ON RELATIVE GORENSTEIN HOMOLOGICAL DIMENSIONS WITH RESPECT TO A DUALIZING MODULE

Maryam Salimi

Abstract. Let $R$ be a commutative Noetherian ring. The aim of this paper is studying the properties of relative Gorenstein modules with respect to a dualizing module. It is shown that every quotient of an injective module is $G_C$-injective, where $C$ is a dualizing $R$-module with $\text{id}_R(C) \leq 1$. We also prove that if $C$ is a dualizing module for a local integral domain, then every $G_C$-injective $R$-module is divisible. In addition, we give a characterization of dualizing modules via relative Gorenstein homological dimensions with respect to a semidualizing module.

1. Introduction

Throughout this paper $R$ is a commutative ring and all modules are unital. The notion of a “semidualizing module” is one of the most central notion in the relative homological algebra. This notion was first introduced by Foxby [6]. Then Vasconcelos [16] and Golod [7] rediscovered these modules using different terminology for different purposes. This notion has been investigated by many authors from different points of view; see for example [1, 4, 8, 14].

Among various research areas on semidualizing modules, one basically focuses on extending the “absolute” classical notion of homological algebra to the “relative” setting with respect to a semidualizing module. For instance, this has been done for the classical and Gorenstein homological dimensions mainly through the works of Golod [7], Holm and Jørgensen [8] and White [17], and (co)homological theories have been extended to the relative setting with respect to a semidualizing module mainly through the works of Takahashi and White [14], Salimi, Tavasoli, Yassemi [11] and Salimi et al. [10].

Following this idea, the present paper aims at studying the properties of relative Gorenstein modules with respect to a dualizing module which actually strengthens the classical results. In particular, in Proposition 3.6, it is shown that every quotient of an injective module is $G_C$-injective, where $C$ is a dualizing $R$-module.
with \( \text{id}_R(C) \leq 1 \). We also prove that if \( C \) is a dualizing module for an integral domain, then every \( G_C \)-injective \( R \)-module is divisible, see Proposition 3.7. In addition, Theorem 3.10 is investigated whether the relative Gorenstein homological dimensions with respect to a semidualizing module have the ability to detect when a semidualizing module is dualizing. Finally, we prove that the \( G_C \)-projective dimension of a finitely generated \( R \)-module is closely related to its depth, see Theorem 3.12.

2. Preliminaries

Throughout this paper \( R \) is a commutative Noetherian ring and \( \mathcal{M}(R) \) denotes the category of \( R \)-modules. We use the term “subcategory” to mean a “full, additive subcategory \( \mathcal{X} \subseteq \mathcal{M}(R) \) such that, for all \( R \)-modules \( M \) and \( N \), if \( M \cong N \) and \( M \in \mathcal{X} \), then \( N \in \mathcal{X} \).” Write \( \mathcal{P}(R) \), \( \mathcal{I}(R) \) and \( \mathcal{F}(R) \) for the subcategories of all projective, injective and flat \( R \)-modules, respectively.

An \( R \)-complex is a sequence
\[
X = \cdots \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} \cdots
\]
of \( R \)-modules and \( R \)-homomorphisms such that \( \partial_{n+1}^X \partial_n^X = 0 \) for each integer \( n \).

**Definition 2.1.** Let \( \mathcal{X} \) be a class of \( R \)-modules and let \( M \) be an \( R \)-module. An \( \mathcal{X} \)-resolution of \( M \) is a complex of \( R \)-modules in \( \mathcal{X} \) of the form
\[
X = \cdots \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} \cdots
\]
such that \( H_0(X) \cong M \) and \( H_n(X) = 0 \) for \( n \geq 1 \). The \( \mathcal{X} \)-projective dimension of \( M \) is the quantity
\[
\mathcal{X} \text{-pd}_R(M) = \inf \{ \sup \{ n \mid X_n \neq 0 \} \mid X \text{ is an } \mathcal{X} \text{-resolution of } M \}.
\]
In particular, \( \mathcal{X} \text{-pd}_R(0) = -\infty \). The modules of \( \mathcal{X} \)-projective dimension zero are the non-zero modules in \( \mathcal{X} \).

