COMMON FIXED POINTS IN *b*-METRIC SPACES ENDOURED WITH A GRAPH

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Abstract. We discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings defined on a *b*-metric space endowed with a graph. Our results improve and supplement several recent results of metric fixed point theory.

1. Introduction

Fixed point theory plays a major role in mathematics and applied sciences such as variational and linear inequalities, mathematical models, optimization, mathematical economics and the like. Different generalizations of the usual notion of a metric space were proposed by several mathematicians. In 1989, Bakhtin [5] introduced *b*-metric spaces as a generalization of metric spaces and generalized the famous Banach contraction principle in metric spaces to *b*-metric spaces. Since then, a series of articles have been dedicated to the improvement of fixed point theory in *b*-metric spaces.

In [17], Jungck introduced the concept of weak compatibility. Several authors have obtained coincidence points and common fixed points for various classes of mappings on a metric space by using this concept.

In recent investigations, the study of fixed point theory endowed with a graph occupies a prominent place in many aspects. In 2005, Echenique [13] studied fixed point theory by using graphs. Espinola and Kirk [14] applied fixed point results in graph theory. Recently, Jachymski [16] proved a sufficient condition for a selfmap *f* of a metric space (*X*, *d*) to be a Picard operator and applied it to the Kelisky-Rivlin theorem on iterates of the Bernstein operators on the space *C*[0, 1].

Motivated by the idea given in some recent work on metric spaces with a graph (see [3,4,6–8]), we reformulate some important fixed point results in metric spaces to *b*-metric spaces endowed with a graph. As some consequences of our results, we obtain Banach contraction principle, Kannan fixed point theorem and Fisher fixed

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point theorem in metric spaces. Finally, some examples are provided to illustrate our results.

2. Some basic concepts

We begin with some basic notations, definitions, and necessary results in $b$-metric spaces.

**Definition 2.1.** [12] Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a $b$-metric on $X$ if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq s(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair $(X, d)$ is called a $b$-metric space.

It seems important to note that if $s = 1$, then the triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of $b$-metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above remarks.

**Example 2.2.** [18] Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then $(X, d)$ is a $b$-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

**Example 2.3.** [19] Let $(X, d)$ be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then $\rho$ is a $b$-metric with $s = 2^{p-1}$.

**Definition 2.4.** [10] Let $(X, d)$ be a $b$-metric space, $x \in X$ and $(x_n)$ be a sequence in $X$. Then

(i) $(x_n)$ converges to $x$ if and only if $\lim_{n \to \infty} d(x_n, x) = 0$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \rightharpoonup x$ ($n \to \infty$).
(ii) $(x_n)$ is a Cauchy sequence if and only if $\lim_{n, m \to \infty} d(x_n, x_m) = 0$.
(iii) $(X, d)$ is complete if and only if every Cauchy sequence in $X$ is convergent.

**Remark 2.5.** [10] In a $b$-metric space $(X, d)$, the following assertions hold:

(i) A convergent sequence has a unique limit.
(ii) Each convergent sequence is Cauchy.
(iii) In general, a $b$-metric is not continuous.
Theorem 2.6. [2] Let \((X, d)\) be a \(b\)-metric space and suppose that \((x_n)\) and \((y_n)\) converge to \(x, y \in X\), respectively. Then, we have
\[
\frac{1}{s^2} d(x, y) \leq \liminf_{n \to \infty} d(x_n, y_n) \leq \limsup_{n \to \infty} d(x_n, y_n) \leq s^2 d(x, y).
\]
In particular, if \(x = y\), then \(\lim_{n \to \infty} d(x_n, y_n) = 0\). Moreover, for each \(z \in X\), we have
\[
\frac{1}{s} d(x, z) \leq \liminf_{n \to \infty} d(x_n, z) \leq \limsup_{n \to \infty} d(x_n, z) \leq s d(x, z).
\]

Let \(T\) and \(S\) be self mappings of a set \(X\). Recall that, if \(y = Tx = Sx\) for some \(x \in X\), then \(x\) is called a coincidence point of \(T\) and \(S\) and \(y\) is called a point of coincidence of \(T\) and \(S\). The mappings \(T, S\) are weakly compatible [17], if for every \(x \in X\), the following holds:
\[
T(Sx) = S(Tx) \text{ whenever } Sx = Tx.
\]

Proposition 2.7. [1] Let \(S\) and \(T\) be weakly compatible selfmaps of a nonempty set \(X\). If \(S\) and \(T\) have a unique point of coincidence \(y = Sx = Tx\), then \(y\) is the unique common fixed point of \(S\) and \(T\).

Definition 2.8. Let \((X, d)\) be a \(b\)-metric space with the coefficient \(s \geq 1\). A mapping \(f : X \to X\) is called expansive if there exists a positive number \(k > s\) such that
\[
d(fx, fy) \geq k d(x, y)
\]
for all \(x, y \in X\).

We next review some basic notions in graph theory.

Let \((X, d)\) be a \(b\)-metric space. We assume that \(G\) is a reflexive digraph where the set \(V(G)\) of its vertices coincides with \(X\) and the set \(E(G)\) of its edges contains no parallel edges. So we can identify \(G\) with the pair \((V(G), E(G))\). \(G\) may be considered as a weighted graph by assigning to each edge the distance between its vertices. By \(G^{-1}\) we denote the graph obtained from \(G\) by reversing the direction of edges, i.e., \(E(G^{-1}) = \{(y, x) \in X \times X : (x, y) \in E(G)\}\). Let \(\tilde{G}\) denote the undirected graph obtained from \(G\) by ignoring the direction of edges. Actually, it will be more convenient for us to treat \(\tilde{G}\) as a digraph for which the set of its edges is symmetric. Under this convention,
\[
E(\tilde{G}) = E(G) \cup E(G^{-1}).
\]

