ARITHMETIC PROPERTIES OF 3-REGULAR BI-PARTITIONS WITH DESIGNATED SUMMANDS

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Abstract. Recently Andrews, Lewis and Lovejoy introduced the partition functions $PD(n)$ defined by the number of partitions of $n$ with designated summands and they found several modulo 3 and 4. In this paper, we find several congruences modulo 3 and 4 for $PBD_3(n)$, which represent the number of 3-regular bi-partitions of $n$ with designated summands. For example, for each $n \geq 1$ and $\alpha \geq 0$ $PBD_3(4 \cdot 3^{\alpha+2}n + 10 \cdot 3^{\alpha+1}) \equiv 0$ (mod 3).

1. Introduction

In 2002 Andrews, Lewis and Lovejoy [1] introduced a new class of partitions, partitions with designated summands which are constructed by taking ordinary partitions and tagging exactly one part among parts with equal size. With a convention that $PD(n) = 0$, for example there are 15 partitions of 5 with designated summands:

$5', \quad 4' + 1', \quad 3' + 2', \quad 3' + 1' + 1, \quad 3' + 1 + 1', \quad 2' + 2 + 1', \quad 2 + 2' + 1', \quad 2' + 1' + 1 + 1, \quad 1 + 1 + 1 + 1 + 1, \quad 1 + 1 + 1 + 1' + 1 + 1', \quad 1 + 1 + 1 + 1' + 1, \quad 1 + 1 + 1 + 1 + 1 + 1'.

The authors [1] derived the following generating function of $PD(n)$.

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1 f_2 f_3}.$$ 

Throughout the paper, we use the standard $q$-series notation, and $f_k$ is defined as

$$f_k := (q^k: q^k)_\infty = \lim_{n \to \infty} \prod_{i=1}^{n} (1 - q^{ik}).$$

For $|ab| < 1$, Ramanujan’s general theta function $f(a, b)$ is defined as

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$ 

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Using Jacobi’s triple product identity [4, Entry 19, p. 35], (2) becomes
\[ f(a, b) = (-a, ab)_\infty (-b, ab)_\infty (ab, ab)_\infty. \]

The most important special cases of \( f(a, b) \) are
\[ \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} = \frac{f_2}{f_1}. \]

\[ f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1. \]

The concept of partitions with designated summands goes back to MacMahon [9]. He considered partitions with designated summands and with exactly \( \ell \) different sizes (see also Andrews and Rose [2]).

Andrews et al. [1] and N. D. Baruah and K. K. Ojah [3] have also studied \( PDO(n) \), the number of partitions of \( n \) with designated summands in which all parts are odd and the generating function is given by
\[ \sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6}{f_1 f_2 f_3 f_{12}}. \]

Thus \( PDO(5) = 8 \) are
\[ 5', \quad 3' + 1' + 1, \quad 3' + 1 + 1', \quad 1' + 1 + 1 + 1 + 1, \quad 1 + 1' + 1 + 1 + 1, \quad 1 + 1 + 1 + 1 + 1 + 1'. \]

Chen, Ji, Jin and Shen [5] have established Ramanujan type identity for the partition function \( PD(3n+2) \) which implies the congruence of Andrews et al. [1] and they also gave a combinatorial interpretation of the congruence for \( PD(3n+2) \) by introducing a rank for partitions with designated summands. Recently Xia [14] extended the work of deriving congruence properties of \( PD(n) \) by employing the generating functions of \( PD(3n) \) and \( PD(3n+2) \) due to Chen et al. [5].

Mahadeva Naika et al. [10] have studied \( PD_3(n) \), the number of partitions of \( n \) with designated summands whose parts are not divisible by 3 and the generating function is given by
\[ \sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_2^3 f_9}{f_1 f_2 f_3 f_{18}}. \]

In [11] Mahadeva Naika et al. have established many congruences for \( PD_2(n) \), the number of bipartitions of \( n \) with designated summands and the generating function is given by
\[ \sum_{n=0}^{\infty} PD_2(n)q^n = \frac{f_2^3 f_2}{f_1^2 f_2 f_3 f_2}. \]

Mahadeva Naika et al. [12] have derived \( PD_{2,3}(n) \), the number of partitions of \( n \) with designated summands in which parts are not multiples of 2 or 3 and generating
function is given by
\[ \sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4^2 f_6 f_{36}}{f_1 f_2 f_{18}}. \]

Motivated by the above work, in this paper, we study \( PBD_3(n) \), the number of 3-regular bi-partitions of \( n \) with designated summands and the generating function is given by
\[ \sum_{n=0}^{\infty} PBD_3(n)q^n = \frac{f_4^2 f_6^2}{f_1^2 f_2 f_8}. \]