Dually, an \( \mathcal{X} \)-coresolution of \( M \) is a complex of \( R \)-modules in \( \mathcal{X} \) of the form
\[
X = \cdots \xleftarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \xleftarrow{\partial_{n-1}^X} \cdots
\]
such that \( H_0(X) \cong M \) and \( H_n(X) = 0 \) for \( n \leq -1 \). The \( \mathcal{X} \)-injective dimension of \( M \) is the quantity
\[
\mathcal{X} \text{-id}_R(M) = \inf \{ \sup \{ n \mid X_n \neq 0 \} \mid X \text{ is an } \mathcal{X} \text{-coresolution of } M \}.
\]
In particular, \( \mathcal{X} \text{-id}_R(0) = -\infty \). The modules of \( \mathcal{X} \)-injective dimension zero are the non-zero modules in \( \mathcal{X} \).

When \( \mathcal{X} \) is the class of projective \( R \)-modules we write \( \text{pd}_R(M) \) for the associated homological dimension and call it the projective dimension of \( M \). Similarly, the injective dimension and flat dimension of \( M \) are denoted \( \text{id}_R(M) \) and \( \text{fd}_R(M) \) respectively.
The notion of semidualizing modules, defined next, goes back at least to Vasconcelos [16], but was rediscovered by others.

**Definition 2.2.** A finitely generated $R$-module $C$ is called **semidualizing** if the natural homothety homomorphism $\chi_C^R : R \to \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}^1_R(C, C) = 0$. An $R$-module $D$ is called **dualizing** if it is semidualizing and has finite injective dimension.

**Fact 2.3** A free $R$-module of rank 1 is semidualizing, and indeed this is the only semidualizing module over a Gorenstein local ring.

For a semidualizing $R$-module $C$, we set

- $\mathcal{P}_C(R) = \{ P \otimes_R C \mid P \text{ is a projective } R\text{-module}\}$,
- $\mathcal{F}_C(R) = \{ F \otimes_R C \mid F \text{ is a flat } R\text{-module}\}$,
- $\mathcal{I}_C(R) = \{ \text{Hom}_R(C, I) \mid I \text{ is an injective } R\text{-module}\}$.

The $R$-modules in $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ are called $C$-projective, $C$-flat and $C$-injective, respectively.

The next definition is due to Holm and Jørgensen [8].

**Definition 2.4.** Let $C$ be a semidualizing $R$-module. A **complete $\mathcal{I}_C$-resolution** is a complex $Y$ of $R$-modules satisfying the following:

(i) $Y$ is exact and $\text{Hom}_R(I, Y)$ is exact for each $I \in \mathcal{I}_C(R)$, and
(ii) $Y_i \in \mathcal{I}_C(R)$ for all $i \geq 0$ and $Y_i$ is injective for all $i < 0$.

An $R$-module $M$ is **$G_C$-injective** if there exists a complete $\mathcal{I}_C$-resolution $Y$ such that $M \cong \text{coker}(\partial_Y^n)$; in this case $Y$ is a **complete $\mathcal{I}_C$-resolution** of $M$. The class of all $G_C$-injective $R$-modules is denoted by $\mathcal{GI}_C(R)$. In the case $C = R$, we use the more common terminology “complete injective resolution” and “Gorenstein injective module” and the notation $\mathcal{GI}(R)$.

A **complete $\mathcal{P}_C$-resolution** is a complex $X$ of $R$-modules such that:

(i) $X$ is exact and $\text{Hom}_R(X, P)$ is exact for each $P \in \mathcal{P}_C(R)$, and
(ii) $X_i$ is projective for all $i \geq 0$ and $X_i \in \mathcal{P}_C(R)$ for all $i < 0$.

