TOTALIZATION OF THE MONTGOMERY IDENTITY

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Abstract. The aim of this note is to define the total value of the Riemann integral that can be used to generalize the well-known Montgomery identity.

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1. Introduction

Let \([a, b]\) be some compact interval in \(\mathbb{R}\). It is an old result that for any function \(F : [a, b] \mapsto \mathbb{R}\), which is differentiable on \([a, b]\), and its derivative \(f\) is Riemann integrable on \([a, b]\), the Montgomery identity holds (see [1])

\[
F(t) = \frac{1}{b-a} \int_a^b F(x) \, dx + \int_a^b P(t, x) f(x) \, dx,
\]

where the Peano kernel \(P(t, x)\) is as follows

\[
P(t, x) = \begin{cases} 
\frac{x-a}{b-a}, & a \leq x < t \\
\frac{x-b}{b-a}, & t < x \leq b 
\end{cases}.
\]

The aim of this note is to define the total value of the Riemann integral that can be used to extend the above mentioned result to any real-valued function \(F\) defined and differentiable on \([a, b] \setminus E\), where \(E\) is a certain subset of \([a, b]\) at whose points \(F\) can take values \(\pm \infty\) or not be defined at all. Unless otherwise stated in what follows, we assume that the endpoints of \([a, b]\) do not belong to \(E\). Now, define point functions \(F_{ex} : [a, b] \mapsto \mathbb{R}\) and \(D_{ex}F : [a, b] \mapsto \mathbb{R}\) by extending \(F\) and its derivative \(f\) from \([a, b] \setminus E\) to \(E\) by \(F_{ex}(x) = 0\) and \(D_{ex}F(x) = 0\) for \(x \in E\) (see [3]), so that

\[
F_{ex}(x) = \begin{cases} 
F(x), & x \in [a, b] \setminus E \\
0, & x \in E
\end{cases}
\]

and

\[
D_{ex}F(x) = \begin{cases} 
f(x), & x \in [a, b] \setminus E \\
0, & x \in E
\end{cases}.
\]
2. Preliminaries

A partition \( P[a,b] \) of \([a,b] \in \mathbb{R}\) is a finite set (collection) of interval-point pairs \( \{([a_i, b_i], x_i) \mid i = 1, \ldots, \nu \} \), such that the subintervals \([a_i, b_i]\) are non-overlapping, \( \bigcup_{i \leq \nu} [a_i, b_i] = [a, b] \) and \( x_i \in [a_i, b_i] \). The points \( \{x_i\}_{i \leq \nu} \) are the tags of \( P[a,b] \). \([1]\). It is evident that a given partition of \([a,b]\) can be tagged in infinitely many ways by choosing different points as tags. If \( E \) is a subset of \([a,b]\), then the restriction of \( P[a,b] \) to \( E \) is a finite collection of \( \{([a_i, b_i], x_i) \in P[a,b] \mid a_i, b_i \} \) such that each pair of sets \([a_i, b_i]\) and \( E \) intersects in at least one point and all \( x_i \) are tagged in \( E \). In symbols, \( P[a,b]|_E = \{([a_i, b_i], x_i) \in P[a,b] \mid a_i, b_i \cap E \neq \emptyset \text{ and } x_i \in E\} \). Let \( \mathcal{P}[a,b] \) be the family of all partitions \( P[a,b] \) of \([a,b]\). Given \( \delta : [a,b] \mapsto \mathbb{R}_+ \), named a gauge, a point-interval pair \( ([a_i, b_i], x_i) \) is called \( \delta \)-fine if \([a_i, b_i] \subseteq (x_i - \delta(x_i), x_i + \delta(x_i))\).

The collection \( \mathcal{I}(a,b) \) is the family of compact subintervals \( I \) of \([a,b]\). The Lebesgue measure of the interval \( I \) is denoted by \(|I|\). Any real-valued function defined on \( \mathcal{I}(a,b) \) is an interval function. For a function \( f : [a,b] \mapsto \mathbb{R} \), the associated interval function of \( f \) is an interval function \( F : \mathcal{I}(a,b) \mapsto \mathbb{R} \), again denoted by \( f \). If \( f \equiv 0 \) on \([a,b]\) then its associated interval function is trivial. The function \( f \) is said to be a null function on \([a,b]\) if the set \( \{x \in [a,b] \mid f(x) \neq 0\} \) is a set of Lebesgue measure zero, see Definition 2.4 in \([2]\).

