + \frac{2M}{\delta^2} T_{mn}g_{uv}. \]

Therefore

\[ |T_{mn}(f; u, v) - f(u, v)T_{mn}(f_0; u, v)| \]

\[ \leq \varepsilon + \varepsilon T_{mn}(f_0; u, v) - f_0(u, v) + \frac{2M}{\delta^2} T_{mn}g_{uv} \]

\[ \leq \varepsilon + \varepsilon T_{mn}f_0 - f_0 + \frac{2M}{\delta^2} T_{mn}g_{uv} \]

In particular, note that

\[ |T_{mn}(f; u, v) - f(u, v)| \]

\[ \leq |T_{mn}(f; u, v) - f(u, v)T_{mn}(f_0; u, v)| + |f(u, v)| |T_{mn}(f_0; u, v) - f_0(u, v)| \]

\[ \leq \varepsilon + (M + \varepsilon) ||T_{mn}f_0 - f_0|| + \frac{2M}{\delta^2} T_{mn}g_{uv} \]

which implies

\[ ||T_{mn}f - f|| \leq \varepsilon + C_3 ||T_{mn}f_3 - f_3|| + C_2 ||T_{mn}f_2 - f_2|| + C_1 ||T_{mn}f_1 - f_1|| + C_0 ||T_{mn}f_0 - f_0||, \]

where \( C_0 = \left\{ \frac{2M}{\delta^2} \left( \left( \frac{A}{1-A} \right)^2 + \left( \frac{B}{1-B} \right)^2 \right) + M + \varepsilon \right\}, C_1 = \frac{4M}{\delta^2} \frac{A}{1-A}, C_2 = \frac{4M}{\delta^2} \frac{B}{1-B} \) and \( C_3 = \frac{2M}{\delta^2} \), i.e.,

\[ ||T_{mn}f - f|| \leq \varepsilon + C \sum_{i=0}^3 ||T_{mn}f_i - f_i||, \quad i = 0, 1, 2, 3, \]

where \( C = \max\{C_0, C_1, C_2, C_3\} \).

For a given \( \gamma > 0 \), choose \( \varepsilon > 0 \) such that \( \varepsilon < \gamma \). Now let

\[ U = \{(m, n) : ||T_{mn}f - f|| \geq \gamma\} \]

and

\[ U_i = \{(m, n) : ||T_{mn}f_i - f_i|| \geq \frac{\gamma - \varepsilon}{4C}\}, \quad i = 0, 1, 2, 3. \]

It follows that \( U \subset \bigcup_{i=0}^3 U_i \) and consequently for all \((j, k) \in \mathbb{N}^2\)

\[ \sum_{(m, n) \in U} a_{jkmn} \leq \sum_{i=0}^3 \sum_{(m, n) \in U_i} a_{jkmn}, \]
which implies that for any \( \sigma > 0 \) and \((m, n) \in U\),
\[
\left\{ (j, k) \in \mathbb{N}^2 : \sum_{(m,n) \in U} a_{jkmn} \geq \sigma \right\} \subseteq \bigcup_{i=0}^{3} \left\{ (j, k) \in \mathbb{N}^2 : \sum_{(m,n) \in U_i} a_{jkmn} \geq \frac{\sigma}{3} \right\}.
\]

Therefore from hypothesis, \( \{ (j, k) \in \mathbb{N}^2 : \sum_{(m,n) \in U} a_{jkmn} \geq \sigma \} \in \mathcal{I} \). This completes the proof of the theorem. \( \blacksquare \)