To be precise by a bipartition with designated summands we mean a pair of partitions \((\mu, \kappa)\) where in partitions \( \mu \) and \( \kappa \) are partitions with designated summands. Thus \( PBD_3(4) = 35 \) are
\[
(4', \emptyset), (2' + 2', \emptyset), (2' + 1', 1), (2' + 1 + 1', 0), (1' + 1 + 1', 0),
(1' + 1' + 1, 0), (1 + 1 + 1 + 1', 0), (2', 2'), (2', 1'+1), (2', 1'+1'),
(1', 1'+1 + 1'), (1', 1' + 1 + 1'), (1' + 1, 1 + 1'), (1' + 1, 1 + 1'),
(1 + 1', 1 + 1'), (2' + 1', 1'), (1'+1', 2'), (1 + 1', 2'),
(1' + 1 + 1, 1'), (1' + 1 + 1'), (1 + 1 + 1', 1'), \emptyset, 4', \emptyset, 2' + 2', \emptyset, 2' + 2', \emptyset, 2' + 1 + 1', \emptyset, 1' + 1 + 1 + 1, (0, 1 + 1 + 1 + 1), \emptyset, 1 + 1 + 1 + 1').
\]

In Section 3, we prove the following theorems.

**Theorem 1.1.** For \( n \geq 0 \) we have
\[
\sum_{n=0}^{\infty} PBD_3(2n)q^n = \frac{f_4^2 f_6^2}{f_1 f_2 f_8} + q \frac{f_4^2 f_6 f_{36}}{f_1 f_2 f_{18}}, \quad (4)
\]
\[
\sum_{n=0}^{\infty} PBD_3(2n + 1)q^n = 2^2 \frac{f_4^2 f_6^2 f_9}{f_1 f_2 f_9}. \quad (5)
\]

**Theorem 1.2.** For each nonnegative integer \( n \) and \( \alpha \geq 0 \), we have
\[
PBD_3\left(4 \times 3^{\alpha+2}n + 10 \times 3^{\alpha+1}\right) \equiv 0 \pmod{3}, \quad (6)
\]
\[
PBD_3\left(8 \times 3^{\alpha+2}n + 8 \times 3^{\alpha+2}\right) \equiv 0 \pmod{3}, \quad (7)
\]
\[
PBD_3\left(2^{\alpha+3}n\right) \equiv 2^n PBD_3(4n) \pmod{3}, \quad (8)
\]
\[
\sum_{n=1}^{\infty} PBD_3(n + 2)q^n \equiv \psi(q)\psi(q^2) \pmod{3}, \quad (9)
\]
\[
\sum_{n=1}^{\infty} PBD_3(8n + 4)q^n \equiv 2\psi(q)\psi(q^3) \pmod{3}. \quad (10)
\]

**Theorem 1.3.** Let \( p \) be a prime with \( \left(\frac{-3}{p}\right) = -1 \). Then for any nonnegative integer \( \alpha \),
\[
\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha}n + 2p^{2\alpha}\right)q^n \equiv \psi(q)\psi(q^3) \pmod{3}, \quad (11)
\]
and for $n \geq 0$, $1 \leq j \leq p - 1$,
\[ PBD_3 \left( 4p^{2\alpha + 1}(pn + j) + 2p^{2\alpha + 2} \right) \equiv 0 \pmod{3}. \] (12)

Theorem 1.4. Let $p$ be a prime with $\left( \frac{-3}{p} \right) = -1$. Then for any nonnegative integer $\alpha$,
\[ \sum_{n=1}^{\infty} PBD_3 \left( 8p^{2\alpha}n + 4p^{2\alpha} \right) q^n \equiv 2\psi(q)\psi(q^3) \pmod{3}, \] (13)
and for $n \geq 0$, $1 \leq j \leq p - 1$,
\[ PBD_3 \left( 8p^{2\alpha + 1}(pn + j) + 4p^{2\alpha + 2} \right) \equiv 0 \pmod{3}. \] (14)

Theorem 1.5. For each $n \geq 0$
\[ PBD_3(12n + 7) \equiv 0 \pmod{4}, \] (15)
\[ PBD_3(12n + 11) \equiv 0 \pmod{4}, \] (16)
\[ PBD_3(24n + 17) \equiv 0 \pmod{4}, \] (17)
\[ PBD_3(36n + 27) \equiv 0 \pmod{4}, \] (18)
\[ PBD_3(72n + 39) \equiv 0 \pmod{4}, \] (19)
\[ PBD_3(72n + 57) \equiv 0 \pmod{4}, \] (20)
\[ PBD_3(216n + 153) \equiv 0 \pmod{4}, \] (21)
\[ \sum_{n=0}^{\infty} PBD_3(72n + 3) \equiv 2f_1 \pmod{4}, \] (22)
\[ \sum_{n=0}^{\infty} PBD_3(72n + 15) \equiv 2f_1f_4 \pmod{4}. \] (23)

Theorem 1.6. For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have
\[ \sum_{n=0}^{\infty} PBD_3 \left( 72p^{2\alpha}n + 3p^{2\alpha} \right) q^n \equiv 2f_1 \pmod{4}. \] (24)