In what follows we will use the following notations: \( F(I) = F(v) - F(u) \), where \( u \) and \( v \) are the endpoints of \( I \),

\[
\Xi_f(P[a,b]) = \sum_{i \leq \nu} f(x_i) \cdot [a_i, b_i]][\text{ and } \Sigma \varphi_f(P[a,b]) = \sum_{i \leq \nu} \varphi([a_i, b_i]) \cdot F([a_i, b_i])].
\]

\textbf{Definition 2.1.} For \( E \subseteq [a,b] \) let \( D_{e,x}F(x) : [a,b] \mapsto \mathbb{R} \) be defined by \((13)\).

Then, the point function \( f \) is said to be Riemann integrable to a real number \( A \) on \([a,b]\) if for every \( \varepsilon > 0 \) there exists a gauge \( \delta_\varepsilon \equiv \delta_\varepsilon = \inf \{\delta(x) \mid x \in [a,b]\} > 0 \), such that \( |\Xi_{D_{e,x}F}(P[a,b]) - A| < \varepsilon \), whenever \( P[a,b]|_E \subset P[a,b] \) and \( P[a,b] \in \mathcal{P}[a,b] \) is a \( \delta_\varepsilon \)-fine partition. In symbols, \( A = \varphi P_j^b a f(x) \, dx \).

\textbf{Definition 2.2.} Let \( \varphi : \mathcal{I}(a,b) \mapsto \mathbb{R} \) and \( E \subseteq [a,b] \). A function \( f : [a,b] \mapsto \mathbb{R} \) is the limit of \( \varphi \) on \([a,b]\) \( \setminus E \) if for every \( \varepsilon > 0 \) there exists a gauge \( \delta_\varepsilon \equiv \delta_\varepsilon \), such that

\[
|\varphi([a_i, b_i]) - f(x_i)| < \varepsilon,
\]

whenever \( ([a_i, b_i], x_i) \in P[a,b] \setminus P[a,b]|_E \) and \( P[a,b] \in \mathcal{P}[a,b] \) is a \( \delta_\varepsilon \)-fine partition.

3. Main results

For a given pair of real-valued point functions \( f \) and \( g \) with the primitives \( F \) and \( G \), respectively, let \( E \subset [a,b] \) be a set of points, of Lebesgue measure zero,
at which they can take values $\pm \infty$ or not be defined at all and $\Delta \phi : I ([a,b]) \mapsto \mathbb{R}$ be an interval function defined by
\begin{equation}
(3.1) \quad \Delta \phi (I) = D_{ex} (FG)(I) - (\varphi (I) D_{ex} G(I) + \gamma (I) D_{ex} F(I)),
\end{equation}
where $D_{ex} (FG)(I)$ denotes an interval function associated with the product of the point functions $D_{ex} F$ and $D_{ex} G$, $\varphi (I) = F_{ex} (I) / |I|$ and $\gamma (I) = G_{ex} (I) / |I|$.

Given $\varepsilon > 0$, we can define a set $\Gamma_\varepsilon$ as follows
\begin{equation}
(3.2) \quad \Gamma_\varepsilon = \{(x, I) \mid x \in [a,b] \text{ is a point of } I \text{ and } |\Delta \phi (I)| < \varepsilon\}.
\end{equation}

From the collection of all $\delta_\varepsilon$-fine point-interval pairs $(x, I) \in \Gamma_\varepsilon$, a subset of $[a,b]$ may be obtained, as follows.

**Definition 3.1.** The set $\{x \in [a,b] \mid$ for every $\varepsilon > 0$ there exists a $\delta_\varepsilon$-fine $(x, I) \in \Gamma_\varepsilon\}$ denoted by $(vp)_\Delta \phi [a,b]$ is said to be the null set of $\Delta \phi$ on $[a,b]$.