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [9,11,15]. If \(x, y\) are vertices of the digraph \(G\), then a path in \(G\) from \(x\) to \(y\) of length \(n\) \((n \in \mathbb{N})\) is a sequence \((x_i)_{i=0}^n\) of \(n + 1\) vertices such that \(x_0 = x, x_n = y\) and \((x_{i-1}, x_i) \in E(G)\) for \(i = 1, 2, \ldots, n\). A graph \(G\) is connected if there is a path between any two vertices of \(G\). \(G\) is weakly connected if \(G\) is connected.
**Definition 2.9.** Let \((X, d)\) be a \(b\)-metric space with the coefficient \(s \geq 1\) and let \(G = (V(G), E(G))\) be a graph. A mapping \(f : X \to X\) is called a Banach \(G\)-contraction or simply \(G\)-contraction if there exists \(\alpha \in (0, \frac{1}{s})\) such that

\[
d(fx, fy) \leq \alpha d(x, y)
\]

for all \(x, y \in X\) with \((x, y) \in E(G)\).

Any Banach contraction is a \(G_0\)-contraction, where the graph \(G_0\) is defined by \(E(G_0) = X \times X\). But it is worth mentioning that a Banach \(G\)-contraction need not be a Banach contraction (see Remark 3.22).

**Definition 2.10.** Let \((X, d)\) be a \(b\)-metric space with the coefficient \(s \geq 1\) and let \(G = (V(G), E(G))\) be a graph. A mapping \(f : X \to X\) is called \(G\)-Kannan if there exists \(k \in (0, \frac{1}{2s})\) such that

\[
d(fx, fy) \leq k[d(fx, x) + d(fy, y)]
\]

for all \(x, y \in X\) with \((x, y) \in E(G)\).

Note that any Kannan operator is \(G_0\)-Kannan. However, a \(G\)-Kannan operator need not be a Kannan operator (see Remark 3.25).

**Definition 2.11.** Let \((X, d)\) be a \(b\)-metric space with the coefficient \(s \geq 1\) and let \(G = (V(G), E(G))\) be a graph. A mapping \(f : X \to X\) is called a Fisher \(G\)-contraction if there exists \(k \in (0, \frac{1}{s(1+s)})\) such that

\[
d(fx, fy) \leq k[d(fx, y) + d(fy, x)]
\]

for all \(x, y \in X\) with \((x, y) \in E(G)\).

If we take \(G = G_0\), then condition (2.1) holds for all \(x, y \in X\) and \(f\) is called a Fisher contraction. The following example shows that a Fisher \(G\)-contraction need not be a Fisher contraction.

**Example 2.12.** Let \(X = [0, \infty)\) and define \(d : X \times X \to \mathbb{R}^+\) by \(d(x, y) = |x - y|^2\) for all \(x, y \in X\). Then \((X, d)\) is a \(b\)-metric space with the coefficient \(s = 2\). Let \(G\) be a digraph such that \(V(G) = X\) and \(E(G) = \Delta \cup \{(4^t x, 4^t(x + 1)) : x \in X\) with \(x \geq 2, t = 0, 1, 2, \ldots\}\), where \(\Delta = \{(x, x) : x \in X\}\). Let \(f : X \to X\) be defined by \(fx = 4x\) for all \(x \in X\).

For \(x = 4^t z, y = 4^t(z + 1), z \geq 2\) with \(k = \frac{16}{125}\), we have

\[
d(fx, fy) = d\left(4^{t+1}z, 4^{t+1}(z + 1)\right) = 4^{2t+2} \\
\leq \frac{16}{125} 4^{2t}(18z^2 + 18z + 17) \\
= \frac{16}{125} \left[d\left(4^{t+1}z, 4^t(z + 1)\right) + d\left(4^{t+1}(z + 1), 4^t z\right)\right] \\
= k[d(fx, y) + d(fy, x)].
\]
Thus, \( f \) is a Fisher \( G \)-contraction. But \( f \) is not a Fisher contraction because, if \( x = 4, y = 0 \), then for any arbitrary positive number \( k < \frac{1}{s(1+s)} \), we have
\[
k[d(fx, y) + d(fy, x)] = k[d(f4, 0) + d(f0, 4)] = k[d(16, 0) + d(0, 4)] = 272k < 256 = d(fx, fy).
\]

**Remark 2.13.** If \( f \) is a \( G \)-contraction (resp., \( G \)-Kannan or Fisher \( G \)-contraction), then \( f \) is both a \( G^{-1} \)-contraction (resp., \( G^{-1} \)-Kannan or Fisher \( G^{-1} \)-contraction) and a \( \tilde{G} \)-contraction (resp., \( \tilde{G} \)-Kannan or Fisher \( \tilde{G} \)-contraction).

### 3. Main results

In this section, we assume that \((X, d)\) is a \( b \)-metric space with the coefficient \( s \geq 1 \), and \( G \) is a reflexive digraph such that \( V(G) = X \) and \( G \) has no parallel edges. Let \( f, g: X \to X \) be such that \( f(X) \subseteq g(X) \). If \( x_0 \in X \) is arbitrary, then there exists an element \( x_1 \in X \) such that \( fx_0 = gx_1 \), since \( f(X) \subseteq g(X) \). Proceeding in this way, we can construct a sequence \((gx_n)\) such that \( gx_n = fx_{n-1} \), \( n = 1, 2, 3, \ldots \).

By \( C_{gf} \) we denote the set of all elements \( x_0 \) of \( X \) such that \((gx_n, gx_m) \in E(\tilde{G})\) for \( m, n = 0, 1, 2, \ldots \). If \( g = I \), the identity map on \( X \), then obviously \( C_{gf} \) becomes \( C_f \) which is the collection of all elements \( x \) of \( X \) such that \((f^nx, f^mx) \in E(\tilde{G})\) for \( m, n = 0, 1, 2, \ldots \).