Theorem 1.7. For any prime $p \geq 5$, $\alpha \geq 0$, $n \geq 0$ and $l = 1, 2, \ldots, p - 1$, we have
\[ \sum_{n=0}^{\infty} PBD_3 \left( 72p^{2\alpha}(pn + l) + 3p^{2\alpha} \right) \equiv 0 \pmod{4}. \] (25)

Theorem 1.8. If $p \geq 5$ is a prime such that $\left( \frac{-4}{p} \right) = -1$. Then for all integers $\alpha \geq 0$,
\[ \sum_{n=0}^{\infty} PBD_3 \left( 72p^{2\alpha}n + 15p^{2\alpha} \right) q^n \equiv 2f_1f_4 \pmod{4}. \] (26)

Theorem 1.9. Let $p \geq 5$ be prime and $\left( \frac{-4}{p} \right) = -1$. Then for all integers $n \geq 0$ and $\alpha \geq 1$,
\[ PBD_3 \left( 72p^{2\alpha}n + p^{2\alpha - 1}(15p + 72j) \right) \equiv 0 \pmod{4}, \] (27)
where $j = 1, 2, \ldots, p - 1$. 
Theorem 1.10. For each $n \geq 0$
\[ PBD_3(18n + 15) \equiv 0 \pmod{6}, \]  
\[ PBD_3(18n + 3) \equiv 4f_1f_3 \pmod{6}. \]  
(28)  
(29) 

Theorem 1.11. If $p \geq 5$ is a prime such that $\left(\frac{-3}{p}\right) = -1$. Then for all integers $\alpha \geq 0$,
\[ \sum_{n=0}^{\infty} PBD_3(18p^{2\alpha}n + 3p^{2\alpha}) q^n \equiv 4f_1f_3 \pmod{6}. \]  
(30) 

Theorem 1.12. Let $p \geq 5$ be prime and $\left(\frac{-3}{p}\right) = -1$. Then for all integers $n \geq 0$ and $\alpha \geq 1$,
\[ PBD_3(18p^{2\alpha}n + p^{2\alpha-1}(3p + 18j)) \equiv 0 \pmod{6}, \]  
where $j = 1, 2, \ldots, p - 1$.  
(31) 

2. Preliminaries

We list a few dissection formulas to prove our main results.

Lemma 2.1. [4, Corollory, p. 49] We have
\[ \psi(q) = f(q^3, q^6) + q\psi(q^9) \]  
(32) 

Lemma 2.2. The following 2-dissections hold:
\[ \frac{f_3^4}{f_1} = \frac{f_3^4f_6^2}{f_2^2f_4} + q\frac{f_{12}^3}{f_4}, \]  
(33) 

\[ \frac{f_1}{f_3^3} = \frac{f_2^4f_6^2}{f_4^2} - q\frac{f_3^4f_6^2}{f_2^2f_4^2}. \]  
(34) 

Hirschhorn, Garvan and Borwein [7] proved equation (33). Replacing $q$ by $-q$ in (33), we obtain (34).

Lemma 2.3. The following 2-dissections hold:
\[ \frac{1}{f_1f_3} = \frac{f_2^2f_{12}^4}{f_2^2f_4f_6f_{24}} + q\frac{f_3^4f_2^2}{f_2^2f_6^2f_{12}}, \]  
(35) 

\[ f_1f_3 = \frac{f_2^2f_{12}^4}{f_1^2f_6f_{24}} - q\frac{f_3^4f_6^2}{f_6^2f_{12}^2}. \]  
(36) 

Equation (35) was proved by Baruah and Ojah [3]. Replacing $q$ by $-q$ in (35) and using the fact that $(-q; -q)_\infty = \frac{f_2^2}{f_1f_4}$, we get (36).
Lemma 2.4. The following 3-dissection holds:
\[ f_1 f_2 = \frac{f_6 f_4^3}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_5^2}. \] (37)

One can see this identity in [8].

Lemma 2.5. The following 2-dissections hold:
\[ f_9 f_1 = f_3 f_1 f_4 f_6 f_{36} + q f_2 f_6 f_{36} f_{12} \] (38)
and
\[ f_1 f_9 = f_4 f_6 f_{18}^2 - q f_4 f_6 f_{36} f_{18} \] (39)

Lemma 2.5 was proved by Xia and Yao [13]. Replacing \( q \) by \(-q\) in (38) and using the relation \((-q; -q)_\infty = f_3 f_4 f_6 f_5\), we obtain (39).

Lemma 2.6. [6, Theorem 2.1] For any odd prime \( p \),
\[ \psi(q) = \sum_{m=0}^{p-3} q^{\frac{m^2 + m}{2}} f \left( q^{\frac{p^2 + (2m+1)p}{2}}, q^{\frac{p^2 - (2m+1)p}{2}} \right) + q^{\frac{p^2-1}{8}} \psi(q^2). \] (40)
Furthermore, \( m^2 + m \not\equiv \frac{p^2-1}{8} \pmod{p} \) for \( 0 \leq m \leq \frac{p-3}{2} \).