We now show that our theorem is stronger than the \( A \)-statistical version \([12]\) (and so the classical version). Let \( \mathcal{I} \) be a non-trivial strongly admissible ideal of \( \mathbb{N} \times \mathbb{N} \). Choose an infinite subset \( C = \{ (p_i, q_i) : i \in \mathbb{N} \} \), from \( \mathcal{I} \) such that \( p_i \neq q_i \) for all \( i, p_1 < p_2 < \cdots \) and \( q_1 < q_2 < \cdots \). Let \( \{ u_{mn} \}_{m,n \in \mathbb{N}} \) be given by
\[
u_{mn} = \begin{cases} 1 & \text{if } m, n \text{ are even} \\ 0 & \text{otherwise}.\end{cases}
\]

Let \( A = (a_{jkmn}) \) be given by
\[
a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i, m = 2p_i, n = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } (j, k) \neq (p_i, q_i), \text{ for any } i, m = 2j + 1, n = 2k + 1 \\ 0 & \text{otherwise}.\end{cases}
\]

Now for \( 0 < \varepsilon < 1 \), \( K_2(\varepsilon) = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : |u_{mn} - 0| \geq \varepsilon \} = \{ (m, n) : m, n \text{ are even} \} \}. \)

Observe that
\[
\sum_{(m,n) \in K_2(\varepsilon)} a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } (j, k) \neq (p_i, q_i), \text{ for any } i \in \mathbb{N}.\end{cases}
\]

Thus for any \( \delta > 0 \),
\[
\left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \sum_{(m,n) \in K_2(\varepsilon)} a_{jkmn} \geq \delta \right\} = C \in \mathcal{I},
\]

which shows that \( \{ u_{mn} \}_{m,n \in \mathbb{N}} \) is \( A^2 \)-statistically convergent to 0. Evidently this sequence is not \( A \)-statistically convergent to 0.