An $R$-module $M$ is **$G_C$-projective** if there exists a complete $\mathcal{P}_C$-resolution $X$ such that $M \cong \text{coker}(\partial_X^n)$; in this case $X$ is a **complete $\mathcal{P}_C$-resolution** of $M$. The class of all $G_C$-projective $R$-modules is denoted by $\mathcal{GP}_C(R)$. In the case $C = R$, we use the more common terminology “complete projective resolution” and “Gorenstein projective module” and the notation $\mathcal{GP}(R)$.

A **complete $\mathcal{F}_C$-resolution** is a complex $Z$ of $R$-modules such that:

(i) $Z$ is exact and $Z \otimes_R I$ is exact for each $I \in \mathcal{I}_C(R)$, and
(ii) $Z_i$ is flat for all $i \geq 0$ and $Z_i \in \mathcal{F}_C(R)$ for all $i < 0$.

An $R$-module $M$ is **$G_C$-flat** if there exists a complete $\mathcal{F}_C$-resolution $Z$ such that $M \cong \text{coker}(\partial_Z^n)$; in this case $Z$ is a **complete $\mathcal{F}_C$-resolution** of $M$. The class of all
On relative Gorenstein homological dimensions

121

$G_C$-flat $R$-modules is denoted by $\mathcal{G}_C(R)$. In the case $C = R$, we use the more
common terminology “complete flat resolution” and “Gorenstein flat module” and
the notation $\mathcal{G}_R$.

3. Main results

In [10, Proposition 5.2] and [14, Theorem 2.11], the authors demonstrated a
strong connection between the classical homological dimensions and relative homo-
logical dimensions with respect to a semidualizing $R$-module which are collected in
the following.

**Fact 3.1.** Let $C$ be a semidualizing $R$-module, and let $M$ be an $R$-module.
Then the following statements hold.
(i) $\mathcal{P}_C$-$\text{pd}_R(M) = \text{pd}_R(\text{Hom}_R(C, M))$.
(ii) $\mathcal{P}_C$-$\text{pd}_R(C \otimes_R M) = \text{pd}_R(M)$.
(iii) $\mathcal{I}_C$-$\text{id}_R(M) = \text{id}_R(C \otimes_R M)$.
(iv) $\mathcal{I}_C$-$\text{id}_R(\text{Hom}_R(C, M)) = \text{id}_R(M)$.
(v) $\mathcal{F}_C$-$\text{pd}_R(M) = \text{fd}_R(\text{Hom}_R(C, M))$.
(vi) $\mathcal{F}_C$-$\text{pd}_R(C \otimes_R M) = \text{fd}_R(M)$.
(vii) $\mathcal{F}_C$-$\text{pd}_R(M) \leq \mathcal{P}_C$-$\text{pd}_R(M)$.

In [15, Proposition 2.4 and Corollary 2.5], Tang showed that in the case $C$ is a
dualizing $R$-module, the connection between the classical homological dimensions
and relative homological dimensions with respect to $C$ is more closed as follows.

**Fact 3.2.** Let $C$ be a dualizing $R$-module with $\text{id}_R(C) \leq n$, and let $M$ be an
$R$-module. Then the following statements hold.
(i) $\mathcal{F}_C$-$\text{pd}_R(M) < \infty \Rightarrow \mathcal{P}_C$-$\text{pd}_R(M) \leq n$.
(ii) $\mathcal{I}_C$-$\text{id}_R(M) \leq n \Leftrightarrow \mathcal{I}_C$-$\text{id}_R(M) < \infty \Leftrightarrow \text{fd}_R(M) < \infty \Leftrightarrow \text{id}_R(M) \leq n$.
(iii) $\mathcal{F}_C$-$\text{pd}_R(M) \leq n \Leftrightarrow \mathcal{F}_C$-$\text{pd}_R(M) < \infty \Leftrightarrow \text{id}_R(M) < \infty \Leftrightarrow \text{id}_R(M) \leq n$.

Using Facts 3.1 and 3.2 we have the following result.

