PARITY RESULTS FOR 13-CORE PARTITIONS

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Abstract. We find some interesting congruences modulo 2 for 13-core partitions.

1. Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of a natural number $n$ is a finite sequence of non-increasing positive integer parts $\lambda_i$ such that $n = \sum_{i=1}^{k} \lambda_i$. The Ferrers-Young diagram of the partition $\lambda$ of $n$ is formed by arranging $n$ nodes in $k$ rows so that the $i^{th}$ row has $\lambda_i$ nodes. The nodes are labeled by row and column coordinates as one would label the entries of a matrix. Let $\lambda'_j$ denote the number of nodes in column $j$. The hook number $H(i,j)$ of the $(i,j)$ node is defined as the number of nodes directly below and to the right of the node including the node itself. That is, $H(i,j) = \lambda_i + \lambda'_j - j - i + 1$. A partition $\lambda$ is said to be a $t$-core if and only if it has no hook numbers that are multiples of $t$. If $a_t(n)$ denotes the number of partitions of $n$ that are $t$-cores, then the generating function for $a_t(n)$ satisfies the identity [9, Equation 2.1]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t; q^t)_\infty}{(q; q)_\infty},$$

where as customary, for any complex numbers $a$ and $q$ with $|q| < 1$,

$$(a; q)_\infty := \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

A number of results on $a_t(n)$ have been proven by various mathematicians. Garvan, Kim and Stanton [9] gave analytic and bijective proofs of the identity $a_5(5n + 4) = 5a_5(n)$. Granville and Ono [10] proved that for $t \geq 4$, every natural number $n$ has a $t$-core, thereby settling a conjecture of Brauer regarding the existence of defect zero characters for finite simple groups. E. X. W. Xia [15] established some new

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Ramanujan-type congruences modulo 2 and 4 for $t$-core partitions, (see [5,6,10,13–15] for further results). In this paper, we prove the following parity results on 13-cores.

**Theorem 1.1.** We have

$$
\sum_{n=0}^{\infty} a_{13}(104n + 6) q^n \equiv (q; q)_\infty^3 (\mod 2)
$$

and

$$
\sum_{n=0}^{\infty} a_{13}(4(26n + i) + 2) q^n \equiv 0 (\mod 2),
$$

where $i = 0$ or $2 \leq i \leq 25$.

**Theorem 1.2.** Let $n \geq 0$. Then for any positive integer $k$ we have

$$
a_{13}(104 \cdot 3^{2k}n + 13 \cdot 3^{2k} - 7) \equiv a_{13}(104n + 6) (\mod 2),
a_{13}\left(104 \cdot 5^{2k}n + 5 \cdot 13 \cdot 5^{2k-1} + 1\right) \equiv a_{13}(104n + 6) (\mod 2),
$$

and

$$
a_{13}\left(104 \cdot 7^{2k}n + 7 \cdot (13 \cdot 7^{2k-1} - 1)\right) \equiv a_{13}(104n + 6) (\mod 2).
$$

**Theorem 1.3.** If $p \geq 5$ is a prime with $\left(\frac{-13}{p}\right) = -1$, then for all nonnegative integers $n$ and $k$ we have

$$
a_{13}\left(4 \cdot p^{2k+1}(pn + j) + 7 \cdot (p^{2k+2} - 1)\right) \equiv 0 (\mod 2),
$$

where $1 \leq j \leq p - 1$.

**Theorem 1.4.** If $p \geq 5$ is a prime with $\left(\frac{-2}{p}\right) = -1$, then for all nonnegative integers $n$ and $k$ we have

$$
a_{13}\left(4 \cdot p^{2k+1}(pn + j) + 13 \cdot p^{2k+2} - 7\right) \equiv 0 (\mod 2),
$$

where $1 \leq j \leq p - 1$.

### 2. Background

For $|ab| < 1$, Ramanujan’s general theta-function $f(a, b)$ is defined by

$$
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.
$$

In this notation, Jacobi’s famous triple product identity [4, p. 35, Entry 19] takes the form

$$
f(a, b) = (a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}.
$$
Two important special cases of the above are
\[ \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)^{\infty}}{(q; q^3)^{\infty}} \]
and
\[ f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)^{\infty}, \quad (2.1) \]
where the last equality in (2.1) is Euler’s famous pentagonal number theorem. We will also need the following results.

