A NOTE ON A SUPPORT OF A LINEAR MAPPING

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Abstract. In this note a notion of the support of a linear mapping from $C_b(T)$ into a locally convex space is introduced. Some of its properties are established.

Introduction

If $E$ is a locally convex space and $P \subset E'$ is a weakly-$*$-bounded set, then $P$ is equicontinuous iff the linear mapping $f$ from $E$ into the Banach space $C_b(P)$, defined by $f(e)(p) = p(e) \ (p \in P, e \in E)$ is continuous. For the case $E = (C_b(T), \beta_T)$, as we will see, some information concerning the continuity of the mapping $f$ is provided by its support.

Preliminaries

All topological spaces considered here are assumed to be completely regular Hausdorff. If $T$ is such a space, then $C_b(T)$ (resp. $C(T)$) denotes the space of bounded (resp. all) real-valued continuous functions on $T$. $\beta T$ is the Stone-Čech compactification of $T$. For each $x \in C_b(T)$ its continuous extension to $\beta T$ is denoted by $x^\beta$. If $x \in C(\beta T)$ and if $A \subset \beta T$, then $x|A$ denotes the restriction of $x$ to $A$. $cl_X A$ is the closure of $A \subset X$.

We denote by $\| \| \sup$ supremum norm on $C_b(T)$, and by $B$ the unit ball $\{x \in C_b(T) : \|x\| \leq 1\}$. $M(T)$ is the Banach space dual to $(C_b(T), \| \|)$. If $H \subset C_b(T)$ (or if $H \subset M(T)$), then $H^+$ denotes the set $\{h \in H : h \geq 0\}$.

For such $H$, if $h \in H$, then $h^+ = \sup\{h, 0\}$, $h^- = \sup\{-h, 0\}$, $|h| = h^+ + h^-$. Let $t_{co}$ be the compact-open topology on $C_b(K)$, i.e. $t_{co}$ is the locally convex topology on $C_b(K)$ defined by the family of seminorms $p_K(x) = \sup\{|x(t)| : t \in K\}$, $K$ runs through the compact subsets of $T$. Then, the strict topology $\beta_t$ on $C_b(K)$ is the finest locally convex topology on $C_b(T)$ coinciding with $t_{co}$ on the unit ball $B$ ([2],[6]). From definition of $\beta_t$ immediately follows that if $f$ is a linear mapping from $C_b(T)$ into a LCS (a locally convex Hausdorff space) then $f$ is $\beta_t$-continuous iff its restriction $f|B$ is $t_{co}$-continuous. $M_t(T)$ denotes the continuous dual of $(C_b(T), \beta_t)$. 

49
Results

A well-known theorem of Nachbin (see [5], III.1.2) says that if \( F \subset C(T) \) is absolutely convex and if \( \varepsilon B \subset F \) for some \( \varepsilon > 0 \), then there is a minimal compact set \( K \subset \beta T \) with the property: if \( x \in C(T) \) and if \( x^0 | K = 0 \), then \( x \in F \). We prove the following variant of Nachbin's theorem.

**Theorem 1.** If \( F \neq \{0\} \) is a non-empty norm-closed absolutely convex subset of \( C_0(T) \), then there is a minimal compact set \( S(F) \subset \beta T \) with the property: if \( x \in C_0(T) \) and if \( x^0 | S(F) = 0 \), then \( x \in F \).

**Proof.** Let \( \mathcal{L} = \{ L \subset \beta T : L \text{ is compact such that } (\forall x \in C_0(T))x^0 | L = 0 \Rightarrow x \in F \} \) and let \( \mathcal{M} = \{ x \in C_0(T) : \text{there exists an open } G \ni K \text{ with } x^0 | G = 0 \} \), for compact \( K \subset \beta T \). Then: (1) \( L \in \mathcal{L} \iff \mathcal{M} \subset F \); (2) if \( L_1, L_2 \in \mathcal{L} \), then \( L_1 \cap L_2 \in \mathcal{L} \); (3) \( S(F) = \bigcap \{ L : L \in \mathcal{L} \} \). Proofs of (2) and (3) are the same as in [5], pp. 63-64. One half of (1) is trivial. To obtain the other half, suppose that \( \mathcal{M} \subset F \) and \( x \in C_0(T) \), \( x^0 | L = 0 \). Let \( y_n(t) = x^0(t) \) if \( |x^0(t)| < 1/n \) and \( y_n(t) = x^0(t)/|x^0(t)| \) if \( |x^0(t)| \geq 1/n \). Then \( y_n \in C(\beta T) \) and \( (x^0 - y_n)G_n = 0 \), for \( G_n = \{ t \in \beta T : |x^0(t)| < 1/n \} \). From \( L \subset G_n \) it follows that \( (x^0 - y_n)|T \in F \) for each \( n = 1, 2, \ldots \). Then \( x \in F \), because \( F \) is closed and \( ||y_n|T|| \leq 1/n \).