Lemma 2.7. [6, Theorem 2.2] For any prime \( p \geq 5 \),
\[ f_1 = \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{\frac{k^2 + k}{2}} f \left( -q^{\frac{3p^2 + (4k+1)p}{2}}, -q^{\frac{3p^2 - (4k+1)p}{2}} \right) + (-1)^{\frac{p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}. \]
Furthermore, for \(- (p-1)/2 \leq k \leq (p-1)/2\) and \( k \not\equiv (\pm 1)/6, \frac{3k^2 + k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p} \).

3. Proofs of main results

3.1 Proof of Theorems 1.1 and 1.2

Substituting (38) into (3), we find that
\[ \sum_{n=0}^{\infty} PBD_3(n)q^n = \frac{f_6 f_4^3}{f_3 f_{18}^2} \left( \frac{f_{12} f_{18}^2}{f_2 f_6 f_{36}} + 2q f_2 f_{12} f_{18} + q^2 \frac{f_6 f_{36}^2}{f_2 f_{12}^2} \right) \]
\[ = \frac{f_6 f_4^3}{f_2 f_6 f_{36}} + 2q f_2 f_{12} f_{18} + q^2 \frac{f_6 f_{36}^2}{f_2 f_{12} f_{18}}. \]

Extracting the terms involving \( q^{2n} \) and \( q^{2n+1} \) from the above equation, we obtain (4) and (5).
By the binomial theorem, it is easy to see that for positive integers $k$ and $m$,
\[ f_{2k}^m \equiv f_k^{2m} \pmod{2}, \quad (41) \]
\[ f_{3k}^m \equiv f_k^{3m} \pmod{3}, \quad (42) \]
and
\[ f_{2k}^{2m} \equiv f_k^{4m} \pmod{4}. \quad (43) \]
Invoking (42) in (4), we find
\[ \sum_{n=0}^{\infty} PBD_3(2n)q^n \equiv 1 + q f_1 f_6 f_2 f_3 \pmod{3}, \quad (44) \]
which implies that
\[ \sum_{n=1}^{\infty} PBD_3(2n)q^n \equiv q f_1 f_6 f_2 f_3 \pmod{3}. \quad (45) \]
Employing (34) into (44), we have
\[ \sum_{n=1}^{\infty} PBD_3(2n)q^n \equiv q f_2 f_1 f_4 f_2 f_3 - q^2 f_3 f_1 f_2^2 f_3 f_6 \pmod{3}. \quad (45) \]
Extracting the terms containing $q^{2n+1}$, dividing throughout by $q$ and then replacing $q^2$ by $q$ from (45) and using the fact that $\psi(q) = f_2^2 f_1$, we get (9).

Substituting (32) into (9), we obtain
\[ \sum_{n=1}^{\infty} PBD_3(4n + 2)q^n \equiv f(q^3, q^6) \psi(q^3) + q \psi(q^3) \psi(q^6) \pmod{3}, \quad (46) \]
implying
\[ \sum_{n=1}^{\infty} PBD_3(12n + 6)q^n \equiv \psi(q) \psi(q^3) \pmod{3}. \quad (47) \]
From equations (9) and (47), we get
\[ PBD_3(12n + 6) \equiv PBD_3(4n + 2) \pmod{3}. \quad (48) \]
By using mathematical induction on $\alpha$ in (48), we have
\[ PBD_3(4 \times 3^{\alpha+1} n + 2 \times 3^{\alpha+1}) \equiv PBD_3(4n + 2) \pmod{3}. \quad (49) \]
Extracting the terms containing $q^{3n+2}$ from (46) we obtain
\[ PBD_3(12n + 10) \equiv 0 \pmod{3}. \quad (50) \]
Using (50) in (49), we find (6).

Extracting the terms containing $q^{2n}$ and replacing $q^2$ by $q$ from (45), we get
\[ \sum_{n=1}^{\infty} PBD_3(4n)q^n \equiv 2q f_1 f_6 f_2 f_3 \pmod{3}. \quad (51) \]
Employing (34) into (51), we obtain
\[ \sum_{n=1}^{\infty} PBD_3(4n)q^n \equiv 2q \frac{f_1 f_6}{f_2 f_3} - 2q^2 \frac{f_2 f_6}{f_3 f_4} \pmod{3}. \quad (52) \]
Congruence (10) is obtained by extracting the terms containing $q^{2n+1}$ from (52) and
using the fact that \( \psi(q) = \frac{f_2}{f_1} \).

