GENERALIZED EIGENVECTOR EXPANSION
FOR WEAKLY PERTURBED DISCRETE OPERATORS

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Abstract. In this paper we consider the expansion theorem in generalized eigenvectors of the operator $A = L + T$, where $L$ is a discrete, positive selfadjoint operator in a separable Hilbert space, and $T$ is a closed operator which is subordinated to $L$ in a certain sense.

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$ and let $L$ be a discrete, positive selfadjoint operator on $\mathcal{H}$. Vector $x \neq 0$ is a generalized eigenvector (for the eigenvalue $\lambda$) if for some $k \geq 1$ $(\lambda - L)^k x = 0$. Denote by $N(\cdot)$ the eigenvalue distribution function of $L$. Let $\mathcal{D}(L)$ and $\mathcal{D}(T)$ denote the domain of the operators $L$ and $T$, respectively.

In this paper we consider the expansion theorem for the operator $A = L + T$, where $T$ is a closed operator which is subordinated to $L$ in a certain sense.

In the case when $T$ is a bounded operator, $L = L^*$ is a discrete operator and $\lambda_{n+1}(L) - \lambda_n(L) \to \infty$ $(n \to \infty)$ the problem was solved in [3].

Theorem 1. Suppose that $T$ is a closed operator on $\mathcal{H}$, $L = L^*$ is a positive discrete operator, $\mathcal{D}(L) \subset \mathcal{D}(T)$, $A = L + T$,
$$||Tx|| \leq C||L^0 x||, \quad x \in \mathcal{D}(L),$$
and numbers $\alpha$ and $\beta$ satisfy one of the following two conditions:

a) $0 < \beta < 1$, $0 < \alpha < \frac{2}{3}(1 - \beta)$ and $N(t) = C_0 t^\alpha (1 + o(1))$ ($t \to +\infty$);

b) $0 < \beta < 1$, $0 < \alpha < 1 - \beta$ and $N(t) = C_0 t^\alpha (1 + O(t^{-\delta}))$, $\alpha < \delta < 1$ ($t \to +\infty$).

Then for every $f \in \mathcal{D}(L)$ we have
$$f = \sum_{k=1}^{\infty} \left( \sum_{s=1}^{n_k} c_{ks} x_{ks} \right),$$
where $x_{ks}$ are generalized eigenvectors of $A$ and $c_{ks} \in \mathbb{C}$.

Proof. Suppose that $\{e_n\}_{n=1}^{\infty}$ is the system of eigenvectors of $L$ ($Le_n = \lambda_n e_n$). Since $L = L^*$, $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Then
$$(L - \lambda)^{-1} = \sum_{n=1}^{\infty} \frac{\langle \cdot, e_n \rangle e_n}{\lambda_n - \lambda}.$$
and

\[ T(L - \lambda)^{-1} = \sum_{n=1}^{\infty} \frac{\langle \cdot, e_n \rangle T e_n}{\lambda_n - \lambda}. \]

From (1) and (3), applying Cauchy’s inequality, we conclude that

\[ \|T(L - \lambda)^{-1}\| \leq C^{1/2} \left( \sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} \right)^{1/2}. \]  

By the following Lemma, the righthandside of this inequality tends to zero if \( \lambda \) belongs to a certain sequence of circles with radii tending to infinity.

**Lemma.** If either of the conditions a) and b) of the Theorem 1 is satisfied, then there exists a sequence of circles \( \Gamma_k = \{ \lambda : |\lambda| = r_k \} \), \( \lim_{k \to \infty} r_k = \infty \), such that

\[ \lim_{k \to \infty} \max_{\lambda \in \Gamma_k} \left( \sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} \right) = 0. \]

Since \( \lim_{n \to \infty} \max_{\lambda \in \Gamma_k} ||T(\lambda - L)^{-1}|| = 0 \) (follows from (4) and the Lemma), it follows from \( (\lambda - A)^{-1} = (\lambda - L)^{-1}(I - T(\lambda - L)^{-1})^{-1} \) that the operator \( A \) is discrete and

\[ \lim_{k \to \infty} \max_{\lambda \in \Gamma_k} ||(\lambda - A)^{-1}|| = 0. \]

From (6) and Naymark’s theorem [4] we obtain the relation (2), for all \( f \in D(L) \), where \( z_{ks} \), \( s = 1, 2, \ldots, n_k \), are the generalized eigenvectors corresponding to eigenvalues lying in the ring \( \{ \lambda : r_k < |\lambda| < r_{k+1} \} \).

**Remark.** In the case when in each interval \( I \) of the fixed length \( l \) the number of eigenvalues \( \lambda \) of \( A \) with property \( \Re \lambda \in I \) is uniformly bounded, the Riesz basis property of the generalized eigenvectors system was proved in [1] (under some additional conditions).

**Proof of the Lemma.** Case a). It follows from \( N(t) = C_0 t^a (1 + o(1)) \) that \( \lambda_n = C_0 t^{-1/a} n^{1/a} (1 + o(1)) \). Let \( q \) be a real number such that

\[ 0 < \alpha q < C_0^{-1/a}. \]

Denote by \( S \) the set of natural numbers \( n \) such that \( \lambda_{n+1} - \lambda_n \geq q n^{1/a-1} \). Suppose that \( S \) is finite, i.e. \( S = \{n_1, n_2, \ldots, n_s \} \). Then we have \( \lambda_{n+1} - \lambda_n < q n^{1/a-1} \) for all \( n > n_s + 1 \) and

\[ \lambda_{n+1} - \lambda_{n+1} < q \sum_{n=n_s+1}^{N} n^{1/a-1} < q \int_{n_s+1}^{N+1} x^{1/a-1} dx = \alpha q [(N+1)^{1/a} - (n_s+1)^{1/a}], \]

i.e.