**Theorem 3.1.** Let \((X, d)\) be a \( b \)-metric space endowed with a graph \( G \) and the mappings \( f, g: X \to X \) satisfy
\[
d(fx, fy) \leq k d(gx, gy) \tag{3.1}
\]
for all \( x, y \in X \) with \((gx, gy) \in E(\tilde{G})\), where \( k \in (0, \frac{1}{s}) \) is a constant. Suppose \( f(X) \subseteq g(X) \) and \( g(X) \) is a complete subspace of \( X \) with the following property:

\( \star \) If \((gx_n)\) is a sequence in \( X \) such that \( gx_n \to x \) and \((gx_n, gx_{n+1}) \in E(\tilde{G})\) for all \( n \geq 1 \), then there exists a subsequence \((gx_{n_i})\) of \((gx_n)\) such that \((gx_{n_i}, x) \in E(\tilde{G})\)
for all \( i \geq 1 \).

Then \( f \) and \( g \) have a point of coincidence in \( X \) if \( C_{gf} \neq \emptyset \). Moreover, \( f \) and \( g \) have a unique point of coincidence in \( X \) if the graph \( G \) has the following property:

\( \star \star \) If \( x, y \) are points of coincidence of \( f \) and \( g \) in \( X \), then \((x, y) \in E(\tilde{G})\).

Furthermore, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point in \( X \).

**Proof.** Suppose that \( C_{gf} \neq \emptyset \). We choose an \( x_0 \in C_{gf} \) and keep it fixed. Since \( f(X) \subseteq g(X) \), there exists a sequence \((gx_n)\) such that \( gx_n = fx_{n-1} \), \( n = 1, 2, 3, \ldots \) and \((gx_n, gx_m) \in E(\tilde{G})\) for \( m, n = 0, 1, 2, \ldots \).

We now show that \((gx_n)\) is a Cauchy sequence in \( g(X) \).

For any natural number \( n \), we have by using condition (3.1) that
\[
d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \leq kd(gx_{n-1}, gx_n). \tag{3.2}
\]
By repeated use of condition (3.2), we get
\[ d(gx_n, gx_{n+1}) \leq k^n d(gx_0, gx_1) \] (3.3)
for all \( n \in \mathbb{N} \). For \( m, n \in \mathbb{N} \) with \( m > n \), using condition (3.3), we have
\[
d(gx_n, gx_m) \leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) \\
\quad + \cdots + s^{m-n-1} d(gx_{m-2}, gx_{m-1}) + s^{m-n-1} d(gx_{m-1}, gx_m) \\
\leq \left[ sk^n + s^2 k^{n+1} + \cdots + s^{m-n-1} k^m + s^{m-n-1} k^m \right] d(gx_0, gx_1) \\
\leq sk^n \left[ 1 + sk + (sk)^2 + \cdots + (sk)^{m-n} + (sk)^{m-n-1} \right] d(gx_0, gx_1) \\
\leq \frac{sk^n}{1-sk} d(gx_0, gx_1) \to 0 \text{ as } n \to \infty.
\]
Therefore, \((gx_n)\) is a Cauchy sequence in \( g(X) \). As \( g(X) \) is complete, there exists an \( u \in g(X) \) such that \( gx_n \to u = gv \) for some \( v \in X \).

As \( x_0 \in C_{gf} \), it follows that \((gx_n, gx_{n+1}) \in E(\tilde{G})\) for all \( n \geq 0 \), and so by property (\( \ast \)), there exists a subsequence \((gx_{n_i})\) of \((gx_n)\) such that \((gx_{n_i}, gv) \in E(\tilde{G})\) for all \( i \geq 1 \). Again, using condition (3.1), we have
\[
d(fv, gv) \leq sd(fv, fx_{n_i}) + sd(fx_{n_i}, gv) \\
\leq skd(gv, gx_{n_i}) + sd(gx_{n_i+1}, gv) \\
\to 0 \text{ as } i \to \infty.
\]
This gives that \( d(fv, gv) = 0 \) and hence, \( fv = gv = u \). Therefore, \( u \) is a point of coincidence of \( f \) and \( g \).

The next is to show that the point of coincidence is unique. Assume that there is another point of coincidence \( u^* \) in \( X \) such that \( fx = gx = u^* \) for some \( x \in X \). By property (\( \ast \ast \)), we have \((u, u^*) \in E(\tilde{G}) \). Then,
\[
d(u, u^*) = d(fv, fx) \leq kd(gv, gx) = kd(u, u^*),
\]
which gives that \( u = u^* \). Therefore, \( f \) and \( g \) have a unique point of coincidence in \( X \).

If \( f \) and \( g \) are weakly compatible, then by Proposition 2.7, \( f \) and \( g \) have a unique common fixed point in \( X \). ■

The following corollary gives fixed point of Banach \( G \)-contraction in \( b \)-metric spaces.

**Corollary 3.2.** Let \((X, d)\) be a complete \( b \)-metric space endowed with a graph \( G \) and the mapping \( f : X \to X \) be such that
\[
d(fx, fy) \leq kd(x, y) \] (3.4)
for all \( x, y \in X \) with \((x, y) \in E(\tilde{G})\), where \( k \in (0, \frac{1}{s}) \) is a constant. Suppose the triple \((X, d, G)\) have the following property:
\( G = \text{be an onto expansive mapping. Then}
\]
\[
(\star') \text{If } (x_n) \text{ is a sequence in } X \text{ such that } x_n \to x \text{ and } (x_n, x_{n+1}) \in E(\tilde{G}) \text{ for all } n \geq 1, \text{ then there exists a subsequence } (x_{n_i}) \text{ of } (x_n) \text{ such that } (x_{n_i}, x) \in E(\tilde{G}) \text{ for all } i \geq 1. \]
Then \( f \) has a fixed point in \( X \) if \( C_f \neq \emptyset \). Moreover, \( f \) has a unique fixed point in \( X \) if the graph \( G \) has the following property:

\( (**') \text{ If } x, y \text{ are fixed points of } f \text{ in } X, \text{ then } (x, y) \in E(\tilde{G}). \)

**Proof.** The proof can be obtained from Theorem 3.1 by considering \( g = I \), the identity map on \( X \). \]

**Corollary 3.3.** Let \( (X, d) \) be a \( b \)-metric space and mappings \( f, g : X \to X \) satisfy (3.1) for all \( x, y \in X \), where \( k \in (0, \frac{1}{2}) \) is a constant. If \( f(X) \subseteq g(X) \) and \( g(X) \) is a complete subspace of \( X \), then \( f \) and \( g \) have a unique point of coincidence in \( X \). Moreover, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point in \( X \).

