SOME HOMOLOGICAL PROPERTIES OF AMALGAMATION

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Abstract. Let $R$ and $S$ be commutative rings, let $J$ be an ideal of $S$ and let $f: R \rightarrow S$ be a ring homomorphism. In this paper, we investigate some homological properties of the amalgamation of $R$ with $S$ along $J$ with respect to $f$ (denoted by $R \bowtie f J$), introduced by D’Anna and Fontana in 2007. In addition, we deal with the strongly cotorsion properties of local cohomology module of $R \bowtie f J$, when $R \bowtie f J$ is a local Noetherian ring.

1. Introduction

Throughout this paper all rings are considered commutative with identity element, and all ring homomorphisms are unital. In [7], D’Anna and Fontana considered a construction obtained involving a ring $R$ and an ideal $I \subset R$ that was denoted by $R \bowtie I$, called amalgamated duplication, and it was defined as the following subring of $R \times R$:

$$R \bowtie I = \{ (r, r+i) \mid r \in R, i \in I \}.$$  

This construction was studied from different points of view in [1, 3, 7, 10, 11, 13]. In [4], a systematic study of a new ring construction is initiated, called the “amalgamation of $R$ with $S$ along $J$ with respect to $f$”, for a given homomorphism of rings $f: R \rightarrow S$ and ideal $J$ of $S$. This construction finds its roots in a paper by J.L. Dorroh appeared in [8] and provides a general frame for studying the amalgamated duplication of a ring along an ideal. The amalgamation of $R$ with $S$ along $J$ with respect to $f$ is a subring of $R \times R$ which is defined as follows:

$$R \bowtie f J = \{ (r, f(r) + j) \mid r \in R, j \in J \}.$$  

This construction is a generalization of the amalgamated duplication of a ring along an ideal and other classical constructions, such as the Nagata’s idealization are strictly related to it [4, Example 2.7 and Remark 2.8]. One of the key tools for studying $R \bowtie f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [4]. This point of view allows to deepen the study initiated in [4] and continued in [5] and to provide an ample description of

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various properties of $R \bowtie^f J$, in connection with the properties of $R$, $J$ and $f$. In [4], necessary and sufficient conditions are provided for $R \bowtie^f J$ to inherit the properties of Noetherian ring, integral domain, and reduced ring and characterized pullbacks that can be expressed as amalgamations. In [5], they provided a complete description of the prime spectrum of $R \bowtie^f J$ and gave bounds for its dimension. In [6], the authors studied in details its prime spectrum and, when $R \bowtie^f J$ is a local Noetherian ring, some of its invariants (like the embedding dimension) and relevant properties (like Cohen-Macaulayness and Gorensteinness). Indeed, in [6, Proposition 5.7], they stated necessary and sufficient conditions for the self-injectivity of $R$. In connection with the strongly cotorsion properties of $R$, Xu in [12] introduced the terminology of strongly cotorsion modules. In [9], introduced the notion of cotorsion modules and as an special case of cotorsion modules, they stated necessary and sufficient conditions for the self-injectivity of $R$ in [5, Proposition 5.7], they provided a complete description of the prime spectrum of $R \bowtie^f J$ in connection with the strongly cotorsion properties of $H^i_{m(R)}(R \bowtie^f J)$ in connection with the strongly cotorsion properties of $H^i_{m(R)}(R)$ and $H^i_{m(R)}(J)$, when $R \bowtie^f J$ is a local Noetherian ring. In addition, we investigate some homological properties of the amalgamation.

2. Main results

Let $R$ and $S$ be commutative rings with unity, let $J$ ba an ideal of $S$ and let $f : R \to S$ be a ring homomorphism. In the following theorem we summarize some properties of $R \bowtie^f J$ from [4] and [6].

**Theorem 2.1.** Let $R$ and $S$ be commutative rings, let $J$ ba an ideal of $S$ and let $f : R \to S$ be a ring homomorphism. The following statements hold.

(i) There exists the natural ring homomorphism $\varphi : R \to R \bowtie^f J$ defined by $\varphi(r) = (r, f(r))$, for all $r \in R$. The map $\varphi$ is an embedding, making $R \bowtie^f J$ a ring extension of $R$. Furthermore, $R$ has $(R \bowtie^f J)$-module structure by the natural projection $p_R : R \bowtie^f J \to R$.