Consider the following Meyer-König and Zeler operators
\[
M_{mn}(f; x, y) = (1 - x)^{m+1}(1 - y)^{n+1} \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} f \left( \frac{k}{m+k+1}, \frac{l}{l+m+1} \right) \binom{m+k}{k} \binom{n+l}{l} x^k y^l,
\]
where \( f \in H_{w}(\mathcal{K}) \) and \( \mathcal{K} = [0, A] \times [0, B] \), \( A, B \in (0,1) \). Then \( M_{mn}(f_0; x, y) = f_0(x, y), M_{mn}(f_1; x, y) = \frac{x}{1-x}, M_{mn}(f_2; x, y) = \frac{y}{1-y} \) and
\[
M_{mn}(f_3; x, y) = \frac{m+2}{m+1} \left( \frac{x}{1-x} \right)^2 + \frac{1}{m+1} \frac{x}{1-x} \left( \frac{y}{1-y} \right)^2 + \frac{n+2}{n+1} \left( \frac{y}{1-y} \right)^2 + \frac{1}{n+1} \frac{y}{1-y}.
\]
Then \( \lim_{m,n} \| M_{mn}f - f \| = 0. \)
Now consider the following positive linear operator \( T_{mn} \) on \( H_w(K) \) defined by 
\[
T_{mn}(f;x,y) = (1 + u_{mn})M_{mn}(f;x,y).
\]
It is easy to observe that \( \|T_{mn}f_i - f_i\| = u_{mn} \) for \( i = 0, 1, 2 \) which imply that \( A^2_{st}\)-lim_{mn,n} \( \|T_{mn}f_i - f_i\| = 0, \ i = 0, 1, 2. \) Again,
\[
\|T_{mn}f_3 - f_3\| \leq D \left\{ \frac{2}{m+1} + \frac{2}{n+1} + u_{mn} \frac{m+3}{m+1} + u_{mn} \frac{n+3}{n+1} \right\},
\]
where \( D = \max \left\{ \left( \frac{A}{1-A} \right)^2, \left( \frac{B}{1-B} \right)^2, \left( \frac{A}{1-A} \right), \frac{A}{1-A} \right\} \). Therefore
\[
\{ (m,n) \in \mathbb{N}^2 : \|T_{mn}(f_3) - f_3\| \geq \varepsilon \}
\]
\[
\subseteq \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \geq \frac{\varepsilon}{4D} \right\}
\]
\[
\cup \left\{ (m,n) \in \mathbb{N}^2 : u_{mn} \frac{m+3}{m+1} + u_{mn} \frac{n+3}{n+1} \geq \frac{\varepsilon}{2D} \right\}
\]
\[
\subseteq \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \geq \frac{\varepsilon}{4D} \right\}
\]
\[
\cup \left\{ (m,n) \in \mathbb{N}^2 : u_{mn} \frac{m+3}{m+1} + u_{mn} \frac{n+3}{n+1} \geq 2 \sqrt{\varepsilon} \right\}
\]
\[
\subseteq \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \geq \frac{\varepsilon}{4D} \right\}
\]
\[
\cup \left\{ (m,n) \in \mathbb{N}^2 : u_{mn} \frac{m+3}{m+1} + u_{mn} \frac{n+3}{n+1} \geq \sqrt{\frac{\varepsilon}{2D}} \right\}
\]
\[
\subseteq \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \geq \frac{\varepsilon}{4D} \right\}
\]
\[
\cup \left\{ (m,n) \in \mathbb{N}^2 : u_{mn} \frac{m+3}{m+1} + u_{mn} \frac{n+3}{n+1} \geq \frac{\varepsilon}{2D} \right\}
\]
\[
\subseteq \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \geq \frac{\varepsilon}{4D} \right\}
\]
\[
\cup \left\{ (m,n) \in \mathbb{N}^2 : u_{mn} \frac{m+3}{m+1} + u_{mn} \frac{n+3}{n+1} \geq \frac{\varepsilon}{6 \sqrt{2D}} \right\}
\]
\[
\cup \left\{ (m,n) \in \mathbb{N}^2 : \frac{m}{m+1} + \frac{n}{n+1} \geq \frac{\varepsilon}{2 \sqrt{2D}} \right\},
\]
which implies that \( A^2_{st}\)-lim_{mn,n} \( \|T_{mn}f_3 - f_3\| = 0. \) Hence from previous theorem it follows that \( A^2_{st}\)-lim_{mn,n} \( \|T_{mn}f - f\| = 0 \) for any \( f \in H_w(K). \) But since \( \{u_{mn}\}_{m,n \in \mathbb{N}} \) is not \( A \)-statistically convergent so the sequence \( \{T_{mn}(f;x,y)\}_{m,n \in \mathbb{N}} \) considered above does not converge \( A \)-statistically to the function \( f \in H_w(K). \)

3. Rate of \( A^2_{st}\)-statistical convergence

In this section we present a way to compute the rate of \( A^2_{st}\)-statistical convergence in Theorem 2.1. We will need the following definitions.
DEFINITION 3.1. Let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix and let $\{\alpha_{mn}\}_{m,n \in \mathbb{N}}$ be a positive non-increasing double sequence. Then a real double sequence $\{x_{mn}\}_{m,n \in \mathbb{N}}$ is said to be $A_2^T$-statistically convergent to a number $L$ with the rate of $o(\alpha_{mn})$ if for every $\varepsilon > 0$ and $\delta > 0$, 

$$\left\{(j,k) \in \mathbb{N}^2 : \frac{1}{\alpha_{jk}} \sum_{(m,n) \in K_2(\varepsilon)} a_{jkmn} \geq \delta \right\} \in \mathcal{I},$$

where $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\}$. In this case, we write $A_2^T$-st-$o(\alpha_{mn})$-$\lim_{m,n} x_{mn} = L$.

DEFINITION 3.2. Let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix and let $\{\alpha_{mn}\}_{m,n \in \mathbb{N}}$ be a positive non-increasing double sequence. Then a real double sequence $\{x_{mn}\}_{m,n \in \mathbb{N}}$ is said to be $A_2^T$-statistically convergent to a number $L$ with the rate of $o(\alpha_{mn})$ if for every $\varepsilon > 0$ and $\delta > 0$, 

$$\left\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n) \in K_2(\varepsilon)} a_{jkmn} \geq \delta \right\} \in \mathcal{I},$$

where $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\alpha_{mn}\}$. In this case, we write $A_2^T$-st-$o(\alpha_{mn})$-$\lim_{m,n} x_{mn} = L$.