\[ \frac{\lambda_{n+1} - \lambda_{n+1}}{N^{1/a}} \leq \alpha q \frac{(N+1)^{1/a} - (n_s+1)^{1/a}}{N^{1/a}} \]

for each \( N > n_s \). When \( N \to \infty \) we obtain \( C_0^{-1/a} \leq \alpha q \), i.e. a contradiction with (7). So, it follows that \( S \) is an infinite set.
Let $\Gamma_\nu = \{ \lambda : |\lambda| = r_\nu = \frac{1}{2}(\lambda_{n+1} + \lambda_n) \}$. We will prove now the relation (5). If $\lambda \in \Gamma_k$, then

$$
\sum_{\nu=1}^{\infty} \frac{\lambda_{n+1}^2}{(r_k - \lambda_{n+1})^2} \leq \sum_{\nu=1}^{\infty} \frac{\lambda_{n+1}^2}{(r_k - \lambda_{n+1})^2} + \sum_{\nu=n+1}^{\infty} \frac{\lambda_{n+1}^2}{(r_k - \lambda_{n+1})^2} + \frac{\lambda_{n+1}^2}{(r_k - \lambda_{n+1})^2}.
$$

As we have $0 < \alpha < \frac{2}{3}(1 - \beta)$, by direct computation we get

$$
\lim_{k \to \infty} \left[ \frac{\lambda_{n+1}^2}{(r_k - \lambda_{n+1})^2} + \frac{\lambda_{n+1}^2}{(r_k - \lambda_{n+1})^2} \right] = 0. \quad (8)
$$

Since the function $\varphi(x) = x^\beta/(r_k - x)$ is nondecreasing on $[0,r_k)$, we obtain

$$
\sum_{\nu=1}^{\infty} \frac{\lambda_{n+1}^2}{(r_k - \lambda_{n+1})^2} \leq \text{const.} \cdot n_k \frac{\lambda_{n+1}^2}{(r_k - \lambda_{n+1})^2} \leq \frac{\text{const}}{n_k^{\frac{2}{3} - \frac{\alpha}{2}}} \to 0 \quad (k \to \infty). \quad (9)
$$

Since

$$
\sum_{\nu=n_k+1}^{\infty} \frac{\lambda_{n+1}^2}{(r_k - \lambda_{n+1})^2} = \int_{\lambda_{n+1}}^{\infty} \frac{\varphi(t)}{(r_k - t)^2} dN(t)
$$

it is enough to prove that

$$
\lim_{k \to \infty} \int_{\lambda_{n+1}}^{\infty} \frac{\varphi(t)}{(r_k - t)^2} dt = 0. \quad (10)
$$

The function $G(x) = \int_{x}^\infty [(\beta - 1)u - \beta]/(u - 1)^3 du \quad (x > 1)$ has the following asymptotic behavior in the neighborhood of $x = 1$: $G(x) \sim \frac{2}{x(1 - x)^2}$. Then

$$
\int_{\lambda_{n+1}}^{\infty} \frac{\varphi(t)}{(r_k - t)^2} dt = 2r_k^{\alpha+2\beta-2}G(c_k) \sim \frac{r_k^{\alpha+2\beta}}{(\lambda_{n+1} - r_k)} \to 0 \quad (k \to \infty),
$$

where $c_k = \lambda_{n+1}/r_k \quad (\alpha > 1)$. From (8), (9) and (10) we obtain (5).

Case b). It follows from b) that

$$
\lambda_n = C_0^{-1/\alpha} n^{1/\alpha} (1 + O(n^{-\delta/\alpha})). \quad (11)
$$

Let $\mu_n = C_0^{-1/\alpha} n^{1/\alpha}$ and $\Gamma_n = \{ \lambda : |\lambda| = r_n = \frac{1}{2}(\mu_n + \mu_{n+1}) \}$. From (11) we get

$$
\sup_{n,p} \left| \frac{\lambda_{n+p} - \mu_{n+p}}{r_n - \lambda_n} \right| < \infty. \quad (12)
$$
If $\lambda \in \Gamma_n$, then from (12) we obtain

$$\sum_{\nu=1}^{\infty} \frac{\lambda_{\nu}^{2\beta}}{|\lambda - \lambda_{\nu}|^2} \leq \text{const} \sum_{\nu=1}^{\infty} \frac{\mu_{\nu}^{2\beta}}{(\tau_n - \mu_{\nu})^2}$$

As in the case a) it can be proved that

$$\sum_{\nu=1}^{\infty} \frac{\mu_{\nu}^{2\beta}}{(\tau_n - \mu_{\nu})^2} \to 0 \quad (n \to \infty)$$

for $0 < \alpha < 1 - \beta$. The Lemma is proved. ■

**Example.** Suppose $m$, $n$, and $r$ are integers, $m \geq 1$, $n \geq 2$, $0 < r < m$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary, $L$ is a formal selfadjoint elliptic differential expression

$$L = (-1)^{m/2} \sum_{|k|=m} a_k(x)D^k$$

with smooth coefficients and $T$ is a linear differential expression

$$T = \sum_{|k| \leq r} b_k(x)D^k$$

with smooth complex functions $b_k$. Let $A : \mathcal{D}(A) \to L^2(\Omega)$ ($\mathcal{D}(A) = W^m_2 \cup \overset{n}{\wedge} W^{m/2}_2$) be a differential operator defined by $A = L + T$. Then we get

**Theorem 2.** If $n/m < \frac{2}{3}(1 - r/m)$, the for $f \in \mathcal{D}(A)$ the expansion theorem in generalized eigenvectors of the operator $A$ holds.

**Proof.** The statement of the theorem is obtained from Theorem 1 for $\alpha = n/m$, $\beta = r/m$ (see [2]). ■

**References**


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