**Remark 2.** If \( F \) is as in Theorem 1 and if \( F \) is norm-bounded, then \( S(F) = \beta T \). In fact, if \( t \in \beta T \setminus S(F) \), then there is \( x \in C_0(T) \) with \( x^0(t) = 1, x^0|S(F) = 0 \). Hence \( nx \in F \), because \( nx^0|S(F) = 0 (n = 1, 2, \ldots) \), i.e. \( F \) is not norm-bounded.

**Definition 3.** Let \( f \neq 0 \) be a norm-continuous linear mapping from \( C_0(T) \) into an LCS \( E \). The big support of \( f \) is \( bsupp f = S(f^{-1}(0)) \) and the support of \( f \) is \( supp f = bsupp f \cap T \).

**Lemma 4.** If \( f \) is a norm-continuous linear functional on \( C_0(T) \), then \( f \) can be identified, via Alexandroff representation theorem ([6], 5.1) with the unique Baire measure \( \mu \) on the minimal algebra which contains all zero sets from \( T \). It is not difficult to see that \( supp f \neq supp \mu \).

In the light of the preceding remark, next result is not new, but we give a proof which is independent from the measure theory.

**Proposition 5.** Let \( f \in M^+(T) \) and \( f \neq 0 \). Then:

(a) If \( x \in C_0^+(T) \) and \( f(x) = 0 \), then \( x^0|bsupp f = 0 \).

(b) The space \( bsupp f \) with the induced topology satisfies the countable chain condition.

**Proof.** (a) Let \( x^0(s) > 0 \) for some \( s \in bsupp f \). Then there exist an open set \( G \subset \beta T \) and \( r > 0 \) with \( x^0(t) > r \) for all \( t \in G \). We will prove that \( bsupp f \) is contained in \( \beta T \setminus G \), which is impossible because \( s \in bsupp f \cap G \). Let \( y_0 \in C_0(T) \), \( y^0|\beta T \setminus G = 0 \) and \( ||y|| < \varepsilon \). From \( x^0(t) > r(y^0)^{-2}(t)/k \) for all \( t \in \beta T \) and from non-negativity of \( f \) it follows that \( f(y^0) = 0 \). Then \( f(y) = f(y^+) - f(y^-) = 0 \). Hence \( bsupp f \subset \beta T \setminus G \), by the minimality of \( bsupp f \).
(b) Let $S = \text{bsupp} f$ and let the functional $g_0$ on $C^+_b(S)$ be defined by $g_0(x) = f(\bar{x})$, where $\bar{x}$ is any non-negative continuous extension of $x \in C^+_b(S)$ on $\beta T$. The functional $g_0$ is well-defined because each two such extensions coincide on $S$. It is trivial to see that $g_0$ is a non-negative additive functional, and by [1, Chap. II, §2, Prop. 2] there is a non-negative linear functional $g$ on $C_b(S)$ that extends $g_0$. By [4, V.5.5], $g \in M^+(S)$.

Let $\{ G_\alpha : \alpha \in A \}$ be a family of non-empty, pairwise disjoint, open subsets of $S$ and let $t_\alpha \in G_\alpha$. Then there are $x_\alpha \in C^+_b(S)$, $x_\alpha \leq 1$, such that $x_\alpha(t_\alpha) = 1$ and $x_\alpha|S \setminus G_\alpha = 0$. From $0 < \sum_\alpha x_\alpha \leq 1$ on $S$ it follows that $0 < \sum_\alpha g(x_\alpha) \leq g(1)$ for all finite $\Phi \subset A$. Then the set $\{ \alpha \in A : g(x_\alpha) \geq g(1)/n \}$ is finite for each $n \in \mathbb{N}$. Countability of $A$ then follows from the inequality $g(x_\alpha) > 0$ (by (a)).