Substituting (32) into (10), we have

\[
\sum_{n=1}^{\infty} PBD_3(8n + 4)q^n \equiv 2f(q^3, q^6) + 2q\psi(q^3)\psi(q^9) \pmod{3}.
\]

Extracting the terms containing \( q^{3n+1} \) and \( q^{3n+2} \) from the above equation, we obtain

\[
\sum_{n=1}^{\infty} PBD_3(24n + 12)q^n \equiv 2\psi(q)\psi(q^3) \pmod{3} \tag{53}
\]

and

\[
PBD_3(24n + 20) \equiv 0 \pmod{3} \tag{54}
\]

In view of the congruences (10) and (53), we get

\[
PBD_3(24n + 12) \equiv PBD_3(8n + 4) \pmod{3} \tag{55}
\]

Utilizing (55) and by mathematical induction on \( \alpha \), we arrive at

\[
PBD_3(24n + 12) \equiv PBD_3(8n + 4) \pmod{3} \tag{56}
\]

Using (54) in (56), we obtain (7).

Extracting the terms containing \( q^{2n} \) by \( q \) from (52), we have

\[
\sum_{n=1}^{\infty} PBD_3(8n)q^n \equiv qf_1f_6 \pmod{3} \tag{57}
\]

In view of the congruences (57) and (51), we obtain

\[
PBD_3(8n) \equiv 2 \cdot PBD_3(4n) \pmod{3} \tag{58}
\]

Utilizing (58) and by mathematical induction on \( \alpha \), we arrive at (8).

\[\square\]

### 3.2 Proof of Theorem 1.3

Equation (9) is the \( \alpha = 0 \) case of (11). If we assume that (11) holds for some \( \alpha \geq 0 \), then, substituting (40) in (11),

\[
\sum_{n=1}^{\infty} PBD_3 \left( 4p^{2\alpha}n + 2p^{2\alpha} \right) q^n \\
\equiv \left( \sum_{m=0}^{p-3} q^{n^2+m} f \left( q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}} \right) + q^{\frac{p^2-1}{2}} \psi(q^{p^2}) \right) \\
\times \left( \sum_{m=0}^{p-3} q^{n^2+m} f \left( q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}} \right) + q^{\frac{p^2-1}{2}} \psi(q^{p^2}) \right) \pmod{3}. \tag{59}
\]

For any odd prime \( p \), and \( 0 \leq m_1, m_2 \leq (p - 3)/2 \), consider the congruence

\[
\frac{m_1^2 + m_1}{2} + \frac{3m_2^2 + m_2}{2} \equiv \frac{4p^2 - 4}{8} \pmod{p},
\]

which implies that

\[
(2m_1 + 1)^2 + 3(2m_2 + 1)^2 \equiv 0 \pmod{p}. \tag{60}
\]
Since \((-\frac{3}{p}) = -1\), the only solution of the congruence (60) is \(m_1 = m_2 = \frac{p-1}{2}\).

Therefore, equating the coefficients of \(q^{p^n+\frac{4p^2-4}{8}}\) from both sides of (59), dividing throughout by \(q^{\frac{4p^2-4}{8}}\) and then replacing \(q^p\) by \(q\), we obtain

\[
\sum_{n=1}^{\infty} PBD_3 \left(4p^{2\alpha} \left(\frac{pn + \frac{4p^2 - 4}{8}}{2}\right) + 2p^{2\alpha}\right) q^n \equiv \psi(q^p)\psi(q^{3p}) \pmod{3}. \tag{61}
\]

Equating the coefficients of \(q^{pn}\) on both sides of (61) and then replacing \(q^p\) by \(q\), we obtain

\[
\sum_{n=1}^{\infty} PBD_3 \left(4p^{2\alpha+2}n + 2p^{2\alpha+2}\right) q^n \equiv \psi(q)\psi(q^3) \pmod{3},
\]

which is the \(\alpha+1\) case of (11). Extracting the terms involving \(q^{pn+j}\) for \(1 \leq j \leq p-1\) in (61), we get (12).

### 3.3 Proof of Theorem 1.4

Equation (10) is the \(\alpha = 0\) case of (13). If we assume that (13) holds for some \(\alpha \geq 0\), then, substituting (40) in (13),

\[
\sum_{n=1}^{\infty} PBD_3 \left(8p^{2\alpha}n + 4p^{2\alpha}\right) q^n \\
= 2 \left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{n^2+m}{2}} f \left(\frac{q^{\frac{2}{3}(2m+1)p}}{, q^{\frac{n^2-(2m+1)p}}\right) + q^{\frac{n^2}{2}-1} \psi(q^p)\right) \tag{62}
\times \left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{n^2+m}{2}} f \left(\frac{q^{\frac{2}{3}(2m+1)p}}{, q^{\frac{n^2-(2m+1)p}}\right) + q^{\frac{n^2+m}{2}-1} \psi(q^p)\right) \pmod{3}.
\]

For any odd prime \(p\), and \(0 \leq m_1, m_2 \leq (p-3)/2\), consider the congruence

\[
m_1^2 + m_1 + 3m_2^2 + m_2 = \frac{4p^2 - 4}{8} \pmod{p},
\]

which implies that

\[
(2m_1 + 1)^2 + 3(2m_2 + 1)^2 \equiv 0 \pmod{p}. \tag{63}
\]

Since \((-\frac{3}{p}) = -1\), the only solution of the congruence (63) is \(m_1 = m_2 = \frac{p-1}{2}\).