(ii) $R \bowtie^f J$ is isomorphic as an $R$-module to $R \oplus J$.

(iii) $R \bowtie^f J$ is a local ring if and only if $R$ is a local ring and $J \subseteq J(S)$, where $J(S)$ is the Jacobson radical of $S$. In particular, if $m$ is the unique maximal ideal of $R$, then $m \bowtie^f J = \{(m, f(m) + j) \mid m \in m, j \in J\}$ is the unique maximal ideal of $R \bowtie^f J$.

(iv) Let $(R, m)$ be a local ring and let $J \subseteq J(S)$ be finitely generated as an $R$-module. Then $\dim R = \dim(R \bowtie^f J) = \dim_R(R \bowtie^f J)$.

(v) Let $(R, m)$ be a local ring and let $J \subseteq J(S)$ be finitely generated as an $R$-module. Then $R \bowtie^f J$ is a Cohen-Macaulay ring if and only if it is a Cohen-Macaulay $R$-module if and only if $J$ is a maximal Cohen-Macaulay module.

(vi) Let $R \bowtie^f J$ be a local ring, where $R$ is a Cohen-Macaulay ring. Assume that $f(R) + J$ satisfies Serre’s condition $(S_1)$ such that $\dim(f(R) + J) = \dim R$, and suppose that $J \neq 0$ such that $f^{-1}(J)$ is a regular ideal of $R$. Then the following conditions are equivalent:

(a) $R \bowtie^f J$ is Gorenstein.
(b) \( f(R) + J \) is a Cohen-Macaulay ring, \( J \) is a canonical module of \( f(R) + J \) and \( f^{-1}(J) \) is a canonical module of \( R \).

Note that Theorem 2.1(vi) provides the necessary and sufficient conditions of self-injectivity of the ring \( R \bowtie^f J \). As a nice generalization of injectivity for modules, Enochs in [9] introduced the notion of cotorsion modules. An \( R \)-module \( M \) is called a cotorsion module if \( \text{Ext}_1^R(F, M) = 0 \) for all flat \( R \)-modules \( F \). Furthermore, as an special case of cotorsion modules Xu in [12] introduced the terminology of strongly cotorsion modules. An \( R \)-module \( M \) is called a strongly cotorsion module if \( \text{Ext}_1^R(F, M) = 0 \) for all \( R \)-modules \( F \) with finite flat dimension. One can easily show that if \( M \) is a strongly cotorsion \( R \)-module, then \( \text{Ext}_i^R(F, M) = 0 \) for all \( i \geq 1 \) and all \( R \)-modules \( F \) with finite flat dimension. In the following theorem we investigate the strongly cotorsion properties of \( H^{\dim R}_m(R \bowtie^f J) \) in connection with the strongly cotorsion properties of \( H^{\dim R}_m(R) \) and \( H^{\dim R}_m(J) \), when \( R \bowtie^f J \) is a local Noetherian ring.

**Theorem 2.2.** We preserve the assumptions of Theorem 2.1, and moreover we assume that \( (R, \mathfrak{m}) \) is a Noetherian local ring with dimension \( d \) and \( 0 \neq J \subseteq J(S) \) is an ideal such that \( J \) is a finitely generated \( R \)-module. Then \( H^{d}_{\mathfrak{m} \bowtie^f J}(R \bowtie^f J) \) is a strongly cotorsion \( R \)-module if and only if \( H^{d}_{\mathfrak{m}}(R) \) and \( H^{d}_{\mathfrak{m}}(J) \) are strongly cotorsion \( R \)-modules.

**Proof.** By Theorem 2.1(iv), \( R \) and \( R \bowtie^f J \) have the same dimension \( d \) and \( R \bowtie^f J \) is a local ring with maximal ideal \( \mathfrak{m}_0 = \mathfrak{m} \bowtie^f J \). Then we have the following \( R \)-isomorphisms:

\[
H^{d}_{\mathfrak{m}_0}(R \bowtie^f J) \cong H^{d}_{\mathfrak{m}}(R) \bowtie^f J \cong H^{d}_{\mathfrak{m}}(R \bowtie J) \cong H^{d}_{\mathfrak{m}}(R) \oplus H^{d}_{\mathfrak{m}}(J).
\]