**Lemma 2.1.** [8, Theorem 2.2] For any prime \( p \geq 5 \),
\[
f(-q) = \sum_{k=0}^{p-1} (-1)^k q^{3k^2+k} \sum_{n=0}^{\infty} (-1)^n (2pn + 2k + 1)q^{n+2k+1} \]
\[ \quad + (-1)^{\frac{p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^p), \quad (2.2) \]
where
\[
\pm \frac{p-1}{6} = \begin{cases} 
\frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}; \\
\frac{p-1}{6}, & \text{if } p \equiv -1 \pmod{6}.
\end{cases}
\]
Furthermore, if \( \frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2} \) and \( k \neq \frac{(\pm p-1)}{6} \), then
\[ \frac{3k^2 + k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}. \]

**Lemma 2.2.** [1] For any prime \( p \geq 5 \), we have
\[
f^3(-q) = \sum_{k=0}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn + 2k + 1)q^{n+2k+1} \]
\[ \quad + p(-1)^{\frac{p-1}{6}} q^{\frac{p^2-1}{24}} f^3(-q^p). \quad (2.3) \]
Furthermore, if \( k \neq \frac{p-1}{2} \) and \( 0 \leq k \leq p-1 \), then
\[ \frac{k^2 + k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}. \]

### 3. Congruences modulo 2 for 13-core partitions

**Theorem 3.1.** We have
\[
\sum_{n=0}^{\infty} a_{13} (4n) q^n \equiv (q; q^{13})^{3} (q^{13}; q^{13})^{3} \pmod{2},
\]
and

\[ \sum_{n=0}^{\infty} a_{13} (4n + 2) q^n \equiv q(q^{26}; q^{26})_\infty^3 \pmod{2}. \] (3.1)

**Proof.** For \( t > 1 \) a partition is called \( t \)-regular if none of its parts is divisible by \( t \), and we denote by \( b_t(n) \) the number of \( t \)-regular partitions of \( n \). Then the generating function for \( b_t(n) \) satisfies the identity

\[ \sum_{n=0}^{\infty} b_t(n) q^n = (q^t; q^t)_\infty / (q; q)_\infty. \] (1.1)

Putting \( t = 13 \) in (1.1), we have

\[ \sum_{n=0}^{\infty} a_{13} (4n + 2) q^n \equiv q(q^{26}; q^{26})_\infty^3 \pmod{2}. \] (3.2)

Using binomial expansion and then taking congruence modulo 2, we have

\[ (q; q)^2_\infty \equiv (q^2; q^2)_\infty \pmod{2}. \] (3.3)

Employing (3.3) in (3.2), we find that

\[ \sum_{n=0}^{\infty} a_{13} (4n + 2) q^n \equiv q(q^{26}; q^{26})_\infty^3 q^3 \pmod{2}. \] (3.4)

From [7, Theorem 2] we recall that

\[ \sum_{n=0}^{\infty} b_{13}(2n) q^n \equiv q^3(q^{26}; q^{26})^3_\infty \pmod{2}. \] (3.5)

Applying (3.6) in (3.5), we obtain

\[ \sum_{n=0}^{\infty} a_{13}(2n) q^n \equiv (q^2; q^2)^3_\infty + q^3(q^{26}; q^{26})^3_\infty \pmod{2}. \] (3.6)

Replacing \( q^2 \) by \( q \) in the above two congruences, we can easily obtain the required result.

**Theorem 3.2.** We have

\[ \sum_{n=0}^{\infty} a_{13} (104n + 6) q^n \equiv (q; q)^3_\infty \pmod{2} \] (3.7)
and
\[ \sum_{n=0}^{\infty} a_{13} (4(26n + i) + 2) q^n \equiv 0 \pmod{2}, \]
where \(i = 0\) or \(2 \leq i \leq 25\).

Proof. This follows directly from the fact that the series on the right hand side of (3.1) only involves powers of \(q\) that are congruent to 1 modulo 26. ■

\[ \text{Theorem 3.3.} \quad \text{Let } n \geq 0. \text{ Then for any positive integer } k \text{ we have} \]
\[ a_{13} \left( 104 \cdot 3^{2k} n + 13 \cdot 3^{2k} - 7 \right) \equiv a_{13}(104n + 6) \pmod{2}, \] \hspace{1cm} (3.8)
\[ a_{13} \left( 104 \cdot 5^{2k} n + 5 \cdot \frac{13 \cdot 5^{2k-1} + 1}{3} \right) \equiv a_{13}(104n + 6) \pmod{2} \] \hspace{1cm} (3.9)
and
\[ a_{13} \left( 104 \cdot 7^{2k} n + 7 \cdot (13 \cdot 7^{2k-1} - 1) \right) \equiv a_{13}(104n + 6) \pmod{2}. \] \hspace{1cm} (3.10)

Proof. Note that for a non-zero integer \(r\) and a nonnegative integer \(n\), the general partition function \(p_r(n)\) is defined as the coefficient of \(q^n\) in the expansion of \((q; q)_\infty^r\). From (3.7), we have
\[ \sum_{n=0}^{\infty} a_{13} (104n + 6) q^n \equiv \sum_{n=0}^{\infty} p_3(n)q^n \pmod{2}. \] (3.11)

From [3], we have
\[ p_3 \left( 3^{2k} n + \frac{3^{2k} - 1}{8} \right) = (-3)^k p_3(n), \]
\[ p_3 \left( 5^{2k} n + \frac{5^{2k} - 1}{24} \right) = 5^k p_3(n) \]
and
\[ p_3 \left( 7^{2k} n + \frac{7^{2k} - 1}{8} \right) = (-7)^k p_3(n). \]