**Proposition 3.3.** Let $C$ be a dualizing $R$-module with $\text{id}_R(C) \leq n$, and let $M$ be an
$R$-module. Then
(i) $\mathcal{I}_C$-$\text{id}_R(M) < \infty \Rightarrow \text{pd}_R(M) \leq n$.
(ii) $\text{pd}_R(M) < \infty \Rightarrow \mathcal{I}_C$-$\text{id}_R(M) \leq n$.
(iii) $\mathcal{P}_C$-$\text{pd}_R(M) < \infty \Rightarrow \text{id}_R(M) \leq n$.
(iv) $\text{id}_R(M) < \infty \Rightarrow \mathcal{P}_C$-$\text{pd}_R(M) \leq n$.

**Proof.** We just prove (i) and (ii).
(i) Let $\mathcal{I}_C$-$\text{id}_R(M) < \infty$. Then Fact 3.2 implies that $\text{fd}_R(M) \leq n$. By Fact
3.1, $\mathcal{F}_C$-$\text{pd}_R(C \otimes_R M) \leq n$, and another use of Fact 3.2 implies that $\mathcal{P}_C$-$\text{pd}_R(C \otimes_R M) \leq n$. Now the assertion follows from Fact 3.1.
(ii) Since $\text{pd}_R(M) < \infty$, we have $\text{fd}_R(M) < \infty$ and the assertion follows from
Fact 3.2.
In the sequel, we show that if $C$ is a dualizing $R$-module, then the class of $G_C$-injective $R$-modules has nice properties as well as the class of Gorenstein modules over Gorenstein rings.

**Theorem 3.4.** Let $C$ be a dualizing $R$-module with $\text{id}_R(C) = n \geq 1$ and let $G$ be an $R$-module. Then $G$ is $G_C$-injective if and only if there exists an exact sequence

$$G_{n-1} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G \rightarrow 0,$$

where $G_{n-1}, \ldots, G_0$ are $G_C$-injective $R$-modules.

**Proof.** The forward implication holds by definition. For the reverse implication, we just prove the case $n = 1$. By assumption there exists a short exact sequence ($\ast$): $0 \rightarrow K \rightarrow G_0 \rightarrow G \rightarrow 0$ where $G_0$ is an $G_C$-injective $R$-module and $K$ is an $R$-module. Let $L$ be an $R$-module with $\mathcal{T}_C-\text{id}_R(L) < \infty$. Then $\text{pd}_R(L) \leq 1$, by Proposition 3.3. By applying the functor $\text{Hom}_R(L, -)$ on the exact sequence ($\ast$), we get that $\text{Ext}^i_R(L, G) \cong \text{Ext}^{i+1}_R(L, K)$ for all $i \geq 1$. Note that $\text{Ext}^{i+1}_R(L, K) = 0$ for all $i \geq 1$, since $\text{pd}_R(L) \leq 1$. So, the assertion follows from the dual of [17, Proposition 2.12].

It is known that $\mathcal{I}_C(R) \subseteq G\mathcal{I}_C(R)$ and $\mathcal{I}(R) \subseteq G\mathcal{I}_C(R)$. So we have the following result.

**Corollary 3.5.** Let $C$ be a dualizing $R$-module with $\text{id}_R(C) = n \geq 1$ and let $G$ be an $R$-module. Then the following statements hold.

(i) $G$ is $G_C$-injective if and only if there exists an exact sequence

$$\text{Hom}_R(C, E_{n-1}) \rightarrow \cdots \rightarrow \text{Hom}_R(C, E_1) \rightarrow \text{Hom}_R(C, E_0) \rightarrow G \rightarrow 0,$$

where $E_{n-1}, \ldots, E_0$ are injective $R$-modules.

(ii) If there exists an exact sequence

$$E_{n-1} \rightarrow \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow G \rightarrow 0,$$

where $E_{n-1}, \ldots, E_0$ are injective $R$-modules, then $G$ is $G_C$-injective.

Note that the dual of Theorem 3.4 and Corollary 3.5 hold too.

**Proposition 3.6.** Let $C$ be a dualizing $R$-module with $\text{id}_R(C) \leq 1$. Then every quotient of an injective module is $G_C$-injective.