Therefore, equating the coefficients of \(q^{p^n+\frac{4p^2-4}{8}}\) from both sides of (62), dividing throughout by \(q^{\frac{4p^2-4}{8}}\) and then replacing \(q^p\) by \(q\), we obtain

\[
\sum_{n=1}^{\infty} PBD_3 \left(8p^{2\alpha} \left(\frac{pn + \frac{4p^2 - 4}{8}}{2}\right) + 4p^{2\alpha}\right) q^n \equiv 2\psi(q^p)\psi(q^{3p}) \pmod{3}. \tag{64}
\]

Equating the coefficients of \(q^{pn}\) on both sides of (64) and then replacing \(q^p\) by \(q\), we obtain

\[
\sum_{n=1}^{\infty} PBD_3 \left(8p^{2\alpha+2}n + 4p^{2\alpha+2}\right) q^n \equiv 2\psi(q)\psi(q^3) \pmod{3},
\]
which is the $\alpha+1$ case of (13). Extracting the terms involving $q^{p_n+j}$ for $1 \leq j \leq p-1$ in (64), we arrive at (14).

### 3.4 Proof of Theorem 1.5

Invoking (43) in (5), we find

$$
\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2 \frac{f_1 f_4}{f_2 f_9} \pmod{8}.
$$

(65)

Employing (39) into (65), we obtain

$$
\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2 \frac{f_0 f_3}{f_2 f_4 f_{18}} - 2q \frac{f_4 f_6 f_9}{f_2 f_4 f_{12} f_{18}} \pmod{8}.
$$

(66)

Extracting the terms containing $q^{2n+1}$, dividing throughout by $q$ and then replacing $q^2$ by $q$ from the above equation, we get

$$
\sum_{n=0}^{\infty} PBD_3(4n+3)q^n \equiv 6 \frac{f_0 f_3^2 f_{18}}{f_1^2 f_6 f_9^2} \pmod{8},
$$

(67)

but

$$
6 \frac{f_0 f_3^2 f_{18}}{f_1^2 f_6 f_9^2} \equiv 6 \frac{f_0 f_3 f_9}{f_1 f_6} \pmod{8}.
$$

(68)

Invoking (41) in (68), we get

$$
\sum_{n=0}^{\infty} PBD_3(4n+3)q^n \equiv 2 f_3 f_6 f_9 \pmod{4}.
$$

(69)

Congruences (15) and (16) follow by extracting the terms containing $q^{3n+1}$ and $q^{3n+2}$ from (69).

Extracting the terms containing $q^{3n}$ and replacing $q^3$ by $q$ from (69), we obtain

$$
\sum_{n=0}^{\infty} PBD_3(12n+3)q^n \equiv 2 f_1 f_2 f_3 \pmod{4}.
$$

(70)

Substituting (37) into (70), we find

$$
\sum_{n=0}^{\infty} PBD_3(12n+3)q^n \equiv 2 \frac{f_0 f_3^2}{f_{18}} - 2 q f_3 f_9 f_{18} \pmod{4}.
$$

(71)

Congruence (18) is obtained by extracting the terms containing $q^{3n+2}$ from (71).

Extracting the terms containing $q^{3n}$ and replacing $q^3$ by $q$ from the above equation we arrive at

$$
\sum_{n=0}^{\infty} PBD_3(36n+3)q^n \equiv 2 \frac{f_2 f_4^2}{f_6} \pmod{4}.
$$

(72)

Using (41) in (72), we obtain

$$
\sum_{n=0}^{\infty} PBD_3(36n+3)q^n \equiv 2 f_2 \pmod{4}.
$$

(73)

Congruences (19) and (22) follow by extracting the terms containing $q^{2n}$ and $q^{2n+1}$.
from (73).

Extracting the terms containing $q^{3n+1}$, dividing throughout by $q$ and then replacing $q^3$ by $q$ from (71), we obtain

$$\sum_{n=0}^{\infty} PBD_3(36n + 15)q^n \equiv 2f_1f_3f_6 \pmod{4}. \tag{74}$$

Employing (36) into (74), we find

$$\sum_{n=0}^{\infty} PBD_3(36n + 15)q^n \equiv 2\frac{f_3^2 f_6^5}{f_3^2 f_6^5} - 2q\frac{f_3^2 f_6^5}{f_3^2 f_6^5} \pmod{4}. \tag{75}$$

Extracting the terms containing $q^{2n}$ and then replacing $q^2$ by $q$ from (75), we obtain

$$\sum_{n=0}^{\infty} PBD_3(72n + 15)q^n \equiv 2f_1f_3 f_6 \pmod{4}. \tag{76}$$

Using (41) in (76), we arrive at (23).