LEMMA 3.1. Let $\{x_{mn}\}_{m,n \in \mathbb{N}}$ and $\{y_{mn}\}_{m,n \in \mathbb{N}}$ be double sequences. Assume that $A = (a_{jkmn})$ is a non-negative RH-regular summability matrix and let $\{\alpha_{mn}\}_{m,n\in\mathbb{N}}$ and $\{\beta_{mn}\}_{m,n\in\mathbb{N}}$ be positive non-increasing double sequences. If 

$$A_2^T$$-st-$o(\alpha_{mn})$-$\lim_{m,n} x_{mn} = L_1 \text{ and } A_2^T$$-st-$o(\beta_{mn})$-$\lim_{m,n} x_{mn} = L_2$$

then we have

(i) $A_2^T$-st-$o(\gamma_{mn})$-$\lim_{m,n} (x_{mn} \pm y_{mn}) = L_1 \pm L_2$ where $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$,

(ii) $A_2^T$-st-$o(\alpha_{mn})$-$\lim_{m,n} \lambda x_{mn} = \lambda L_1$ for any real number $\lambda$.

Proof. The proof is straightforward and so is omitted. 

LEMMA 3.2. Let $\{x_{mn}\}_{m,n \in \mathbb{N}}$ and $\{y_{mn}\}_{m,n \in \mathbb{N}}$ be double sequences. Assume that $A = (a_{jkmn})$ is a non-negative RH-regular summability matrix and let $\{\alpha_{mn}\}_{m,n \in \mathbb{N}}$ and $\{\beta_{mn}\}_{m,n \in \mathbb{N}}$ be positive non-increasing double sequences. If 

$$A_2^T$$-st-$o(\alpha_{mn})$-$\lim_{m,n} x_{mn} = L_1 \text{ and } A_2^T$$-st-$o(\beta_{mn})$-$\lim_{m,n} x_{mn} = L_2$$

then we have

(i) $A_2^T$-st-$o(\gamma_{mn})$-$\lim_{m,n} (x_{mn} \pm y_{mn}) = L_1 \pm L_2$ where $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$,

(ii) $A_2^T$-st-$o(\alpha_{mn})$-$\lim_{m,n} \lambda x_{mn} = \lambda L_1$ for any real number $\lambda$. 


**Proof.** The proof is straightforward and so is omitted. ■

Now we prove the following theorem.

**Theorem 3.1.** Let \{T_{mn}\}_{m,n \in \mathbb{N}} be a sequence of positive linear operators from \(H_w(K)\) into \(C(K)\). Let \(A = (a_{jkmn})\) be a non-negative RH-regular summability matrix and \{\alpha_{mn}\}_{m,n \in \mathbb{N}} and \{\beta_{mn}\}_{m,n \in \mathbb{N}} be positive non-increasing double sequences. Assume that the following conditions hold

\[
\begin{align*}
(i) & \quad A_T^2 \text{-st-o}(\alpha_{mn}) \lim_{m,n} \|T_{mn}f_0 - f_0\| = 0, \\
(ii) & \quad A_T^2 \text{-st-o}(\beta_{mn}) \lim_{m,n} w(f; \delta_{mn}) = 0,
\end{align*}
\]

where \(\delta_{mn} = \sqrt{\|T_{mn}(\psi)\|}\) with \(\psi(u,v) = \left(\frac{u}{1-u} - \frac{x}{1-x}\right)^2 + \left(\frac{v}{1-v} - \frac{y}{1-y}\right)^2\). Then for any \(f \in H_w(K)\),

\[
A_T^2 \text{-st-o}(\gamma_{mn}) \lim_{m,n} \|T_{mn}f - f\| = 0,
\]

where \(\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}\) for each \((m,n) \in \mathbb{N}^2\).