**Proof.** Let ($\ast$): $0 \rightarrow M \rightarrow E \rightarrow E/M \rightarrow 0$ be a short exact sequence of $R$-modules such that $E$ is injective. Let $L$ be an $R$-module such that $\text{pd}_R(L) < \infty$. Using Proposition 3.3, we conclude that $\text{pd}_R(L) \leq 1$. By applying the functor $\text{Hom}_R(L, -)$ on the sequence ($\ast$), we have the following long exact sequence

$$0 \rightarrow \text{Hom}_R(L, M) \rightarrow \text{Hom}_R(L, E) \rightarrow \text{Hom}_R(L, E/M) \rightarrow \cdots.$$  

Therefore we get $\text{Ext}^i_R(L, E/M) \cong \text{Ext}^{i+1}_R(L, M) = 0$ for all $i \geq 1$. By dual of [17, Proposition 2.12] and Proposition 3.3, we get the assertion.■
It is known that over an integral domain $R$, every injective $R$-module is divisible. In [2, Lemma 5], it is shown that over local Gorenstein integral domain $R$ of Krull dimension at most one, an $R$-module is Gorenstein injective if and only if it is divisible. In the following proposition we prove the relative counterpart of this result.

**Proposition 3.7.** Let $R$ be an integral domain and let $C$ be a dualizing $R$-module. Then every $G_C$-injective $R$-module is divisible.

**Proof.** Let $M$ be a $G_C$-injective $R$-module and let $0 \neq r \in R$. Then $\text{pd}_R(R/rR) \leq 1$. By dual of [17, Proposition 2.12] and Proposition 3.3, we have $\text{Ext}^1_R(R/rR, M) = 0$. Hence $M \xrightarrow{r} M \rightarrow 0$ is exact and therefore $M$ is divisible. $lacksquare$

It is known that in local regular rings, every module has finite homological dimensions. In [12, Corollary 3.2], it is shown that the $I_C$-injective dimension and $\mathcal{P}_C$-projective dimension have the ability to detect the regularity of $R$, where $C$ is a semidualizing $R$-module. In addition, finiteness of Gorenstein homological dimensions characterizes Gorenstein local rings as follows.

**Theorem 3.8.** [5, Theorem 2.19 and Corollary 3.23] Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following statements are equivalent:

(i) $R$ is Gorenstein.

(ii) $\text{Gpd}_R(M) < \infty$ for all $R$-modules $M$.

(iii) $\text{Gpd}_R(k) < \infty$.

(iv) $\text{Gid}_R(M) < \infty$ for all $R$-modules $M$.

(v) $\text{Gid}_R(k) < \infty$.

In the following theorem, we show that the relative Gorenstein homological dimensions with respect to a semidualizing module have also the ability to detect when a semidualizing module is dualizing. First, we recall the notion of trivial extension of the ring $R$ by an $R$-module. If $M$ is an $R$-module, then the direct sum $R \oplus M$ can be equipped with the product:

$$(a, m)(a', m') = (aa', am' + a'm),$$

where $a, a' \in R$ and $m, m' \in M$. This turns $R \oplus M$ into a ring which is called the trivial extension of $R$ by $M$ and denoted $R \ltimes M$. There are canonical ring homomorphisms $R \xrightarrow{=} R \ltimes M$, which enable us to view $R$-modules as $(R \ltimes M)$-modules and vice versa.

Let $C$ be a semidualizing module. In [8], it is shown that the three $G_C$-dimensions always agree with the changed ring dimensions as follows.

**Fact 3.9.** [8, Theorem 2.16] Let $C$ be a semidualizing $R$-module. The following statements hold for every $R$-module $M$.

(i) $\mathcal{G}I_C \dashv \text{id}_R(M) = \text{Gid}_{R \ltimes C}(M)$.

(ii) $\mathcal{G}\mathcal{P}_C \dashv \text{pd}_R(M) = \text{Gpd}_{R \ltimes C}(M)$.