**Theorem 6.** Let $E$ be a metrizable LCS, let $(U_n)$ be its neighborhood basis of origin consisting of absolutely convex sets with $2U_{n+1} \subset U_n$, and let $f \neq 0$ be a norm-continuous linear mapping from $C_b(T)$ into $E$. Then $f$ is $\beta$-continuous if and only if there are compact sets $L_n \subset T (n \in \mathbb{N})$ with the property that $f(x) \in U_n$, whenever $x \in B^+$ and $x|L_n = 0$. Moreover, $L_n$’s may be chosen such that $\text{supp } f = \text{cl}_T(\bigcup_{n=1}^{\infty} L_n)$.

**Proof.** $\Rightarrow$ The restriction $f|B$ is $t_{co}$-continuous. Then there is an increasing sequence of compact sets $K_n \subset T$ and a decreasing sequence $\varepsilon_n$ of positive numbers with the property: if $x \in B$ and $p_{K_n}(x) < \varepsilon_n$, then $f(x) \in U_n$. We will first prove that $\bigcup_{n=1}^{\infty} K_n \cap \text{bsupp } f \neq \emptyset$. Suppose the contrary. Then, there are $x_n \in B^+$ such that $x_n|K_n = 0$, $x_n|\text{bsupp } f = 1$. There is $u \in B$ such that $f(u) \neq 0$. From $(ux_n)^{\beta}|K_n = 0,$ $(ux_n)^{\beta}|\text{bsupp } f = u^{\beta}|\text{bsupp } f$ it follows that $f(u) = f(ux_n) \in U_n$ for all $n$, which is in contradiction with $f(u) \neq 0$.

Hence, there is $k \in \mathbb{N}$ such that $K_k \cap \text{bsupp } f$ is non-empty for all $n \geq k$. Let $L_k = K_{n+k} \cap \text{bsupp } f$, $\delta_n = \varepsilon_{n+k}$ and let $x \in B^+$, $x|L_k = 0$. If $K_{n+k} \subset G_n = \{ t \in \beta T : x^\beta(t) < \delta_n \}$ then $f(x) \in U_{n+k} \subset U_n$. If $K_{n+k} \not\subset G_n$, then from $L_k \subset G_n$ it follows that there is $y \in B^+$ with $y^\beta|K_{n+k} \cap (T \setminus G_n) = 0$, $y^\beta|\text{bsupp } f = 1$. Since $K_{n+k} = (L_k \cup (K_{n+k} \setminus (T \setminus G_n))) \cup (K_{n+k} \cap (G_n \setminus L_k))$, then $p_{K_{n+k}}(xy) < \delta_n$. From this and from the fact that $x^\beta$ and $(xy)^\beta$ coincide on $\text{bsupp } f$ it follows that $f(xy) \in U_{n+k} \subset U_n$.

For the equality $\text{supp } f = \text{cl}_T(\bigcup_{n=1}^{\infty} L_n)$, only inclusion $\text{bsupp } f \subset \text{cl}_T(\bigcup_{n=1}^{\infty} L_n)$ needs a proof. If $x \in C_b(T)$ and $x^\beta|\text{cl}_T(\bigcup_{n=1}^{\infty} L_n) = 0$, then $(x^\beta/|z|)|L_n = 0$. It follows that $f(z^\beta) = 0$ for all $n$, i.e. $f(z) = 0$. By the minimality of $\text{bsupp } f$, the proof is finished.

$\Leftarrow$ Since $f$ is norm-continuous, we may choose positive numbers $a_n < 1$ so that $4f(a_n B) \subset U_{n+1}$ for each $n$. We will show that $f(V_n \cap B) \subset U_n$, where $V_n$ is the set $\{ x \in C_b(T) : p_{L_{n+2}}(x) < a_n \}$. Let $x \in V_n \cap B$ and let

$$y^+(t) = \begin{cases} x^+(t), & \text{if } x^+(t) < a_n, \\ a_n, & \text{if } x^+(t) \geq a_n. \end{cases}$$

$$y^-(t) = \begin{cases} x^-(t), & \text{if } x^-(t) < a_n, \\ a_n, & \text{if } x^-(t) \geq a_n. \end{cases}$$

Then $x^\pm - y^\pm \in B^+$, $y^\pm \in a_n B$, $(x^\pm - y^\pm)|L_{n+2} = 0$, and so $f(x^\pm - y^\pm) \in U_{n+2}$ and $4f(y^\pm) \in U_{n+1}$. From this it follows that $f(x) = f(x^+) - f(x^-) = (f(x^+) - f(y^+))$. 


