DIVISIBLE LINEARLY ORDERED TOPOLOGICAL SPACES

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Abstract. We prove that a ccc linearly ordered topological space is metrizable if and only if it is divisible.

Let $X$ be a topological space and $A$ a subset of $X$. We will say that a family $D_A$ of subsets of $X$ is a divisor for $A$ if for every $x \in A$ and every $y \in X \setminus A$ there exists $D \in D_A$ such that $x \in D$ and $y \notin D$ [1]. If all members of $D_A$ are closed (open, compact, \ldots) in $X$, then we say that $D_A$ is a closed (open, compact, \ldots) divisor for $A$. In [1], A. Arhangel'skii defined a space $X$ to be divisible if for every $A \subset X$ there is a countable closed divisor for $A$. The divisibility degree $dv(X)$ of a space $X$ is defined to be the smallest cardinal $\tau$ such that for every $A \subset X$ there exists a closed divisor for $A$ having cardinality $\leq \tau$ [5], [6].

A family $\mathcal{U}$ of open subsets of a space $X$ is called a pseudobase for $X$ if for every $x \in X$ we have $\{x\} = \bigcap \{U \in \mathcal{U} \mid x \in U\}$. The pseudoweight of $X$, denoted by $pw(X)$, is defined by $pw(X) = \omega \cdot \min \{ |\mathcal{U}| : \mathcal{U} \text{ is pseudobase for } X \}$.

Obviously, $dv(X) \leq pw(X)$.

We use the usual topological terminology and notation following [2]; for definitions and results on cardinal functions we refer to [4]. $w$, $pw$, $L$, $c$, $\psi$ denote the weight, pseudoweight, Lindelöf number, cellularity and pseudocardenal, respectively. All cardinals in this note are infinite.

Recall that a family $\gamma$ of subsets of a set $S$ is said to be point separating if for any $p, q \in S$, $p \neq q$, there is some $A \in \gamma$ such that $p \in A$ and $q \notin A$. We need the following known lemma:

1. Lemma. If $S$ is a set of cardinality $\leq 2^\omega$, then there exists a point separating family $\gamma$ of subsets of $S$ having cardinality $\leq \gamma$.

In [5] (see also [6], [7]), the following result was shown:

2. Theorem. Every divisible compact Hausdorff space is metrizable.

Here we prove that a ccc LOTS (= linearly ordered topological space) is divisible if and only if it is metrizable. In general, this result is not true for GO-spaces (= subspaces of LOTS's).
3. Theorem. For any LOTS X we have \( w(X) = c(X)dv(X) \).

Proof. Let \( c(X)dv(X) = \tau \). Since X is a LOTS then, as is well known [4], \( |X| \leq c(X) \leq 2^\tau \). According to Lemma 1 there exists a point separating family \( \{ S_\alpha \mid \alpha \in \tau \} \) of subsets of X. For every \( \alpha \in \tau \) choose a closed divisor \( D_\alpha \) for \( S_\alpha \) with \( |D_\alpha| \leq \tau \) and put \( D = \bigcup \{ D_\alpha \mid \alpha \in \tau \} \). Then, \( |D| \leq \tau \) and \( D \) is a point separating family of closed subsets of X. Therefore, family \( B = \{ X \setminus D \mid D \in D \} \) is a pseudobase for X of cardinality \( \leq \tau \), i.e. \( pw(X) \leq \tau \). By a result of K. P. Hart [3] (concerning LOTS's) we have \( w(X) = c(X)pw(X) \leq \tau \). The opposite inequality \( c(X)dv(X) \leq w(X) \) is always true and the theorem is proved. \( \square \)

4. Corollary. A ccc LOTS X is divisible if and only if it is a separable metrizable space.

Using this result we can once again get one known fact:

5. Example. The lexicographically ordered unit square is not a ccc space. Otherwise, this would mean that it is metrizable; but it is known that this is not true.

6. Remark. (1) The previous result is not valid in general for GO-spaces. The Sorgenfrey line \( S \) is a divisible (since \( pw(X) \leq \omega \)) ccc space, but \( S \) is not metrizable (even it is not developable).

(2) It is well known [2], [4] that every ccc LOTS is Lindelöf. So, it is natural to ask whether the conclusions of Corollary 4 can be extended to the class of Lindelöf LOTS's. It is not possible. In [3] there is an example of a LOTS \( X \) with \( pw(X) = L(X) = \omega \) which is not metrizable. Of course, this space is divisible, because of \( pw(X) = \omega \).

However, we have the following result.

7. Theorem. If for every subset \( A \) of a LOTS \( X \) there exists a countable divisor consisting of closed Lindelöf \( G_\delta \)-sets, then \( X \) is metrizable.

Proof. First of all we prove that \( X \) is a Lindelöf space. Let \( X \) be any element in \( X \). Let \( D_X = \{ L_i \mid i \in \omega \} \) be a countable divisor for \( X \setminus \{ x \} \) consisting of closed Lindelöf \( G_\delta \)-sets. By the definition of a divisor we have \( X \setminus \{ x \} = \bigcup \{ L_i \mid i \in \omega \} \), so that \( X \setminus \{ x \} \) is a Lindelöf space. Thus \( X \) is also Lindelöf. It is known that every Lindelöf \( T_1 \)-space in which for every \( A \subset X \) there exists a countable divisor consisting of closed \( G_\delta \)-sets has a \( G_\delta \)-diagonal [1]. On the other hand, by a result of D. Lutzer [2], every LOTS with a \( G_\delta \)-diagonal is metrizable. \( \square \)

References

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