(iii) $\mathcal{G}\mathcal{F}_C \dashv \text{pd}_R(M) = \text{Gfd}_{R \ltimes C}(M)$. 
For an $R$-module $M$, Reiten and Foxby in [6] and [9] proved that $R \ltimes M$ is Gorenstein if and only if $R$ is Cohen-Macaulay and $M$ is a dualizing module. Now Theorem 3.8 and Fact 3.9 imply the following result.

**Proposition 3.10.** Let $(R, \mathfrak{m}, k)$ be a local ring and let $C$ be a semidualizing $R$-module. Then the following statements are equivalent:

(i) $C$ is dualizing.

(ii) $\mathcal{G}C \cdot \text{pd}_R(M) < \infty$ for all $R$-modules $M$.

(iii) $\mathcal{G}C \cdot \text{id}_R(k) < \infty$.

(iv) $\mathcal{G}\mathcal{T}_C \cdot \text{id}_R(M) < \infty$ for all $R$-modules $M$.

(v) $\mathcal{G}\mathcal{T}_C \cdot \text{id}_R(k) < \infty$.

The projective dimension of a finitely generated $R$-module is closely related to its depth. This is captured by the Auslander-Buchsbaum Formula [3, Theorem 1.3.3], which states that for every finitely generated $R$-module $M$ of finite projective dimension there is an equality $\text{pd}_R(M) = \text{depth} R - \text{depth}_R M$. The Gorenstein counterpart actually strengthens the classical result; this is a recurring theme in Gorenstein homological algebra as follows.

**Theorem 3.11.** [5, Theorem 1.25 and Proposition 2.16] Let $R$ be a local ring and let $M$ be a finitely generated $R$-module with finite Gorenstein projective dimension. Then

$$\text{Gpd}_R(M) = \text{depth} R - \text{depth}_R M.$$ 

In the following theorem, we show that the $G_C$-projective dimension of a finitely generated $R$-module is also closely related to its depth.

**Theorem 3.12.** Let $C$ be a semidualizing module for local ring $R$ and let $M$ be a finitely generated $R$-module with finite $G_C$-projective dimension. Then

$$\mathcal{G}C \cdot \text{pd}_R(M) = \text{depth} R - \text{depth}_R M.$$ 

**Proof.** By Fact 3.9, we have $\mathcal{G}C \cdot \text{pd}_R(M) = \text{Gpd}_{R \ltimes C}(M)$ and Theorem 3.11 implies that $\mathcal{G}C \cdot \text{pd}_R(M) = \text{depth}(R \ltimes C) - \text{depth}_{R \ltimes C}(M)$. Note that by [3, Exercise 1.2.26], $\text{depth}_{R \ltimes C}(M) = \text{depth}_R M$ and by [13, Theorem 2.2.6], $\text{depth}(R \ltimes C) = \text{min}\{\text{depth} R, \text{depth}_R C\} = \text{depth} R$, which implies the assertion. 

**Proposition 3.13.** Let $R$ be a local ring and let $C$ be a dualizing $R$-module. If $M$ is a finitely generated $R$-module, then $M$ is $G_C$-projective if and only if $M$ is maximal Cohen-Macaulay.

**Proof.** Note that $R$ is Cohen-Macaulay, since $R$ has a finitely generated module of finite injective dimension. For the forward implication, $0 = \mathcal{G}C \cdot \text{pd}_R(M) = \text{depth} R - \text{depth}_R M$. So, $\text{depth}_R M = \text{depth} R = \dim R$ which implies that $M$ is maximal Cohen-Macaulay. For the reverse implication, we have $\mathcal{G}C \cdot \text{pd}_R(M) < \infty$ by Proposition 3.10. Now the assertion follows from Theorem 3.12. 

On relative Gorenstein homological dimensions

REFERENCES


(Received 14.07.2016; in revised form 15.01.2017; available online 26.01.2017)

Department of Mathematics, East Tehran Branch, Islamic Azad University, Tehran, Iran
E-mail: maryamsalimi@ipm.ir