ON THE CONVERGENCE OF A MULTICOMPONENT THREE LEVEL ALTERNATING DIRECTION DIFFERENCE SCHEME

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Abstract. A multicomponent alternating direction finite difference scheme for solving the wave equation with several variables is considered. Its stability and convergence are investigated in the case when the solution of the initial-boundary value problem belongs to a Sobolev space.

We consider the first initial-boundary value problem (IBVP) for the wave equation

\[ \frac{\partial^2 u}{\partial t^2} - \Delta u + f, \quad (x, t) \in Q = \Omega \times (0, T) = (0, 1)^n \times (0, T), \]

\[ u(x, 0) = u_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = u_1(x), \quad x \in \Omega, \quad (1) \]

\[ u(x, t) = 0, \quad x \in \Gamma = \partial \Omega, \quad t \in (0, T). \]

We assume that the generalized solution of IBVP (1) belongs to the Sobolev space \( W^n_2(Q), \ s \geq 2 \) [4]. In this case there exists a trace \( u|_{t=T} \in W^{n-1/2}_2(\Gamma) \subset L_2(\Omega) \).

We also assume that the solution \( u \) can be oddly extended in space variables outside the domain \( \Omega \), preserving the Sobolev class.

Let \( \Phi \) be a uniform mesh in \( \overline{\Omega} \) with the step size \( h \). Let us set \( \omega = \overline{\Phi} \cap \Omega, \gamma = \overline{\Phi} \setminus \omega \) and \( \omega_i = \omega \cup \{x = (x_1, \ldots, x_n) \in \gamma | x_i = 0\} \). Let \( \overline{\Phi} \) be a uniform mesh on \([-\tau/2, T]\) with the step size \( \tau \) and \( \theta = \overline{\Phi} \cap (0, T) \). Finally, let \( Q_{h, \tau} = \overline{\Phi} \times \overline{\Theta} \). For a function \( v \) defined on the mesh \( Q_{h, \tau} \), we introduce the finite-difference operators \( v_x, \ v_r, \ v_t \) and \( v_\tau \) in a usual manner [5]. Let us denote \( v = v(x, t), \ \tilde{v} = v(x, t + \tau) \) and \( \hat{v} = v(x, t - \tau) \).

Let \( H_h \) be the set of discrete functions defined on the mesh \( \Phi \), which vanish on \( \gamma \). Let us denote

\[ A_i v = \begin{cases} -v_{x_i}, & x \in \omega, \\ 0, & x \in \gamma \end{cases} \quad \text{and} \quad A v = \sum_{i=1}^n A_i v. \]

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The unit operator on $H_h$ will be denoted by $I$.

We introduce the following discrete inner product

$$(v, w)_\omega = h^n \sum_{x \in \omega} v(x) w(x)$$

and norms

$$||v||_\omega = (v, v)_\omega^{1/2} = \left( h^n \sum_{x \in \omega} v^2(x) \right)^{1/2} \quad \text{and} \quad ||v||_{\omega_i} = \left( h^n \sum_{x \in \omega_i} v^2(x) \right)^{1/2}.$$ 

For a linear, selfadjoint and positive operator $L$ on $H_h$ with $||v||_L$ we denote so called "energy" norm

$$||v||_L = (L v, v)_\omega^{1/2}.$$ 

In particular

$$||v||_{A_i} = (A_i v, v)_\omega^{1/2} = ||v_i||_{\omega_i}.$$ 

With $T_i$ and $T^*_i$ we denote the Steklov averaging operators in space variables $x_i$ and time variable $t$ (see [2])

$$T_i f(x, t) = \frac{1}{h} \int_{x_{i-1} - h/2}^{x_{i+1} + h/2} f(x_1, \ldots, x'_i, \ldots, x_n, t) \, dx'_i,$$

$$T^*_i f(x, t) = \frac{1}{\tau} \int_{t - \tau/2}^{t + \tau/2} f(x_1, \ldots, x_n, t') \, dt'.$$

Finally, $C$ will stand for a positive generic constant, independent of $h$ and $\tau$.

We approximate IBVP (1) with the following alternating direction finite-difference scheme (FDS) [1]

$$(I + \sigma^2 A_i) v^t_{\Omega u} + \sum_{j=1}^n A_j v^t = \tilde{f} \equiv T_1 \cdots T_n T_i f, \quad t \in \theta,$$

$$v^t|_{t=\pm \tau/2} = T_1 \cdots T_n (u_0 \mp 0.5 \tau u_1), \quad i = 1, 2, \ldots, n \quad \tag{2}$$

where $\sigma$ is a free weight parameter. FDS (2) represents a system of $n$ unknown mesh functions $v^t$. They can be determined parallelly, contrary to the other variant of the alternating direction method, such as the factorized scheme