Employing the above three identities in (3.11), we can easily obtain (3.8), (3.9) and (3.10). ■

\[ \text{Theorem 3.4.} \quad \text{If } p \geq 5 \text{ is a prime with } \left( \frac{-13}{p} \right) = -1, \text{ then for all nonnegative integers } k \text{ we have} \]
\[ \sum_{n=0}^{\infty} a_{13} \left( 4 \cdot p^{2k} n + 7 \cdot (p^{2k} - 1) \right) q^n \equiv (q; q^3)_\infty(q^{13}; q^{13})^3 \pmod{2}. \] (3.12)

Proof. Note first that (3.1) is the \(k = 0\) case of (3.12). Now suppose (3.12) holds for some \(k \geq 0\), and consider the congruence
\[ \frac{(\ell^2 + \ell)}{2} + 13 \cdot \frac{(m^2 + m)}{2} \equiv 14 \cdot \frac{(p^2 - 1)}{8} \pmod{p}, \] (3.13)
for $0 \leq \ell, m \leq p - 1$. Since the above congruence is equivalent to
\[(2\ell + 1)^2 + 13 \cdot (2m + 1)^2 \equiv 0 \pmod{p},\]
and $\left(\frac{-13}{p}\right) = -1$, it follows that (3.13) has only one solution, namely $k = m = (p - 1)/2$. Therefore, extracting the terms involving $q^{pn + 7(\ell^2 - 1)/4}$ from both sides of (3.12), by (2.3) we deduce that
\[
\sum_{n=0}^{\infty} a_{13} \left(4 \cdot p^{2k} (pn + 7(\frac{p^2 - 1}{4})) + 7 \cdot (p^{2k} - 1)\right) q^n \equiv (q^p; q^p)^3 (q^{13p}; q^{13p})^3 \pmod{2}.
\]
Again, extracting terms involving $q^{pn}$ from both sides of the above congruence and replacing $q^p$ by $q$, we obtain
\[
\sum_{n=0}^{\infty} a_{13} \left(4 \cdot p^{2k+2n} + 7 \cdot (p^{2k+2} - 1)\right) q^n \equiv (q; q)^3 (q^{13}; q^{13})^3 \pmod{2},
\]
which is the $k+1$ case of (3.12). 

Theorem 3.5. If $p \geq 5$ is a prime with $\left(\frac{-13}{p}\right) = -1$, then for all nonnegative integers $k$ we have
\[
a_{13} \left(4 \cdot p^{2k+1}(pn + j) + 7 \cdot (p^{2k+2} - 1)\right) \equiv 0 \pmod{2},
\]
where $1 \leq j \leq p - 1$.

Theorem 3.6. If $p \geq 5$ is a prime with $\left(\frac{-2}{p}\right) = -1$, then for all nonnegative integers $k$ we have
\[
\sum_{n=0}^{\infty} a_{13} \left(104 \cdot p^{2k} n + 13 \cdot p^{2k} - 7\right) q^n \equiv (q; q)^3 (q^{13}; q^{13})^3 \pmod{2}.
\]

Proof. Note that (3.7) is the $k = 0$ case of (3.15). Now suppose (3.15) holds for some $k \geq 0$, and consider the congruence
\[
\frac{(3\ell^2 + \ell)}{2} + 2 \cdot \frac{(3m^2 + m)}{2} \equiv 3 \cdot \frac{(p^2 - 1)}{24} \pmod{p},
\]
for $0 \leq \ell, m \leq p - 1$. The above congruence is equivalent to
\[(6\ell + 1)^2 + 2 \cdot (6m + 1)^2 \equiv 0 \pmod{p},
\]
and $\left(\frac{-2}{p}\right) = -1$, it follows that (3.16) has only one solution, namely $\ell = m = (\pm p - 1)/6$. Therefore, extracting the terms involving $q^{pn + (\ell^2 - 1)/8}$ from both sides of (3.15), by (2.2) we deduce that
\[
\sum_{n=0}^{\infty} a_{13} \left(104 \cdot p^{2k} (pn + \frac{p^2 - 1}{8}) + 13 \cdot p^{2k} - 7\right) q^n \equiv (q^p; q^p)^3 (q^{13p}; q^{13p})^3 \pmod{2}.
\]
Extracting the terms involving $q^{pn}$ from both sides of (3.17) and replacing $q^p$ by $q$, we obtain

$$\sum_{n=0}^{\infty} a_{13} \left( 104 \cdot p^{2k+2} n + 13 \cdot p^{2k+2} - 7 \right) q^n \equiv (q;q)_\infty (q^2;q^2)_\infty \pmod{2},$$

which is the $k+1$ case of (3.15).

From (3.17), we can easily obtain the following result.

**Theorem 3.7.** If $p \geq 5$ is a prime with $\left( \frac{-2}{p} \right) = -1$, then for all nonnegative integers $k$ we have

$$a_{13} \left( 4 \cdot p^{2k+1}(pn + j) + 13 \cdot p^{2k+2} - 7 \right) \equiv 0 \pmod{2},$$

where $1 \leq j \leq p - 1$.

REFERENCES