**Proof.** The proof follows from Theorem 3.1 by taking \( G = G_0 \), where \( G_0 \) is the complete graph \( (X, X \times X) \). \]

The following corollary is the \( b \)-metric version of Banach contraction principle.

**Corollary 3.4.** Let \( (X, d) \) be a complete \( b \)-metric space and a mapping \( f : X \to X \) be such that (3.4) holds for all \( x, y \in X \), where \( k \in (0, \frac{1}{2}) \) is a constant. Then \( f \) has a unique fixed point \( u \) in \( X \) and \( f^n x \to u \) for all \( x \in X \).

**Proof.** It follows from Theorem 3.1 by putting \( G = G_0 \) and \( g = I \).

From Theorem 3.1, we obtain the following corollary concerning the fixed point of expansive mapping in \( b \)-metric spaces.

**Corollary 3.5.** Let \( (X, d) \) be a complete \( b \)-metric space and let \( g : X \to X \) be an onto expansive mapping. Then \( g \) has a unique fixed point in \( X \).

**Proof.** The conclusion of the corollary follows from Theorem 3.1 by taking \( G = G_0 \) and \( f = I \).

**Corollary 3.6.** Let \( (X, d) \) be a complete \( b \)-metric space endowed with a partial ordering \( \preceq \) and the mapping \( f : X \to X \) be such that (3.4) holds for all \( x, y \in X \) with \( x \preceq y \) or \( y \preceq x \), where \( k \in (0, \frac{1}{2}) \) is a constant. Suppose the triple \( (X, d, \preceq) \) has the following property:

\( (\dagger) \text{ If } (x_n) \text{ is a sequence in } X \text{ such that } x_n \to x \text{ and } x_n, x_{n+1} \text{ are comparable for all } n \geq 1, \text{ then there exists a subsequence } (x_{n_i}) \text{ of } (x_n) \text{ such that } x_{n_i}, x \) are comparable for all \( i \geq 1. \)

If there exists \( x_0 \in X \) such that \( f^m x_0, f^n x_0 \) are comparable for \( m, n = 0, 1, 2, \ldots \), then \( f \) has a fixed point in \( X \). Moreover, \( f \) has a unique fixed point in \( X \) if the following property holds:

\( (\dagger\dagger) \text{ If } x, y \text{ are fixed points of } f \text{ in } X, \text{ then } x, y \text{ are comparable.} \)
Proof. The proof can be obtained from Theorem 3.1 by taking \( g = I \) and \( G = G_2 \), where the graph \( G_2 \) is defined by \( E(G_2) = \{ (x, y) \in X \times X : x \leq y \text{ or } y \leq x \} \). □

**Theorem 3.7.** Let \((X, d)\) be a \( b\)-metric space endowed with a graph \( G \) and the mappings \( f, g : X \to X \) satisfy
\[
d(f(x), f(y)) \leq kd(f(x), gx) + ld(f(y), gy) \tag{3.5}
\]
for all \( x, y \in X \) with \((gx, gy) \in E(G)\), where \( k, l \) are positive numbers with \( k+l < \frac{1}{s} \). Suppose \( f(X) \subseteq g(X) \) and \( g(X) \) is a complete subspace of \( X \) with the property (\( * \)). Then \( f \) and \( g \) have a point of coincidence in \( X \) if \( C_{gf} \neq \emptyset \). Moreover, \( f \) and \( g \) have a unique point of coincidence in \( X \) if the graph \( G \) has the property (\( ** \)). Furthermore, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point in \( X \).

Proof. As in the proof of Theorem 3.1, we can construct a sequence \((gx_n)\) such that \( gx_n = fx_{n-1}, n = 1, 2, 3, \ldots \) and \((gx_n, gx_m) \in E(G)\) for \( m, n = 0, 1, 2, \ldots \). We shall show that \((gx_n)\) is a Cauchy sequence in \( g(X) \).

For any natural number \( n \), we have by using condition (3.5) that
\[
d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1}) \leq kd(fx_n, gx_n) + ld(fx_{n-1}, gx_{n-1})
= kd(gx_{n+1}, gx_n) + ld(gx_n, gx_{n-1}),
\]
which gives that
\[
d(gx_{n+1}, gx_n) \leq \alpha d(gx_n, gx_{n-1}) \tag{3.6}
\]
where \( \alpha = \frac{l}{1-k} \in (0, \frac{1}{s}) \). By repeated use of condition (3.6), we obtain
\[
d(gx_{n+1}, gx_n) \leq \alpha^n d(gx_1, gx_0), \tag{3.7}
\]
for all \( n \in \mathbb{N} \). For \( m, n \in \mathbb{N} \), using conditions (3.5) and (3.7), we have
\[
d(gx_m, gx_n) = d(fx_{m-1}, fx_{n-1})
\leq kd(fx_{m-1}, gx_{m-1}) + ld(fx_{n-1}, gx_{n-1})
= kd(gx_m, gx_{m-1}) + ld(gx_n, gx_{n-1})
\leq k\alpha^{m-1} d(gx_1, gx_0) + l\alpha^{n-1} d(gx_1, gx_0)
\to 0 \text{ as } m, n \to \infty.
\]
Therefore, \((gx_n)\) is a Cauchy sequence in \( g(X) \). As \( g(X) \) is complete, there exists an \( u \in g(X) \) such that \( gx_n \to u = gv \) for some \( v \in X \). As \( x_0 \in C_{gf} \), it follows that \((gx_n, gx_{n+1}) \in E(G)\) for all \( n \geq 0 \), and so by property (\( * \)), there exists a subsequence \((gx_{n_i})\) of \((gx_n)\) such that \((gx_{n_i}, gv) \in E(G)\) for all \( i \geq 1 \).