Extracting the terms containing $q^{2n}$ and replacing $q^2$ by $q$ from (66), we get

$$\sum_{n=0}^{\infty} PBD_3(4n + 1)q^n \equiv 2\frac{f_3^3 f_4}{f_3^3 f_4} \pmod{4}. \tag{77}$$

Using (41) in (77), we have

$$\sum_{n=0}^{\infty} PBD_3(4n + 1)q^n \equiv 2\frac{f_3^3 f_4}{f_3^3 f_4} \pmod{4}. \tag{78}$$

Substituting (33) into (78), we arrive at

$$\sum_{n=0}^{\infty} PBD_3(4n + 1)q^n \equiv 2\frac{f_3^3 f_4}{f_3^3 f_4} + 2q\frac{f_3^3 f_4}{f_3^3 f_4} \pmod{4}. \tag{79}$$

Extracting the terms containing $q^{2n}$ and replacing $q^2$ by $q$ from (79), we obtain

$$\sum_{n=0}^{\infty} PBD_3(8n + 1)q^n \equiv 2\frac{f_3^3 f_6}{f_3^3 f_6} \pmod{4},$$

but

$$\frac{f_3^3 f_6}{f_3^3 f_6} \equiv \frac{f_3^3 f_6}{f_3^3 f_6} \pmod{2}. \tag{mod 2}$$

This yields

$$\sum_{n=0}^{\infty} PBD_3(8n + 1)q^n \equiv 2\frac{f_3^3 f_6}{f_3^3 f_6} \pmod{4}. \tag{80}$$

Using Jacobi’s triple product identity and $\psi(q) = \frac{f_3^3}{f_1}$ in (32), we arrive at

$$\frac{f_3^3}{f_3^3} = \frac{f_3^3}{f_3 f_1} + q\frac{f_3^3}{f_3} \tag{81}$$

Employing (81) into (80), we get

$$\sum_{n=0}^{\infty} PBD_3(8n + 1)q^n \equiv 2\frac{f_3 f_6}{f_3} + 2q\frac{f_3 f_6}{f_3} \pmod{4}. \tag{82}$$
Congruence (17) is obtained by extracting the terms containing $q^{3n+2}$ from the above equation.

Extracting the terms containing $q^{3n+1}$, dividing throughout by $q$ and then replacing $q^3$ by $q$ from (82), we obtain

$$\sum_{n=0}^{\infty} PBD_3(24n + 9)q^n \equiv 2 \frac{f_1 f_2 f_6}{f_3^2} \pmod{4}. \quad (83)$$

Using (41) in (83), we have

$$\sum_{n=0}^{\infty} PBD_3(24n + 9)q^n \equiv 2f_1 f_2 f_6 \pmod{4}. \quad (84)$$

Substituting (37) into (84), we obtain

$$\sum_{n=0}^{\infty} PBD_3(72n + 9)q^n \equiv 2 \frac{f_2^2 f_4}{f_1 f_6} \pmod{4}. \quad (85)$$

Congruence (20) follows from (85) and extracting the terms containing $q^{3n}$ and replacing $q^3$ by $q$ from the above equation, we find

$$\sum_{n=0}^{\infty} PBD_3(72n + 9)q^n \equiv 2 \frac{f_2^2 f_4}{f_1 f_6} \pmod{4}. \quad (86)$$

Using (41) in (86), we get

$$\sum_{n=0}^{\infty} PBD_3(72n + 9)q^n \equiv 2 \frac{f_2^2}{f_1} \equiv 2\psi(q) \pmod{4}. \quad (87)$$

Substituting (32) into (87) and extracting the terms containing $q^{3n+2}$, we arrive at (21).

3.5 Proof of Theorem 1.6

Employing Lemma (2.7) into (22), it can be see that

$$\sum_{n=0}^{\infty} PBD_3 \left( 72 \left( pn + \frac{p^2 - 1}{24} \right) + 3 \right) q^n \equiv 2f_p \pmod{4}, \quad (88)$$

which implies that

$$\sum_{n=0}^{\infty} PBD_3 \left( 72p^2n + 3p^3 \right) q^n \equiv 2f_1 \pmod{4}.$$

Therefore, $PBD_3 \left( 72p^2n + 3p^3 \right) \equiv PBD_3(72n + 3) \pmod{4}$.

Using the above relation and by induction on $\alpha$, we arrive at (24).

3.6 Proof of Theorem 1.7

Combining (88) with Theorem (1.6), we derive that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3 \left( 72p^{2\alpha+1}n + 3p^{3\alpha} \right) \equiv 2f_p \pmod{4}.$$
Therefore, it follows that
\[ \sum_{n=0}^{\infty} PBD_3 \left( 72p^{2\alpha+1}(pn + l) + 3p^{3\alpha} \right) \equiv 0 \pmod{4}, \]
where \( l = 1, 2, ..., p - 1 \), and we obtain (25).