**Definition 3.2.** The set $[a,b] \setminus (vp)_\Delta \phi [a,b]$ denoted by $(vs)_\Delta \phi [a,b]$ is said to be the residual set of $\Delta \phi$ on $[a,b]$.

Accordingly, we are now in a position to define the notion of a residue of an interval function $F : I ([a,b]) \mapsto \mathbb{R}$ at $x \in [a,b]$.

**Definition 3.3.** An interval function $F : I ([a,b]) \mapsto \mathbb{R}$ is said to have a residue at $x \in [a,b]$ with residual value $R (x)$ if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon \equiv \delta_\varepsilon$, such that
\begin{equation}
(3.3) \quad |F (I) - R (x)| < \varepsilon,
\end{equation}
whenever $(x, I)$ is a $\delta_\varepsilon$-fine point-interval pair and $x$ is a point of $I \in I ([a,b])$.

A real-valued point function $R$, which is the limit of $F$ on $[a,b]$, is called a residual function of $F$ on $[a,b]$.

**Definition 3.4.** For $F : I ([a,b]) \mapsto \mathbb{R}$ let $E \subset [a,b]$ be its residual set. Then, the residual function $R$ of $F$ is said to be basically summable ($BS \delta_e$) on $E$ with the sum $R \in \mathbb{R}$, if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon \equiv \delta_\varepsilon$, such that $|\Sigma_F (P [a,b] |_E) - R| < \varepsilon$, whenever $P [a,b] |_E \subset P [a,b]$ and $P [a,b] \in \mathcal{P} [a,b]$ is a $\delta_\varepsilon$-fine partition. The residual function $R$ of $F$ is $BS \delta_e$ on $E$ if $E$ can be written as a countable union of sets on each of which $F$ is $BS \delta_e$. In symbols, $R = \sum_{x \in E} R (x)$.

**Remark 3.5.** By Definition 5.11 in [2], if $\Re = 0$ above, then $F$ has negligible variation on $E$. However, if there is a set $E \subset [a,b]$ of variation zero, then $F$ does not satisfy the variational Strong Lusin condition on $[a,b]$. Here $E$ is of variation zero if, given $\varepsilon > 0$ there is a gauge $\delta_\varepsilon \equiv \delta_\varepsilon$ such that $|\Sigma_I (P [a,b] |_E)| < \varepsilon$, whenever $P [a,b] |_E \subset P [a,b]$ and $P [a,b] \in \mathcal{P} [a,b]$ is $\delta_\varepsilon$-fine partition, [3]; on which $R$ of $F$ is $BS \delta_e$ with $\Re \neq 0$. On the other hand, since for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon$ such that $|F (I)| < \varepsilon$, whenever $(x, I)$ is a $\delta_\varepsilon$-fine point-interval pair tagged in $(vp)_F [a,b]$ and $x$ is a point of $I \in I ([a,b])$, it follows
immediately that $\mathcal{R} (x) \equiv 0$ on $(vp)_F [a, b]$. In addition, for a given pair of functions $F$ and $\mathcal{R}$, if $F$ is an additive function, and $\mathcal{R}$ vanishes identically on the whole interval $[a, b]$, then $F ([a, b]) = \sum_{x \in [a, b]} \mathcal{R} (x)$. So, if $F_{ex} : [a, b] \mapsto \mathbb{R}$ is the primitive defined by (1.3), then using the Newton-Leibniz formula we may obtain that for any compact interval $I \subset [a, b] \setminus E$

$$\sum_{x \in I} \mathcal{R} (x) = F (I) = \int_I f \, dx.$$ 

Therefore, if $E \subset [a, b]$ is a set of points of Lebesgue measure zero at which a real-valued function $F$ can take values $\pm \infty$ or not be defined at all and, in addition, $E$ is the residual set of the interval function $F_{ex} : \mathcal{I} ([a, b]) \mapsto \mathbb{R}$ associated to the point function $F_{ex} : [a, b] \mapsto \mathbb{R}$ ($E = (vs)_{F_{ex}} [a, b]$), then we can divide the infinite sum of all values of the null function $\mathcal{R}$ as a residual function of $F_{ex}$ on $[a, b]$ into two sums $\sum_{x \in (vp)_{F_{ex}} [a, b]} \mathcal{R} (x) = vp \int_a^b f \, dx$ and $\sum_{x \in E} \mathcal{R} (x)$, so that

$$F ([a, b]) = \sum_{x \in [a, b]} \mathcal{R} (x) = vp \int_a^b f \, dx + \sum_{x \in E} \mathcal{R} (x).$$