Now using conditions (3.5) and (3.7), we find
\[
d(fv, gv) \leq sd(fv, fx_{n_i}) + sd(fx_{n_i}, gv)
\leq [skd(fv, gv) + sd(fx_{n_i}, gx_{n_i})] + sd(gx_{n_i+1}, gv)
= skd(fv, gv) + sd(gx_{n_i+1}, gx_{n_i}) + sd(gx_{n_i+1}, gv),
\]
which yields

\[
\begin{align*}
d(fv, gv) & \leq \frac{sl}{1 - sk} d(gx_{n+1}, gx_n) + \frac{s}{1 - sk} d(gx_{n+1}, gv) \\
& \leq \frac{sla^n}{1 - sk} d(gx_1, gx_0) + \frac{s}{1 - sk} d(gx_{n+1}, gv) \\
& \to 0 \text{ as } i \to \infty.
\end{align*}
\]

This gives that, \(fv = gv = u\). Therefore, \(u\) is a point of coincidence of \(f\) and \(g\).

Finally, to prove the uniqueness of the point of coincidence, suppose that there is another point of coincidence \(u^*\) in \(X\) such that \(fx = gx = u^*\) for some \(x \in X\).

By property (**)\(\ast\), we have \((u, u^*) \in E(\tilde{G})\). Then,

\[
d(u, u^*) = d(fv, fx) \leq kd(fv, gv) + ld(fx, gx) = 0,
\]

which gives that \(u = u^*\). Therefore, \(f\) and \(g\) have a unique point of coincidence in \(X\).

If \(f\) and \(g\) are weakly compatible, then by Proposition 2.7, \(f\) and \(g\) have a unique common fixed point in \(X\). \(\blacksquare\)

**Corollary 3.8.** Let \((X, d)\) be a complete \(b\)-metric space endowed with a graph \(G\) and the mapping \(f : X \to X\) be such that

\[
d(fx, fy) \leq kd(fx, x) + ld(fy, y) \tag{3.8}
\]

for all \(x, y \in X\) with \((x, y) \in E(\tilde{G})\), where \(k, l\) are positive numbers with \(k + l < \frac{1}{s}\). Suppose the triple \((X, d, G)\) has the property (**)\(\ast\). Then \(f\) has a fixed point in \(X\) if \(C_f \neq \emptyset\). Moreover, \(f\) has a unique fixed point in \(X\) if the graph \(G\) has the property (**)\(\ast\).

**Proof.** The proof can be obtained from Theorem 3.7 by putting \(g = I\). \(\blacksquare\)

**Remark 3.9.** In particular (i.e., taking \(k = l\)), the above corollary gives fixed points of \(G\)-Kannan operators in \(b\)-metric spaces.

**Corollary 3.10.** Let \((X, d)\) be a \(b\)-metric space and the mappings \(f, g : X \to X\) satisfy (3.5) for all \(x, y \in X\), where \(k, l\) are positive numbers with \(k + l < \frac{1}{s}\). If \(f(X) \subseteq g(X)\) and \(g(X)\) is a complete subspace of \(X\), then \(f\) and \(g\) have a unique point of coincidence in \(X\). Moreover, if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point in \(X\).

**Proof.** It can be obtained from Theorem 3.7 by taking \(G = G_0\). \(\blacksquare\)

**Corollary 3.11.** Let \((X, d)\) be a complete \(b\)-metric space and \(f : X \to X\) be a mapping such that (3.8) holds for all \(x, y \in X\), where \(k, l\) are positive numbers with \(k + l < \frac{1}{s}\). Then \(f\) has a unique fixed point \(u\) in \(X\) and \(f^n x \to u\) for all \(x \in X\).

**Proof.** The proof follows from Theorem 3.7 by putting \(G = G_0\) and \(g = I\). \(\blacksquare\)
Remark 3.12. In particular (i.e., taking $k = l$), the above corollary is the $b$-metric version of Kannan fixed point theorem.

Corollary 3.13. Let $(X, d)$ be a complete $b$-metric space endowed with a partial ordering $\preceq$ and the mapping $f : X \to X$ be such that (3.8) holds for all $x, y \in X$ with $x \preceq y$ or $y \preceq x$, where $k, l$ are positive numbers with $k + l < \frac{1}{s}$. Suppose the triple $(X, d, \preceq)$ has the property $(\dagger)$. If there exists $x_0 \in X$ such that $f^m x_0, f^n x_0$ are comparable for $m, n = 0, 1, 2, \ldots$, then $f$ has a fixed point in $X$. Moreover, $f$ has a unique common fixed point in $X$ if the property $(\dagger\dagger)$ holds.

Proof. The proof can be obtained from Theorem 3.7 by taking $g = I$ and $G = G_2$. ■

Theorem 3.14. Let $(X, d)$ be a $b$-metric space endowed with a graph $G$ and the mappings $f, g : X \to X$ satisfy
\[
d(fx, fy) \leq k d(fx, gy) + ld(fy, gx)
\] (3.9)
for all $x, y \in X$ with $(gx, gy) \in E(G)$, where $k, l$ are positive numbers with $sk < \frac{1}{1+s}$ or $sl < \frac{1}{1+s}$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of $X$ with the property $(\ast)$. Then $f$ and $g$ have a point of coincidence in $X$ if $C_{gf} \neq \emptyset$. Moreover, $f$ and $g$ have a unique point of coincidence in $X$ if the graph $G$ has the property $(\ast\ast)$ and $k + l < 1$. Furthermore, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point in $X$.