### 3.7 Proof of Theorem 1.8

For a prime \( p \geq 5 \) and \(- (p - 1)/2 \leq k, m \leq (p - 1)/2\), consider
\[ \frac{3k^2 + k}{2} + 4 \times \frac{3m^2 + m}{2} \equiv \frac{5p^2 - 5}{24} \pmod{p}. \]
This is equivalent to \((6k + 1)^2 + 4(6m + 1)^2 \equiv 0 \pmod{p}\). Since \( \left( \frac{-4}{p} \right) = -1 \), the only solution of the above congruence is \( k = m = (\pm p - 1)/6 \). Therefore, from Lemma 2.7,
\[ \sum_{n=0}^{\infty} PBD_3 \left( 72 \left( p^2n + 5 \times \frac{p^2 - 1}{24} \right) + 15 \right) q^n \equiv 2f_1f_4 \pmod{4}. \]
Using (23), (89), and induction on \( \alpha \), we get (26).

### 3.8 Proof of Theorem 1.9

From Lemma 2.7 and Theorem 1.8, for each \( \alpha \geq 0 \),
\[ \sum_{n=0}^{\infty} PBD_3 \left( 72 \left( p^2n + 5 \times \frac{p^2 - 1}{24} \right) + 15 \right) q^n \equiv 2f_1f_4 \pmod{4}. \]
That is,
\[ \sum_{n=0}^{\infty} PBD_3 \left( 72p^{2\alpha+1}n + 15p^{2\alpha+2} \right) q^n \equiv 2f_pf_{4p} \pmod{4}. \]
Since there are no terms on the right of (90) where the powers of \( q \) are congruent to 1, 2, \ldots, \( p - 1 \) modulo \( p \),
\[ PBD_3 \left( 72p^{2\alpha+1}(pn + j) + 15p^{2\alpha+2} \right) \equiv 0 \pmod{4}, \]
for \( j = 1, 2, \ldots, p - 1 \). Therefore, for \( j = 1, 2, \ldots, p - 1 \) and \( \alpha \geq 1 \), we arrive at (27).

### 3.9 Proof of Theorem 1.10

By the binomial theorem, it is easy to see that for positive integers \( k \) and \( m \),
\[ f_{3k}^{3m} \equiv f_k^{m} \pmod{9}. \]
Invoking (91) in (5), we have
\[ \sum_{n=0}^{\infty} PBD_3(2n + 1)q^n \equiv 2f_3f_4^2f_3f_4^2 \pmod{18}. \]
Employing (37) into (92) and extracting the terms containing $q^{3n+1}$, dividing throughout by $q$ and then replacing $q^3$ by $q$ from (92), we obtain
\[ \sum_{n=0}^{\infty} PBDA_{3}(6n+3)q^n \equiv 14\frac{f_3^3f_4^4}{f_6} + 8q\frac{f_3^3f_8^8}{f_3^3} \pmod{18}. \] (93)

Invoking (42) in (93), we see that
\[ \sum_{n=0}^{\infty} PBDA_{3}(6n+3)q^n \equiv 4f_3^4 + 4q f_8^8 \pmod{6}. \] (94)

Congruence (28) follows by extracting the terms containing $q^{3n+2}$ from the above equation.

Extracting the terms containing $q^{3n}$ and replacing $q^3$ by $q$ from (94), we arrive at
\[ \sum_{n=0}^{\infty} PBDA_{3}(18n+3)q^n \equiv 4f_1^1 f_3^3 \pmod{6}. \] (95)

Invoking (42) in (95) we get (29). \hfill \Box

3.10 Proof of Theorem 1.11

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider
\[ \frac{3k^2+k}{2} + 3 \times \frac{3m^2+m}{2} \equiv 4p^2 - 4 \pmod{p}. \]
This is equivalent to $(6k+1)^2 + 3(6m+1)^2 \equiv 0 \pmod{p}$.

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p-1)/6$.

Therefore, from Lemma 2.7,
\[ \sum_{n=0}^{\infty} PBDA_{3} \left(18 \left(p^2n + 4 \times \frac{p^2-1}{24}\right) + 3\right) q^n \equiv 4f_1f_3 \pmod{6}. \] (96)

Using (29), (96), and induction on $\alpha$, we arrive at (30). \hfill \Box

3.11 Proof of Theorem 1.12

From Lemma 2.7 and Theorem 1.11, for each $\alpha \geq 0$,
\[ \sum_{n=0}^{\infty} PBDA_{3} \left(18 \left(p^2n + 4 \times \frac{p^2-1}{24}\right) + 3\right) q^n \equiv 4f_1f_3 \pmod{6}. \]

That is,
\[ \sum_{n=0}^{\infty} PBDA_{3} (18p^{2\alpha+1}n + 3p^{2\alpha+2}) q^n \equiv 4f_1f_3 \pmod{6}. \] (97)

Since there are no terms on the right of (97) where the powers of $q$ are congruent to 1, 2, \ldots, $p-1$ modulo $p$,
\[ PBDA_{3} (18p^{2\alpha+1}(pn + j) + 3p^{2\alpha+2}) \equiv 0 \pmod{6}, \]
for $j = 1, 2, \ldots, p - 1$. Therefore, for $j = 1, 2, \ldots, p - 1$ and $\alpha \geq 1$, we obtain (31).

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