In what follows, we will prove the lemma that gives us this result explicitly. Clearly, if $vp \int_a^b f \, dx$ does not exist, then the right-hand side of the previous equation is reduced to the so-called indeterminate expression $\infty - \infty$ that actually have, in this situation, the real numerical value of $F ([a, b])$.

Now, we are in a position to define the total value $(vt)$ of the Riemann integral of a real-valued function $f$ with the primitive $F$ ($f$ is the limit of the interval function $\varphi : \mathcal{I} ([a, b]) \mapsto \mathbb{R}$ on $[a, b] \setminus E$, defined by $\varphi (I) = F_{ex} (I) / |I|$, where $E$ be a non-empty subset of $[a, b]$ of Lebesgue measure zero), [3].

**Definition 3.6.** For a compact interval $[a, b] \in \mathbb{R}$ let $E_f \subset [a, b]$ and $E_g \subset [a, b]$ be non-empty sets of Lebesgue measure zero, such that $E_f \cap E_g = \emptyset$. In addition, let $D_{ex} F : [a, b] \mapsto \mathbb{R}$ and $D_{ex} G : [a, b] \mapsto \mathbb{R}$ be defined according to (1.3) via any pair of real-valued functions $f$ and $g$, with their primitives $F$ and $G$, respectively, each of which is the limit of the corresponding interval function $\varphi (I) = F_{ex} (I) / |I|$ or $\gamma (I) = G_{ex} (I) / |I|$ on the corresponding set of points $[a, b] \setminus E_f$ or $[a, b] \setminus E_g$, respectively. The function $f$ is said to be totally Riemann integrable, with respect to $dG = gdx$, to a real number $\Im$ on $[a, b]$, if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon \equiv \delta_\varepsilon$, such that $|\Sigma_{\varphi G_{ex}} (P [a, b]) - \Im| < \varepsilon$, whenever $P [a, b] |_{E_f \cup E_g} \subset P [a, b]$ and $P [a, b] \in \mathcal{P} [a, b]$ is a $\delta_\varepsilon$-fine partition. In symbols, $\Im = vt \int_a^b f \, dG$.

Clearly, if $G = x$, then $vt \int_a^b f \, dx = F ([a, b])$, that is,

$$vt \int_a^b f \, dx = vp \int_a^b f \, dx + \sum_{x \in E_f} \mathcal{R} (x).$$

Our main result reads as follows.
exists a gauge $\xi$ if $P$ such that

$$\forall \xi > 0 \text{ there exists a gauge } \xi \text{ on } [a, b] $$

above. The constant function

$$\phi$$

Proof. Let $P$ be a non-empty subset of $[a, b]$ and $\phi$ be a non-empty subset of $[a, b]$ such that $\phi$ is Riemann integrable on $[a, b]$ and

$$f \text{ is Riemann integrable on } [a, b]$$

then $f$ is Riemann integrable on $[a, b]$ and

$$\phi$$

In addition, $\phi$ is Riemann integrable on $[a, b]$ and

$$f$$

Before starting with the proof we give the following lemma.

**Lemma 3.8.** Let $E$ be a non-empty subset of $[a, b]$. If a function $f$ with primitive $F$ (both are extended from $[a, b] \setminus E$ to $[a, b]$ by $D_{ex} F : [a, b] \mapsto \mathbb{R}$ and $F_{ex} : [a, b] \mapsto \mathbb{R}$, respectively) is totally Riemann integrable to the real number $\exists$ on $[a, b]$ and the null function $R$, as a residual function of $F_{ex} : \mathcal{I}([a, b]) \mapsto \mathbb{R}$ on $[a, b]$, is basically summable $(BS_{\delta_e})$ to the sum $R$ on $E$, then $f$ is Riemann integrable on $[a, b]$ and