$$(I + \sigma^2 A_1) \cdots (I + \sigma^2 A_n) v_{\Omega u} + Av = f.$$ 

The errors defined as $z^t = T_1 \cdots T_n u - v^t$ satisfy the FDS

$$(I + \sigma^2 A_i) z^t_{\Omega u} + \sum_{j=1}^n A_j z^j = \varphi^t, \quad t \in \theta,$$

$$z^t|_{t=\pm \tau/2} = \eta, \quad 0.5 (z^t + \hat{z}^t)|_{t=\pm \tau/2} = \xi, \quad i = 1, 2, \ldots, n \quad \tag{3}$$
where

$$
\varphi^i = T_1 \cdots T_n \left[ (u_t - T_i \frac{\partial^2 u}{\partial x^2} + \sum_{j=1}^n \left( T_i \frac{\partial^2 u}{\partial x_j^2} - u_{x_j} z_j \right) - \sigma \tau u_{x_j} z_j, \right] ;
$$

$$
\eta = T_1 \cdots T_n \left( T_i \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} \right) \bigg|_{t=0},
$$

$$
\xi = 0.5 T_1 \cdots T_n \left( u_{|t=-\tau/2} - 2 u_{|t=0} + u_{|t=\tau/2} \right).
$$

To prove the stability and the convergence of the FDS (2) we represent equation (3) in matrix form

$$
(I + \sigma \tau^2 A) z_{t\tau} + E A z = f, \quad t \in \theta,
$$

$$
z_{|t=-\tau/2} = b, \quad 0.5 (z + \tilde{z})_{|t=-\tau/2} = d, \quad (4)
$$

where $z = (z^1, \ldots, z^n)^T$, $f = (\varphi^1, \ldots, \varphi^n)^T$, $b = (\eta, \ldots, \eta)^T$, $d = (\xi, \ldots, \xi)^T$, $I = \text{diag}(I, \ldots, I)$, $A = \text{diag}(A_1, \ldots, A_n)$ and

$$
E = \begin{pmatrix}
I & I & \cdots & I \\
I & I & \cdots & I \\
\vdots & \vdots & \ddots & \vdots \\
I & I & \cdots & I
\end{pmatrix}.
$$

Let us also define the inner product and norm of vector–functions

$$
(z, w) = \sum_{i=1}^n (z^i, w^i), \quad ||z|| = (z, z)^{1/2}.
$$

Applying $A$ to (4) we obtain a FDS in canonical form (see [5])

$$
C z_{t\tau} + A z = g, \quad (5)
$$

where $A = E A = A^* \geq 0$, $C = A + \sigma \tau^2 A^2 = A^* > 0$ and $g = A f$. According to Samarski’s stability theory [5] the FDS (5) is stable when

$$
C - 0.25 \tau^2 A > 0.
$$

For $\sigma \geq n/4$ we have

$$
\left( (C - 0.25 \tau^2 A) z, z \right) = (A z, z) + \sigma \tau^2 (A z, A z) - 0.25 \tau^2 (E A z, A z)
$$

$$
= \sum_{i=1}^n (A_i z^i, z^i) + \sigma \tau^2 \sum_{i=1}^n (A_i z^i, A_i z^i) - 0.25 \tau^2 \left\| \sum_{i=1}^n A_i z^i \right\|^2_0
$$

$$
= \sum_{i=1}^n \left\| z^i \right\|^2_{A_i} + \left( \sigma - n/4 \right) \tau^2 \sum_{i=1}^n \left\| A_i z^i \right\|^2_0 + 0.25 \tau^2 \sum_{i=2}^n \sum_{j=1}^{i-1} \left\| A_i z^i - A_j z^j \right\|^2_0
$$

$$
\geq \sum_{i=1}^n \left\| z^i \right\|^2_{A_i} = ||z||^2_A,
$$

On the convergence of a multicomponent three level FDS
which means that
\[
\mathbf{C} - 0.25 \tau^2 \mathbf{A} \geq \Lambda > 0,
\]
and, consequently, FDS (5) is stable.

Using energy method, multiplying (5) by \( \hat{z} - \hat{z} \), we obtain a priori estimate
\[
\max_{t \in \theta} N(z) \leq N(z)|_{t=-\tau/2} + \tau \sum_{t \in \theta} \|g\|_{\mathbf{C} - 0.25 \tau^2 \mathbf{A}}^{-1},
\]
where
\[
N^2(z) = \left\| z \right\|_{\mathbf{C} - 0.25 \tau^2 \mathbf{A}}^2 + \frac{1}{2} \left\| \frac{z + \hat{z}}{2} \right\|_{\mathbf{A}}^2.
\]
Other standard a priori estimates (see [3]) do not hold because operators \( \mathbf{A} \) and \( \mathbf{C} \) do not commute.