The first isomorphism follows from [2, Theorem 4.2.1] and the second one follows from Theorem 2.1(ii). Now assume that \( H^{d}_{\mathfrak{m}_0}(R \bowtie^f J) \) is a strongly cotorsion \( R \)-module. Therefore, for any \( R \)-module \( F \) with finite flat dimension we have

\[
0 = \text{Ext}_1^R(F, H^d_{\mathfrak{m}_0}(R \bowtie^f J)) \cong \text{Ext}_1^R(F, H^d_{\mathfrak{m}}(R) \bowtie J) \cong \text{Ext}_1^R(F, H^d_{\mathfrak{m}}(R)) \oplus \text{Ext}_1^R(F, H^d_{\mathfrak{m}}(J)).
\]

Hence, \( \text{Ext}_1^R(F, H^d_{\mathfrak{m}}(J)) = \text{Ext}_1^R(F, H^d_{\mathfrak{m}}(R)) = 0 \) for any \( R \)-module \( F \) with finite flat dimension and this implies that \( H^d_{\mathfrak{m}}(R) \) and \( H^d_{\mathfrak{m}}(J) \) are strongly cotorsion \( R \)-modules. The converse can be proven in a similar way.

Let \( R \) be a ring and let \( I \) be an ideal of \( R \). The amalgamated duplication of \( R \) along \( I \), denoted by \( R \bowtie I \), is the special case of \( R \bowtie^f I \) where \( f : R \to R \) is an identity homomorphism, see [7]. Note that if \( (R, \mathfrak{m}) \) is a Noetherian local ring of dimension \( d \), then \( R \bowtie I \) is a Noetherian local ring with maximal ideal \( \mathfrak{m} \bowtie I = \{(m, m + i) \mid m \in \mathfrak{m}, i \in I\} \) of dimension \( d \), see [7, Corollary 3.3 and Theorem 3.5]. Therefore we have the following result.

**Corollary 2.3.** Let \( (R, \mathfrak{m}) \) be a Noetherian local ring of dimension \( d \) and let \( 0 \neq I \) be an ideal of \( R \). Then \( H^{d}_{\mathfrak{m} \bowtie I}(R \bowtie I) \) is a strongly cotorsion \( R \)-module if and only if \( H^{d}_{\mathfrak{m}}(R) \) and \( H^{d}_{\mathfrak{m}}(I) \) are strongly cotorsion \( R \)-modules.
In the sequel we investigate some homological properties of the amalgamation.

**Proposition 2.4.** Let \( f : R \to S \) be a ring homomorphism and let \( J \) be a non-zero ideal of \( S \) which is a flat \( R \)-module. Then the following statements hold for any \( R \)-module \( M \).

(i) \( \text{fd}_R(M) = \text{fd}_{R \otimes_R J}(M \otimes_R (R \otimes_R J)) \).

(ii) \( \text{pd}_R(M) = \text{pd}_{R \otimes_R J}(M \otimes_R (R \otimes_R J)) \).

**Proof.** By Theorem 2.1(ii), the \( R \)-module \( R \otimes_R J \) is faithfully flat since \( J \) is flat as an \( R \)-module. First, suppose that \( \text{fd}_R(M) \leq n \) (resp. \( \text{pd}_R(M) \leq n \)) and pick an \( n \)-step flat (resp. projective) resolution of \( M \) over \( R \) as follows:

\[
(*) : 0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0.
\]

Applying the functor \( - \otimes_R (R \otimes_R J) \) to \((*)\), we obtain the exact sequence of \((R \otimes_R J)\)-modules:

\[
0 \to F_n \otimes_R (R \otimes_R J) \to \cdots \to F_0 \otimes_R (R \otimes_R J) \to M \otimes_R (R \otimes_R J) \to 0.
\]