Proof. As in the proof of Theorem 3.1, we can construct a sequence $(gx_n)$ such that $gx_n = fx_{n-1}, n = 1, 2, \ldots$ and $(gx_n, gx_m) \in E(G)$ for $m, n = 0, 1, 2, \ldots$. We shall show that $(gx_n)$ is Cauchy in $g(X)$. We assume that $sk < \frac{1}{1+s}$.

For any natural number $n$, we have by using condition (3.9) that
\[
d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1})
\]
\[
\leq k d(fx_n, gx_{n-1}) + l d(fx_{n-1}, gx_n)
\]
\[
= k d(gx_{n+1}, gx_{n-1})
\]
\[
\leq sk d(gx_{n+1}, gx_n) + sk d(gx_n, gx_{n-1}),
\]
which gives that,
\[
d(gx_{n+1}, gx_n) \leq \alpha d(gx_n, gx_{n-1})
\] (3.10)
where $\alpha = \frac{sk}{1-sk} \in (0, \frac{1}{s})$, since $sk < \frac{1}{1+s}$. By repeated use of condition (3.10), we obtain
\[
d(gx_{n+1}, gx_n) \leq \alpha^n d(gx_1, gx_0), \quad \text{for all } n \in \mathbb{N}.
\]

By an argument similar to that used in Theorem 3.1, it follows that $(gx_n)$ is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, there exists an $u \in g(X)$ such that $gx_n \to u = gv$ for some $v \in X$. Since $x_0 \in C_{gf}$, it follows that $(gx_n, gx_{n+1}) \in E(G)$ for all $n \geq 0$, and so by property $(\ast)$, there exists a subsequence $(gx_{n_i})$ of $(gx_n)$ such that $(gx_{n_i}, gv) \in E(G)$ for all $i \geq 1$. 

Common fixed points in $b$-metric spaces
Now using condition (3.9), we find
\[
d(fv, gv) \leq s[d(fv, fx_n) + d(x_n, gv)] + s[d(gx_{n+1}, gv)]
\]
\[
\leq s^2 k d(fv, gv) + s^2 k d(gv, x_n) + s(l + 1) d(gx_{n+1}, gv)
\]
which gives that
\[
d(fv, gv) \leq s^2 k d(gv, x_n) + s(l + 1) d(gx_{n+1}, gv) \rightarrow 0 \text{ as } i \rightarrow \infty.
\]
This proves that \(fv = gv = u\). Therefore, \(u\) is a point of coincidence of \(f\) and \(g\).

Finally, to prove the uniqueness of point of coincidence, suppose that there is another point of coincidence \(u^*\) in \(X\) such that \(fx = gx = u^*\) for some \(x \in X\). By property (**) we have \((u, u^*) \in E(\tilde{G})\). Then,
\[
d(u, u^*) = d(fv, fx) \leq kd(fv, gx) + ld(fx, gv)
\]
\[
= (k + l)d(u^*, u).
\]
If \(k + l < 1\), then it must be the case that \(d(u, u^*) = 0\) i.e., \(u = u^*\). Therefore, \(f\) and \(g\) have a unique point of coincidence in \(X\).

If \(f\) and \(g\) are weakly compatible, then by Proposition 2.7, \(f\) and \(g\) have a unique common fixed point in \(X\). □

**Corollary 3.15.** Let \((X, d)\) be a complete \(b\)-metric space endowed with a graph \(G\) and the mapping \(f : X \rightarrow X\) be such that
\[
d(fx, fy) \leq kd(fx, y) + ld(fy, x) \quad (3.11)
\]
for all \(x, y \in X\) with \((x, y) \in E(\tilde{G})\), where \(k, l\) are positive numbers with \(sk < \frac{1}{1+s}\) or \(sl < \frac{1}{1+s}\). Suppose the triple \((X, d, G)\) has the property (**) Then \(f\) has a fixed point in \(X\) if \(C_f \neq \emptyset\). Moreover, \(f\) has a unique fixed point in \(X\) if the graph \(G\) has the property (***) and \(k + l < 1\).

**Proof.** The proof can be obtained from Theorem 3.14 by putting \(g = I\). □

**Remark 3.16.** In particular (i.e., taking \(k = l\)), the above corollary gives fixed points of Fisher \(G\)-contraction in \(b\)-metric spaces.

**Corollary 3.17.** Let \((X, d)\) be a \(b\)-metric space and the mappings \(f, g : X \rightarrow X\) satisfy (3.9) for all \(x, y \in X\), where \(k, l\) are positive numbers with \(sk < \frac{1}{1+s}\) or \(sl < \frac{1}{1+s}\). If \(f(X) \subseteq g(X)\) and \(g(X)\) is a complete subspace of \(X\), then \(f\) and \(g\) have a point of coincidence in \(X\). Moreover, if \(k + l < 1\), then \(f\) and \(g\) have a unique point of coincidence in \(X\). Furthermore, if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point in \(X\).

**Proof.** The proof can be obtained from Theorem 3.14 by taking \(G = G_0\). □
The following corollary is [18, Theorem 5]. In particular (when \( k = l \)), it is the \( b \)-metric version of Fisher’s theorem.

**Corollary 3.18.** Let \( (X, d) \) be a complete \( b \)-metric space and let \( f : X \to X \) be a mapping such that (3.11) holds for all \( x, y \in X \), where \( k, l \) are positive numbers with \( sk < \frac{1}{1+s^2} \) or \( sl < \frac{1}{1+s^2} \). Then \( f \) has a fixed point in \( X \). Moreover, if \( k + l < 1 \), then \( f \) has a unique fixed point \( u \in X \) and \( f^n x \to u \) for all \( x \in X \).