$$\phi$$

$$f$$

Proof. Given $\varepsilon > 0$ we will construct a gauge for $f$ as follows. Since $f$ is the limit of $\phi$ on $[a, b] \setminus E$ it follows from Definition that for every $\varepsilon > 0$ there exists a gauge $\delta^*_\varepsilon \equiv \delta^*_{\varepsilon}$ on $[a, b]$ such that

$$f \text{ is Riemann integrable on } [a, b]$$

In addition, $f$ is totally Riemann integrable to the real number $\exists$ on $[a, b]$, so that for every $\varepsilon > 0$ there exists a gauge $\delta^*_\varepsilon \equiv \delta^*_{\varepsilon}$ on $[a, b]$ such that

$$\phi$$

choose a gauge $\delta^*_\varepsilon \equiv \delta^*_{\varepsilon}$ as required in Definition above. The constant function $\delta^*_\varepsilon(x) \equiv \delta^*_\varepsilon = \min \left( \delta^*_{\varepsilon} \right)$ is a gauge on $[a, b]$.

We now let $P[a, b] = \{(x_i, [a_i, b_i]) \mid i = 1, ..., \nu \}$ be a $\delta^*_\varepsilon$-fine partition of $[a, b]$ such that $P[a, b] \setminus E \subset P[a, b]$. It is readily seen that (remember $D_{ex} F(x) = 0$ if $x \in E$)
\[ \leq |\Xi_f (P[a,b] \setminus P[a,b]|_E) - (\Sigma_{F_{\varepsilon}} P[a,b] - \Sigma_{F_{\varepsilon}} P[a,b]|_E)| + \\
+ |[\Sigma_{F_{\varepsilon}} P[a,b] - \exists]| + |\Sigma_{F_{\varepsilon}} (P[a,b]|_E) - \Re| < (|[a,b]| + 2) \varepsilon . \]

Therefore, \( f \) is Riemann integrable on \([a,b]\) and

\[ \nu P \int_a^b f \, dx = \exists - \Re. \]

Remark 3.9. For an illustration of (3.7) we consider the Heaviside unit function defined by

\[ F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\
1, & \text{otherwise} \end{cases} . \]

Since \( \Sigma_{F} (P[a,b]) \equiv 1 \), whenever \( P[a,b] \in \mathcal{P}[a,b] \), it follows from Definition 3.6 that \( \nu P \int_a^b f \, dx = 1 \), where

\[ f(x) = \begin{cases} +\infty, & \text{if } x = 0 \\
0, & \text{otherwise} \end{cases} \]

is the derivative of \( F \) and \([a,b]\) is a compact interval within which is the null point. In addition, \( \nu P \int_a^b f \, dx = 0 \), so that \( \Re(0) = 1 \).

Let \([a,b]\) be as above. Consider the real-valued function \( F(x) = 1/x \) that is differentiable to \( f(x) = -\left(1/x^2\right) \) at all but the exceptional set \( \{0\} \) of \([a,b]\). In spite of the fact that \( f \) is not integrable (in the sense of the generalized Riemann integrals) on \([a,b]\), it follows from Definition 3.6 that \( \nu P \int_a^b f \, dx = (a-b)/(ab) \).

The residual function \( \Re \) of \( F \) is not defined at the point \( x = 0 \), that is

\[ \Re(x) = \begin{cases} +\infty, & \text{if } x = 0 \\
0, & \text{otherwise} \end{cases} . \]

Now, \( \nu P \int_a^b f \, dx \) is reduced to the so-called indeterminate expression \( \infty - \infty \) (here \( \nu P \int_a^b f \, dx = -\infty \)) that actually have, in this situation, the real numerical value of \((a-b)/(ab)\).