**Proof.** Let \{T_{mn}\}_{m,n \in \mathbb{N}} be a sequence of positive linear operators from \(H_w(K)\) into \(C(K)\) and let \(A = (a_{jkmn})\) be a non-negative RH-regular summability matrix and \(N = \|f\|\). Then for any \(f \in H_w(K)\),

\[
|T_{mn}(f; u, v) - f(u, v)| \\
\leq T_{mn}(|f(x, y) - f(u, v); u, v| + |f(u, v)|)|T_{mn}(f_0; u, v) - f_0(u, v)|
\]

\[
\leq w(f; \delta)T_{mn} \left(1 + \sqrt{\left(\frac{u}{1-u} - \frac{x}{1-x}\right)^2 + \left(\frac{v}{1-v} - \frac{y}{1-y}\right)^2} \right); u, v)
\]

\[
= w(f; \delta)T_{mn}(f_0; u, v) + w(f; \delta)T_{mn} \left(\sqrt{\left(\frac{u}{1-u} - \frac{x}{1-x}\right)^2 + \left(\frac{v}{1-v} - \frac{y}{1-y}\right)^2} \right); u, v)
\]

\[
= w(f; \delta)T_{mn}(f_0; u, v) - w(f; \delta)f_0(u, v) + w(f; \delta) + \frac{w(f; \delta)}{\delta^2} T_{mn}(\psi; u, v)
\]

\[
+ N|T_{mn}(f_0; u, v) - f_0(u, v)|
\]

\[
= w(f; \delta)T_{mn}(f_0; u, v) - w(f; \delta)f_0(u, v) + w(f; \delta) + \frac{w(f; \delta)}{\delta^2} T_{mn}(\psi; u, v)
\]

\[
+ N|T_{mn}(f_0; u, v) - f_0(u, v)|
\]

\[
\leq w(f; \delta)|T_{mn}(f_0; u, v) - f_0(u, v)| + w(f; \delta) + \frac{w(f; \delta)}{\delta^2} T_{mn}(\psi; u, v)
\]

\[
+ N|T_{mn}(f_0; u, v) - f_0(u, v)|.
\]

Taking supremum over \((u, v) \in K\),

\[
\|T_{mn}f - f\| \leq w(f; \delta)||T_{mn}f_0 - f_0|| + w(f; \delta) + \frac{w(f; \delta)}{\delta^2} \|T_{mn}\psi\| + N\|T_{mn}f_0 - f_0\|.
\]

If we take \(\delta = \delta_{mn} = \sqrt{\|T_{mn}\psi\|}\) then

\[
\|T_{mn}f - f\| \leq w(f; \delta)||T_{mn}f_0 - f_0|| + 2w(f; \delta) + N\|T_{mn}f_0 - f_0\|
\]

\[
\leq M\{w(f; \delta)||T_{mn}f_0 - f_0|| + w(f; \delta) + \|T_{mn}f_0 - f_0\|\},
\]
where $M = \max\{2, N\}$. Let $\mu > 0$ be given. Now consider the following sets

\[ U = \{(m, n) : \|T_{mn}f - f\| \geq \mu\}, \]
\[ U_1 = \{(m, n) : w(f; \delta) \geq \frac{\mu}{3M}\}, \]
\[ U_2 = \{(m, n) : \|T_{mn}f_0 - f_0\| \geq \frac{\mu}{3M}\}, \]
\[ U_3 = \{(m, n) : w(f; \delta)\|T_{mn}f_0 - f_0\| \geq \frac{\mu}{3M}\}. \]