**Proof.** The proof can be obtained from Theorem 3.14 by considering \( G = G_0 \) and \( g = I \).

**Corollary 3.19.** Let \( (X, d) \) be a complete \( b \)-metric space endowed with a partial ordering \( \preceq \) and the mapping \( f : X \to X \) be such that (3.11) holds for all \( x, y \in X \) with \( x \preceq y \) or \( y \preceq x \), where \( k, l \) are positive numbers with \( sk < \frac{1}{1+s^2} \) or \( sl < \frac{1}{1+s^2} \). Suppose the triple \( (X, d, \preceq) \) has the property (†). If there exists \( x_0 \in X \) such that \( f^n x_0, f^n x_0 \) are comparable for \( m, n = 0, 1, 2, \ldots \), then \( f \) has a fixed point in \( X \). Moreover, \( f \) has a unique fixed point in \( X \) if the property (††) holds and \( k + l < 1 \).

**Proof.** The proof can be obtained from Theorem 3.14 by taking \( g = I \) and \( G = G_2 \).

We furnish some examples in favour of our results.

**Example 3.20.** Let \( X = \mathbb{R} \) and define \( d : X \times X \to \mathbb{R}^+ \) by \( d(x, y) = |x - y|^2 \) for all \( x, y \in X \). Then \( (X, d) \) is a complete \( b \)-metric space with the coefficient \( s = 2 \). Let \( G \) be a digraph such that \( V(G) = X \) and \( E(G) = \Delta \cup \{(0, \frac{1}{2^2}) : n = 0, 1, 2, \ldots \} \).

Let \( f, g : X \to X \) be defined by

\[
f(x) = \begin{cases} \frac{2}{5}, & \text{if } x \neq \frac{2}{5}, \\ 1, & \text{if } x = \frac{2}{5}, \end{cases}
\]

and \( gx = 3x \) for all \( x \in X \). Obviously, \( f(X) \subseteq g(X) = X \).

If \( x = 0, y = \frac{1}{3.5^2} \), then \( gx = 0, gy = \frac{1}{3^2} \) and so \((gx, gy) \in E(G)\).

For \( x = 0, y = \frac{1}{3.5^2} \), we have

\[
d(f(x), f(y)) = d \left( 0, \frac{1}{3.5^{n+1}} \right) = \frac{1}{9.5^{2n+2}} \\
< \frac{1}{9} \cdot \frac{1}{5^{2n}} = kd(gx, gy), \text{ where } k = \frac{1}{9}.
\]

Therefore, \( d(f(x), f(y)) \leq kd(gx, gy) \) holds for all \( x, y \in X \) with \((gx, gy) \in E(G)\), where \( k = \frac{1}{9} \in (0, \frac{1}{s}) \) is a constant. We can verify that \( 0 \in C_{gf} \). In fact, \( gx_n = f x_{n-1}, n = 1, 2, 3, \ldots \) gives that \( gx_1 = f0 = 0 \Rightarrow x_1 = 0 \) and so \( gx_2 = f x_1 = 0 \Rightarrow x_2 = 0 \). Proceeding in this way, we get \( gx_n = 0 \) for \( n = 0, 1, 2, \ldots \) and hence \((gx_n, gx_m) = (0, 0) \in E(G)\) for \( m, n = 0, 1, 2, \ldots \).
Also, any sequence \((gx_n)\) with the property \((gx_n, gx_{n+1}) \in E(\tilde{G})\) must be either a constant sequence or a sequence of the following form

\[ gx_n = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \frac{1}{2^n}, & \text{if } n \text{ is even}, \end{cases} \]

where the words ‘odd’ and ‘even’ are interchangeable. Consequently it follows that property (\(\ast\)) holds. Furthermore, \(f\) and \(g\) are weakly compatible. Thus, we have all the conditions of Theorem 3.1 and 0 is the unique common fixed point of \(f\) and \(g\) in \(X\).

We now show that the weak compatibility condition in Theorem 3.1 cannot be relaxed.

**Remark 3.21.** In Example 3.20, if we take \(gx = 3x - 14\) for all \(x \in X\) instead of \(gx = 3x\), then \(5 \in C_{gf}\) and \(f(5) = g(5) = 1\) but \(g(f(5)) \neq f(g(5))\) i.e., \(f\) and \(g\) are not weakly compatible. However, all other conditions of Theorem 3.1 are satisfied. We observe that 1 is the unique point of coincidence of \(f\) and \(g\) without being a common fixed point.

**Remark 3.22.** In Example 3.20, \(f\) is a Banach \(G\)-contraction with constant \(k = \frac{1}{25}\) but it is not a Banach contraction. In fact, for \(x = \frac{2}{5}, y = 1\), we have

\[ d(fx, fy) = d \left( 1, \frac{1}{5} \right) = \frac{16}{25} = \frac{16}{9} \cdot \frac{9}{25} > kd(x, y), \]

for any \(k \in (0, \frac{1}{s})\). This implies that \(f\) is not a Banach contraction.

The next example shows that the property (\(\ast\)) in Theorem 3.1 is necessary.

**Example 3.23.** Let \(X = [0, 1]\) and define \(d : X \times X \rightarrow \mathbb{R}^+\) by \(d(x, y) = |x - y|^2\) for all \(x, y \in X\). Then \((X, d)\) is a complete \(b\)-metric space with the coefficient \(s = 2\). Let \(G\) be a digraph such that \(V(G) = X\) and \(E(G) = \{(0, 0)\} \cup \{(x, y) : (x, y) \in (0, 1) \times (0, 1), x \geq y\}\). Let \(f, g : X \rightarrow X\) be defined by

\[ fx = \begin{cases} \frac{2}{5}, & \text{if } x \in (0, 1), \\ 1, & \text{if } x = 0, \end{cases} \]

and \(gx = x\) for all \(x \in X\). Obviously, \(f(X) \subseteq g(X) = X\).