Thus, \( \text{fd}_{R \otimes_R J}(M \otimes_R (R \otimes_R J)) \leq n \) (resp. \( \text{pd}_{R \otimes_R J}(M \otimes_R (R \otimes_R J)) \leq n \)). Conversely, suppose that \( \text{fd}_{R \otimes_R J}(M \otimes_R (R \otimes_R J)) \leq n \) (resp. \( \text{pd}_{R \otimes_R J}(M \otimes_R (R \otimes_R J)) \leq n \)). Since \( R \otimes_R J \) is a flat \( R \)-module, we conclude that for any \( R \)-module \( N \) and each \( i \geq 1 \) we have:

\[
(1) : \text{Tor}^R_i(M, N \otimes_R (R \otimes_R J)) \cong \text{Tor}^R_{i+1}(M \otimes_R (R \otimes_R J), N \otimes_R (R \otimes_R J))
\]

\[
(2) : \text{Ext}^R_i(M, N \otimes_R (R \otimes_R J)) \cong \text{Ext}^R_{i+1}(M \otimes_R (R \otimes_R J), N \otimes_R (R \otimes_R J))
\]

Furthermore, \( \text{Tor}^R_i(M, N) \) and \( \text{Ext}^R_i(M, N) \) are direct summands of \( \text{Tor}^R_i(M, N \otimes_R (R \otimes_R J)) \) and \( \text{Ext}^R_i(M, N \otimes_R (R \otimes_R J)) \) respectively. Then, we conclude that \( \text{fd}_R(M) \leq n \) (resp. \( \text{pd}_R(M) \leq n \)).

**Proposition 2.5.** Let \( f : R \to S \) be a ring homomorphism and let \( J \) be a non-zero ideal of \( S \) which is a flat \( R \)-module. Then the following statements hold for every \( R \)-module \( M \).

(i) \( \text{id}_R(M) = \text{id}_R(M \otimes_R (R \otimes R J)) \)

(ii) \( \text{fd}_R(M) = \text{fd}_R(M \otimes_R (R \otimes R J)) \)

**Proof.** Note that \( R \otimes R J \) is a faithfully flat \( R \)-module. (i) follows from [13, Corollary 2.9] and (ii) follows from [13, Corollary 2.11].

**Corollary 2.6.** We preserve the assumptions of Proposition 2.5. For every \( R \)-module \( M \), we have

\[
\text{fd}_R(M) = \text{fd}_{R \otimes_R J}(M \otimes_R (R \otimes_R J)) = \text{fd}_R(M \otimes_R (R \otimes_R J)).
\]

**Proof.** By Proposition 2.4, we have \( \text{fd}_R(M) = \text{fd}_{R \otimes_R J}(M \otimes_R (R \otimes_R J)) \), and by Proposition 2.5, \( \text{fd}_R(M) = \text{fd}_R(M \otimes_R (R \otimes_R J)) \).

**Proposition 2.7.** Let \( f : R \to S \) be a ring homomorphism and let \( J \) be a non-zero ideal of \( S \). Then the following statements hold.

...
If $M$ is a (faithfully) injective $R$-module, then $\operatorname{Hom}_R(R \bowtie_f J, M)$ is a (faithfully) injective $(R \bowtie_f J)$-module.

(ii) Every injective $(R \bowtie_f J)$-module is a direct summand of the $R$-module $\operatorname{Hom}_R(R \bowtie_f J, M)$, where $M$ is an injective $R$-module.

**Proof.** (i) The following sequence of $(R \bowtie_f J)$-isomorphisms makes clear that if $M$ is a (faithfully) injective $R$-module, then $\operatorname{Hom}_R(R \bowtie_f J, M)$ is a (faithfully) injective $(R \bowtie_f J)$-module.

$$\operatorname{Hom}_{R \bowtie_f J}(-, \operatorname{Hom}_R(R \bowtie_f J, M)) \cong \operatorname{Hom}_R((R \bowtie_f J) \otimes_{R \bowtie_f J} -, M) \cong \operatorname{Hom}_R(-, M).$$

Note that in the above sequence, the first isomorphism follows from Hom-tensor adjointness, and the second isomorphism is induced by tensor cancellation.

(ii) Let $E$ be an injective $(R \bowtie_f J)$-module. It is enough to show that $E$ is embedded into an $R$-module of the form $\operatorname{Hom}_R(R \bowtie_f J, M)$ where $M$ is an injective $R$-module. Consider $E$ as an $R$-module and embed it into an injective $R$-module $M$. Then use isomorphisms in part (i), to convert the monomorphism of $R$-modules $E \hookrightarrow M$ to a monomorphism of $(R \bowtie_f J)$-modules $E \hookrightarrow \operatorname{Hom}_R(R \bowtie_f J, M)$.

**REFERENCES**


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