Further
\[
N^2(z) = \left\| z \right\|_{\mathbf{C} - 0.25 \tau^2 \mathbf{A}}^2 + \left\| \frac{z + \hat{z}}{2} \right\|_{\mathbf{A}}^2 + \left\| \Lambda \frac{z + \hat{z}}{2} \right\|_{E}^2
\]
\[
= \sum_{i=1}^{n} \| z_i \|_{A_i}^2 + \sum_{i=1}^{n} A_i \| z_i + \hat{z}_i \|^2_{\omega} + \| g \|_{(\mathbf{C} - 0.25 \tau^2 \mathbf{A})^{-1}} \leq \| f \|_{\Lambda} = \left( \sum_{i=1}^{n} \| \varphi_i^j \|_{A_i}^2 \right)^{1/2}.
\]

From here and (6), for \( \tau \sim h \) (i.e. \( C_1 h \leq \tau \leq C_2 h \)), we obtain
\[
\max_{t \in \theta} \| z \|_2 \leq \sum_{i=1}^{n} \left( \| \eta_i \|_{\omega_i} + \| \xi_i \|_{\omega_i} + \tau \sum_{t \in \theta} \| \varphi_i^j \|_{\omega_i} \right).
\]

To prove the convergence of FDS (2) we must estimate the terms \( \varphi_i^j, \eta_i, \) and \( \xi_i \). That can be done using the Bramble–Hilbert lemma, in the same way as in [2]. For \( \tau \sim h \) in such a way we obtain the following convergence rate estimate
\[
\max_{t \in \theta} \| z \|_2 \leq C h^{s-3} \| u \|_{W_2^s(Q)}, \quad 3 \leq s \leq 5.
\]

Remark. In some cases the assumption \( \tau \sim h \) can be omitted. For example, for \( s = 5 \) terms \( \varphi_i^j, \eta_i, \) and \( \xi_i \), can be represented in integral form, wherefrom directly follows
\[
\max_{t \in \theta} \| z \|_2 \leq C (h^2 + \tau^2) \| u \|_{W_2^s(Q)}.
\]

Another group of convergence rate estimates can be obtained in the following way. Applying \( A_t (I + \sigma \tau A_t)^{-1} \) to (3), after summation on \( i \) we obtain
\[
z_{tt} + Az = \psi, \quad t \in \theta,
\]
\[
z_{t}\big|_{t=-\tau/2} = \eta, \quad 0.5 (z + \hat{z})\big|_{t=-\tau/2} = \xi,
\]

where
\[
N^2(z) = \left\| z \right\|_{\mathbf{C} - 0.25 \tau^2 \mathbf{A}}^2 + \frac{1}{2} \left\| \frac{z + \hat{z}}{2} \right\|_{\mathbf{A}}^2.
\]
where
\[ z = A^{-1} \sum_{i=1}^{n} A_i z_i, \quad A = \sum_{i=1}^{n} A_i (I + \sigma \tau A_i)^{-1}, \quad \psi = A^{-1} \sum_{i=1}^{n} A_i \varphi_i. \]

For \( \sigma \geq n/[4 (1 - \alpha)] \), \( 0 < \alpha < 1 \), we have \( 0 < \alpha I \leq I - 0.25 \tau^2 A \leq I \), so the FDS (9) is absolutely stable.

The operators \( A \) and \( \Lambda \) satisfy the relations \( I \leq (I - 0.25 \tau^2 A)^{-1} \leq \alpha^{-1} I \)
and \( \Lambda \leq A \). In the case when \( \tau \sim h \) we also have \( \beta A \leq A \), \( 0 < \beta < 1 \). Using these relations and the energy method [5] we obtain the a priori estimate
\[
\max_{t \in \theta} \|z\|_1 \equiv \max_{t \in \theta} \left( \|z\|_\omega^2 + \frac{z + \bar{z}}{2} \right)^{1/2} \\
\leq C \left( \|\eta\|_\omega + \sum_{i=1}^{n} \|\xi_i\|_\omega + \tau \sum_{t \in \theta} \|\varphi^i\|_\omega \right) .
\] (10)

Similarly, applying operator \( A^{k-1} \) (\( k = 2, 3, \ldots \)) to (9) and repeating the same procedure, we obtain
\[
\max_{t \in \theta} \|z\|_k \equiv \max_{t \in \theta} \left( \|z\|_{A^{k-1}}^2 + \frac{z + \bar{z}}{2} \right)^{1/2} \\
\leq C \left( \|\eta\|_{A^{k-1}} + \|\xi\|_{A^{k}} + \tau \sum_{t \in \theta} \|\varphi^i\|_{A^{k-1}} \right) .
\] (11)

In such a way, the problem of deriving the convergence rate estimate for the FDS (9), or (2), is now reduced to estimation of the right hand side terms in (10) and (11). Using the Bramble–Hilbert lemma, in the same manner as in the previous case, from (10–11) we obtain
\[
\max_{t \in \theta} \|z\|_k \leq CH^{s-k-1} \|u\|_{W^2(Q)} , \quad k + 1 \leq s \leq k + 3; \quad k = 1, 2, \ldots
\] (12)

Analogous results for the parabolic case are obtained in [3].

REFERENCES

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