For \(x, y \in X\) with \((gx, gy) \in E(\tilde{G})\), we have \(d(fx, fy) = \frac{1}{2^\alpha}d(gx, gy)\), where \(\alpha = \frac{1}{5} \in (0, \frac{1}{s})\) is a constant. We see that \(f\) and \(g\) have no point of coincidence in \(X\). We now verify that the property (\(\ast\)) does not hold. In fact, \((gx_n)\) is a sequence in \(X\) with \(gx_n \rightarrow 0\) and \((gx_n, gx_{n+1}) \in E(\tilde{G})\) for all \(n \in \mathbb{N}\) where \(x_n = \frac{1}{n}\). But there exists no subsequence \((gx_{n_i})\) of \((gx_n)\) such that \((gx_{n_i}, 0) \in E(\tilde{G})\).

The following example supports our Theorem 3.7.

**Example 3.24.** Let \(X = [0, \infty)\) and define \(d : X \times X \rightarrow \mathbb{R}^+\) by \(d(x, y) = |x - y|^2\) for all \(x, y \in X\). Then \((X, d)\) is a complete \(b\)-metric space with the coefficient \(s = 2\). Let \(G\) be a digraph such that \(V(G) = X\) and \(E(G) = \Delta \cup \{(3^t x, 3^t(x + 1)) : x \in X \text{ with } x \geq 2, t = 0, 1, 2, \ldots \}\).
Let \( f, g : X \to X \) be defined by \( fx = 3x \) and \( gx = 9x \) for all \( x \in X \). Clearly, \( f(X) = g(X) = X \).

If \( x = 3^{t-2}z, y = 3^{t-2}(z + 1) \), then \( gx = 3^t z, gy = 3^t(z + 1) \) and so \( (gx, gy) \in E(\hat{G}) \) for all \( z \geq 2 \).

For \( x = 3^{t-2}z, y = 3^{t-2}(z + 1), z \geq 2 \) with \( k = l = \frac{1}{52} \), we have
\[
\begin{align*}
    d(fx, fy) &= d(3^{t-1}z, 3^{t-1}(z + 1)) = 3^{2t-2} \\
    &\leq \frac{1}{52} 3^{2t-2}(8z^2 + 8z + 4) \\
    &= \frac{1}{52} \left[ d(3^{t-1}z, 3^t z) + d(3^{t-1}(z + 1), 3^t(z + 1)) \right] \\
    &= k[d(fx, gx) + ld(fy, gy)].
\end{align*}
\]
Thus, condition (3.5) is satisfied. It is easy to verify that \( 0 \in C_{gf} \).

Also, any sequence \( (gx_n) \) with \( gx_n \to x \) and \((gx_n, gx_{n+1}) \in E(\hat{G}) \) must be a constant sequence and hence property (*) holds. Furthermore, \( f \) and \( g \) are weakly compatible. Thus, we have all the conditions of Theorem 3.7 and 0 is the unique common fixed point of \( f \) and \( g \) in \( X \).

**Remark 3.25.** In Example 3.23, \( f \) is a G-Kannan operator with constant \( k = \frac{9}{52} \). But \( f \) is not a Kannan operator because, if \( x = 3, y = 0 \), then for any arbitrary positive number \( k < \frac{1}{52} \), we have
\[
k[d(fx, x) + d(fy, y)] = k[d(f3, 3) + d(f0, 0)] = 36k < 81 = d(fx, fy).
\]

**Example 3.26.** Let \( X = \mathbb{R} \) and define \( d : X \times X \to \mathbb{R}^+ \) by \( d(x, y) = |x - y|^p \) for all \( x, y \in X \), where \( p > 1 \) is a real number. Then \( (X, d) \) is a complete \( b \)-metric space with the coefficient \( s = 2^{p-1} \). Let \( f, g : X \to X \) be defined by
\[
    fx = \begin{cases} 
        1, & \text{if } x \neq 4, \\
        2, & \text{if } x = 4,
    \end{cases}
\]
and \( gx = 2x - 1 \) for all \( x \in X \). Obviously, \( f(X) \subseteq g(X) = X \).

Let \( G \) be a digraph such that \( V(G) = X \) and \( E(G) = \Delta \cup \{(1, 2), (2, 4)\} \). If \( x = 1, y = \frac{3}{2} \), then \( gx = 1, gy = 2 \) and so \( (gx, gy) \in E(\hat{G}) \). Again, if \( x = \frac{3}{2}, y = \frac{5}{2} \), then \( gx = 2, gy = 4 \) and so \( (gx, gy) \in E(\hat{G}) \).

It is easy to verify that condition (3.9) of Theorem 3.14 holds for all \( x, y \in X \) with \((gx, gy) \in E(\hat{G}) \). Furthermore, \( 1 \in C_{gf} \), i.e., \( C_{gf} \neq \emptyset \), \( f \) and \( g \) are weakly compatible, and the triple \((X, d, G)\) have property (*). Thus, all the conditions of Theorem 3.14 are satisfied and 1 is the unique common fixed point of \( f \) and \( g \) in \( X \).

**Remark 3.27.** In Example 3.26, \( f \) is not a Fisher G-contraction for \( p = 5 \). In fact, for \( x = 2, y = 4 \) and \( p = 5 \), we have
\[
k[d(fx, y) + d(fy, x)] = k[2 + 2] = 4k < \frac{243}{272} < 1 = d(fx, fy),
\]
for arbitrary positive number $k$ with $k < \frac{1}{\sqrt{1+s^2}}$. This implies that $f$ is not a Fisher $G$-contraction for $p = 5$. However, we can verify that $f$ is a Fisher $G$-contraction for $p = 4$.

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