Then $U \subset U_1 \cup U_2 \cup U_3$. Now define

\[ U'_3 = \{(m, n) : w(f; \delta) \geq \sqrt{\frac{\mu}{3M}}\}, \]
\[ U''_3 = \{(m, n) : \|T_{mn}f_0 - f_0\| \geq \sqrt{\frac{\mu}{3M}}\}. \]

Then $U \subset U_1 \cup U_2 \cup U'_3 \cup U''_3$. Now since $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$ for each $(m, n) \in \mathbb{N}^2$ then for all $(j, k) \in \mathbb{N}^2$,

\[
\frac{1}{\gamma_{jk}} \sum_{(m, n) \in U} a_{jkmn} \leq \frac{1}{\beta_{jk}} \sum_{(m, n) \in U_1} a_{jkmn} + \frac{1}{\alpha_{jk}} \sum_{(m, n) \in U_2} a_{jkmn} + \frac{1}{\beta_{jk}} \sum_{(m, n) \in U'_3} a_{jkmn} + \frac{1}{\alpha_{jk}} \sum_{(m, n) \in U''_3} a_{jkmn}.
\]

Then for any $\sigma > 0$

\[
\left\{(j, k) \in \mathbb{N}^2 : \frac{1}{\gamma_{jk}} \sum_{(m, n) \in U} a_{jkmn} \geq \sigma\right\} \subseteq \left\{(j, k) \in \mathbb{N}^2 : \frac{1}{\beta_{jk}} \sum_{(m, n) \in U_1} a_{jkmn} \geq \frac{\sigma}{4}\right\} \cup \left\{(j, k) \in \mathbb{N}^2 : \frac{1}{\alpha_{jk}} \sum_{(m, n) \in U_2} a_{jkmn} \geq \frac{\sigma}{4}\right\} \cup \left\{(j, k) \in \mathbb{N}^2 : \frac{1}{\beta_{jk}} \sum_{(m, n) \in U'_3} a_{jkmn} \geq \frac{\sigma}{4}\right\} \cup \left\{(j, k) \in \mathbb{N}^2 : \frac{1}{\alpha_{jk}} \sum_{(m, n) \in U''_3} a_{jkmn} \geq \frac{\sigma}{4}\right\}.
\]

Now from hypothesis the sets on the right-hand side belong to $\mathcal{I}$ and consequently

\[
\left\{(j, k) \in \mathbb{N}^2 : \frac{1}{\gamma_{jk}} \sum_{(m, n) \in U} a_{jkmn} \geq \sigma\right\} \in \mathcal{I}
\]

for any $\sigma > 0$. This completes the proof. \end{proof}

The proof of the following theorem is analogous to the proof of Theorem 3.1 and so is omitted.

\textbf{Theorem 3.2.} Let $\{T_{mn}\}_{m,n \in \mathbb{N}}$ be a sequence of positive linear operators from $H_w(K)$ into $C(K)$. Let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix and $\{\alpha_{mn}\}_{m,n \in \mathbb{N}}$ and $\{\beta_{mn}\}_{m,n \in \mathbb{N}}$ be positive non-increasing double sequences.
Assume that the following conditions hold

(i) $A^2_{st} \alpha_{mn}(\alpha_{mn}) \lim_{m,n} \|T_{mn}f_0 - f_0\| = 0$,

(ii) $A^2_{st} \beta_{mn}(\beta_{mn}) \lim_{m,n} w(f; \delta_{mn}) = 0$,

where $\delta_{mn} = \sqrt{\|T_{mn}(\psi)\|}$ with $\psi(u, v) = (\frac{x}{1-x} - \frac{u}{1-u})^2 + (\frac{y}{1-y} - \frac{v}{1-v})^2$. Then for any $f \in H_w(K)$,

$A^2_{st} \gamma_{mn}(\gamma_{mn}) \lim_{m,n} \|T_{mn}(f) - f\| = 0$,

where $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$ for each $(m, n) \in \mathbb{N}^2$.

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