$y^+ + f(y^+) = (f(x^- - y^-) + f(y^-)) \in 2U_{n+2} + \frac{1}{2}U_{n+1} \subset U_n$, which completes the proof of the theorem.

**Remark 7.** If $E$ is a non-metrizable LCS, then supp $f$ need not be the closure of a $\sigma$-compact subset of $T$, as the following example shows. Let $T$ be the discrete space, card $T = c$. Then $T$ is a realcompact ([3,11.D.(a)]) metrizable space. By [5,III.3.5 and III.4.3] $E = (C(T), t_{\infty})$ is a bornological barrelled complete LCS. The inclusion mapping $i$ from $C_0(T)$ into $E$ is $\beta_\tau$-continuous and from remark 2 it follows that supp $i = T$. Each compact subset of $T$ is finite, hence $\text{cl}_T \bigcup_{n=1}^\infty L_n = \bigcup_{n=1}^\infty L_n \neq T$ for all compact $L_n$'s.

**Remark 8.** In the proof of theorem 6 we showed also that $\text{cl}_T \bigcup_{n=1}^\infty L_n$ is dense in bsupp $f$. Hence, supp $f$ is dense in bsupp $f$.

The next lemma is well-known and we omit the proof.

**Lemma 9.** Let $f_n \in M_1^+(T)$, $\|f_n\| \leq 1$ and let $f = \sum_{n=1}^\infty 2^{-n} f_n$. Then $f \in M_1^+(T)$, $\|f\| \leq 1$ and supp $f = \text{cl}_T \bigcup_{n=1}^\infty \text{supp}(f_n)$.

**Theorem 10.** Let $f \neq 0$ be a weakly continuous linear mapping from $(C_0(T), \beta_\tau)$ into an LCS $E$. Then:

(a) If $F \subset E'$ is weakly-*dense in $E'$, then $\text{supp} f = \text{cl}_T \bigcup_{w \in F} \text{supp}(wf)$.

(b) $\text{supp} f$ is dense in bsupp $f$.

(c) If $E'$ is weakly-*separable, then there is $\mu \in M_1^+(T)$, $\|\mu\| \leq 1$ such that supp $f = \text{supp} \mu$.

(d) If $E'$ is weakly-*separable, then $\text{supp} f$ satisfies the countable chain condition.

**Proof.** (a) From $f^{-1}(0) \subset (wf)^{-1}(0)$ and the theorem 1 it follows that bsupp$(wf) \subset \text{bsupp} f$ for each $w \in F$. On the other hand, if $x^0 \mid \text{cl}_T \bigcup_{w \in F} \text{supp}(wf) \neq 0$, then by the remark 8, $x^0 \mid \text{bsupp}(wf) = 0$. This gives that $wf(x) = 0$ for all $w \in F$. Hence $f(x) = 0$. From the theorem 1 it follows that $\text{bsupp} f \subset \text{cl}_T \bigcup_{w \in F} \text{supp}(wf)$.

(b) Immediately follows from (a).

(c) Let $\{ w_n : n \in \mathbb{N} \}$ be weakly-*dense in $E'$. Then $\text{supp} f = \text{cl}_T \bigcup_{n=1}^\infty \text{supp}(w_nf)$, by (a). If $\mu = \sum_{n \in N_1} 2^{-n} (|w_n f|/|w_n f|)$, where $N_1 = \{ n : w_n f \neq 0 \}$, then $\mu \in M_1^+(T)$ and $\text{supp} f = \text{supp} \mu$, by the lemma 9.

(d) From (b) and (c) it follows that $\text{supp} f$ is dense in bsupp $\mu$. Then, by the proposition 5, from [3, 2.J.(d)] it follows that $\text{supp} f$ satisfies the countable chain condition.

**Remark 11.** Assertions in (c), (d) are not true if we omit the separability condition, even if $E$ is a Banach space. For example, let $T$ be the compact space $\beta\mathbb{N} \setminus \mathbb{N}$ and let $f$ be the identity mapping on $(C(T), \| \|)$. Then $\text{supp} f = T$, but $T$ does not satisfy the countable chain condition [3, 3.6. Example 2].

Applications of our results will be given in a subsequent paper.
REFERENCES


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