Let \( C : [0,1] \rightarrow \mathbb{R} \) be the Cantor function, [2]. Its derivative \( c \) is not defined on the Cantor set \( \mathcal{C} \). Since the Riemann integral of \( c \) (\( c_{\varepsilon} \) vanishes identically on \([0,1]\)) is equal to zero on \([0,1]\) (\( \nu P \int_0^1 c \, dx = 0 \)), it follows from Definition 5.1 and (3.7) that

\[ \Re = \nu P \int_0^1 c \, dx = C([a,b]) = 1, \]

where \( \Re = \sum_{x \in \mathcal{C}} \Re(x) \). So, the sum of the changes in the value of \( C \) over \( \mathcal{C} \) is reduced to the so-called indeterminate expression \( \infty \cdot 0 \) (the residue function \( \Re \) of \( C \) vanishes identically on \([0,1]\) because \( C \) is continuous on \([0,1]\)), that actually have, in this situation, the real numerical value of 1 (it means that \( C \) is not absolutely continuous and has no negligible variation on \( \mathcal{C} \)). Let’s prove it once more. For the Cantor function with the total length of 2 on \([0,1]\) the
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total length of all line segments contained within \([0, 1] \setminus C\), on each of which \(C\) is constant, is as follows

\[
\frac{1}{2} \sum_{n=1}^{+\infty} \left(\frac{2}{3}\right)^n = \frac{1}{2} (3 - 1) = 1.
\]

Hence, the sum of the changes in the value of \(C\) over \(C\), is equal to \(2 - 1\), meaning that \(\sum_{x \in C} R(x) = 1\).

We now turn to the proof of Theorem 3.7.

Proof. Fix some \(\varepsilon > 0\). By Definition 3.1, there exists a constant gauge \(\delta_\varepsilon^* (x) = \delta_\varepsilon^*\) such that \(|\Sigma_{\Delta \phi}(P [a, b] \setminus P [a, b] |E)| < \varepsilon\), whenever \(P [a, b] |E \subset P [a, b]\) and \(P [a, b] \in \mathcal{P} [a, b]\) is a \(\delta_\varepsilon\)-fine partition. If \(\delta_\varepsilon (x) \equiv \delta_\varepsilon^* = \min (\delta_\varepsilon^*, \delta_\varepsilon^*),\) where \(\delta_\varepsilon^*\) is a gauge as required in Definition 3.4, then \(|\Sigma_{\Delta \phi}(P [a, b]) - \mathcal{R}| < 2\varepsilon\), whenever \(P [a, b] |E \subset P [a, b]\) and \(P [a, b] \in \mathcal{P} [a, b]\) is a \(\delta_\varepsilon\)-fine partition. Therefore, it follows from Definition 3.6 that \(d\phi\) being the limit of \(\Delta \phi\) is totally Riemann integrable on \([a, b]\) and

\[
vt \int_a^b d\phi = \mathcal{R}.
\]

Finally, based on the result of Lemma 3.8.

\[
vp \int_a^b d\phi = 0.
\]

Remark 3.10. It is easy to see that the total Riemann integral has the linearity property. Hence, if \(\Delta \phi\) has negligible variation on \(E\), then

\[
vt \int_a^b d\phi = 0,
\]

that is,

\[
(3.9) \quad vt \int_a^b (fg) = vt \int_a^b gdf + vt \int_a^b fdg.
\]

Let \(f\) be the Peano kernel \(P (t, x)\) defined by (1.2) and let \(F\) be a real-valued function with the primitive \(F\). The corresponding interval function \(\Delta \phi\), defined by (3.1) for this pair of functions, has negligible variation on \(E \cup \{t\}\), where \(E \subset (a, b) \setminus \{t\}\), as the residual set of \(F\), is a set of points of Lebesgue measure zero, at which \(F\) can take values \(\pm \infty\) or not be defined at all. Since \(vt \int_a^b d(PF) = P (t, b) F (b) - P (t, a) F (a) = 0\) it follows that

\[
vt \int_a^b FdP + vt \int_a^b PdF = 0.
\]

By Definitions 3.4 and 3.6.
\[ vt \int_a^b F dP = \frac{1}{b-a} vt \int_a^b F dx - F(t), \]
taking into consideration the fact that the residue of the interval function \((FP)_{ex}(I)\) at the point \(t\) is \(-F(t)\). Hence,

\[(3.10) \quad F(t) = \frac{1}{b-a} vt \int_a^b F dx + vt \int_a^b P dF,\]

that represents totalization of the Montgomery identity \([\|1\|]\).

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**References**


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